Exercise 9

BODE DIAGRAM IDENTIFICATION PROCEDURE

1. Aim of the exercise

Determination of the transfer function of a system on the basis of Bode diagrams, which are obtained by the experimental frequency response measurement.

2. Theoretical introduction

Regardless of the fact whether the frequency response of a dynamic system is obtained experimentally or through knowledge of its sinusoidal transfer function, it is useful to depict graphically the response in an appropriate manner. One of the most useful techniques for depicting the frequency response of a system has been developed by Bode. The basic idea of the so-called **Bode diagrams** is to represent the frequency response of a system as two separate graphs. The first graph shows the logarithm of the response amplitude, the second one **-** the phase angle. Both are functions of the input frequency.

2.1. Minimum phase transfer function in the corner frequency factored form

The rational transfer function $G(s)$ is defined to be the minimum phase if all its poles and zeros are situated in the left half plane ($Re(s) \leq 0$). The minimum phase transfer function in the corner frequency factored form can be presented as follows:

$$G(s) = k\, s^n \frac{m_1(s)\, m_2(s) \ldots}{a_1(s)\, a_2(s) \ldots} \tag{9.1}$$

where: k – gain factor,

- n – integer representing the number of system zeros (if $n > 0$) or poles (if $n < 0$) at $s = 0$,
- $m_i(s)$ – zero factor,
- $a_i(s)$ – pole factor.

The corresponding sinusoidal transfer function has the form:

$$G(j\omega) = k\,(j\omega)^n \frac{m_1(j\omega)\, m_2(j\omega) \ldots}{a_1(j\omega)\, a_2(j\omega) \ldots} \tag{9.2}$$

Each complex term $m_i(j\omega)$ and $a_i(j\omega)$ in (9.2) can be expressed in the exponential form:

$$m_i(j\omega) = \left| m_i(j\omega) \right| e^{j\alpha_i}, \qquad a_i(j\omega) = \left| a_i(j\omega) \right| e^{j\beta_i}. \tag{9.3}$$

The low frequency factor $k\,(j\omega)^n$ can be expressed as:

$$k\,(j\omega)^n = k\omega^n j^n = k\,\omega^n e^{jn\frac{\pi}{2}} \tag{9.4}$$

Substituting (9.3) and (9.4) into (9.2), one receives:

$$G(j\omega) = \left| G(j\omega) \right| e^{j\phi} = k\,\omega^n \frac{\left| m_1(j\omega) \right| \left| m_2(j\omega \right| \ldots}{\left| a_1(j\omega) \right| \left| a_2(j\omega) \right| \ldots} e^{j(n\frac{\pi}{2} + \alpha_1 + \alpha_2 + \ldots - \beta_1 - \beta_2 - \ldots\ldots)} \tag{9.5}$$

Using Eq. (9.5), one can assign the amplitude of the sinusoidal transfer function in decibels as:

$$|G(j\omega)=|_{dB} \quad 20\log|G(j\omega)| = 20\log k + 20n\log\omega + 20 + 20\log|m_1(j\omega)| + \\ + 20\log|m_2(j\omega)| + \ldots - 20\log|a_1(j\omega)| - 20\log|a_2(j\omega)| - \ldots \qquad (9.6)$$

and the phase angle as:

$$\phi = n\frac{\pi}{2} + \alpha_1 + \alpha_2 + \ldots - \beta_1 - \beta_2 - \ldots \qquad (9.7)$$

Equations (9.6) and (9.7) express the amplitude and the phase angle of the frequency response of a system defined by the minimum phase transfer function in the form of a linear combination of the terms that are mutually independent.

2.2. Bode diagrams of low frequency, first order, and second order factors

Equation (9.4) shows the exponential form of the sinusoidal transfer function of the low frequency factor. The amplitude of (9.4) expressed in decibels is as follows:

$$|G_0(j\omega)|_{dB} = 20\log k\omega^n = 20\log k + 20n\log\omega \qquad (9.8)$$

This amplitude assumes the values shown in Table 9.1. Therefore, on the log scale, $|G_o(j\omega)|_{dB}$ is plotted as a straight line with the slope 20n dB/dec (Fig. 9.1). The phase angle of the low frequency factor $\phi_0 = n\pi/2$.

The first-order factor in the corner frequency ω_1 factored form is expressed as:

$$G_1(s) = (Ts+1) = \left(\frac{s}{\omega_1} + 1\right). \qquad (9.9)$$

Table 9.1. Amplitude values of the low frequency factor

Frequency ω	Amplitude $\lvert G(j\omega)\rvert_{dB}$
0.01	20 log k – 40 n
0.1	20 log k – 20 n
1.0	20 log k
10.0	20 log k + 20 n
100.0	20 log k + 40 n

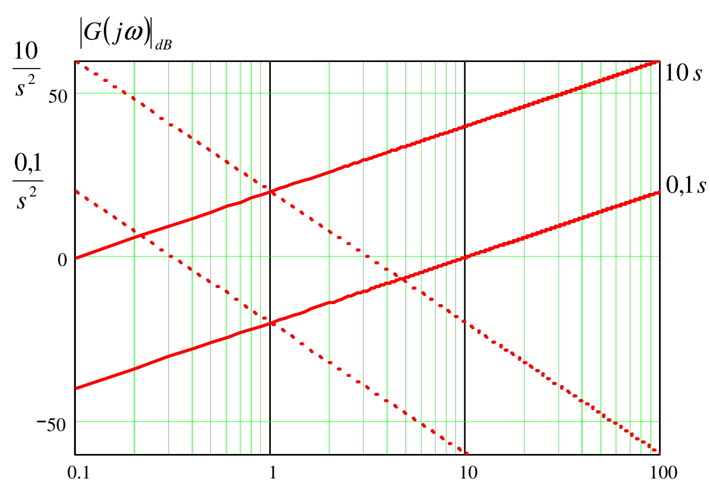

Fig. 9.1. Amplitude Bode diagrams of $G_o(s) = ks^n$ when k = 0.1 and 10 for both n = 1 (solid lines) and n = -2 (dotted lines)

The corresponding sinusoidal transfer function in the exponential form is as follows:

$$G_1(j\omega)=G_1(s)\big|_{s=j\omega}=\left|G_1(j\omega)\right|e^{j\phi_1} = \sqrt{1+\frac{\omega^2}{\omega_1^2}}\,e^{j(\tan^{-1}\frac{\omega}{\omega_1})} \qquad (9.10)$$

In this case, $|G_1(j\omega)|_{\text{dB}}$ can be approximated by two straight line segments that are called the asymptotes of the actual amplitude graph. The low frequency asymptote (when $\omega << \omega_1$) is:

$$\left|G_1(j\omega)\right|=1 \quad \Rightarrow \quad \left|G_1(j\omega)\right|_{dB}=20\log 1=0\ dB\,. \qquad (9.11)$$

Thus, the low frequency asymptote of the first-order factor plots the horizontal line at 0 dB (Fig. 9.2). The high frequency asymptote (when $\omega >> \omega_1$):

$$\left|G_1(j\omega)\right|=\frac{\omega}{\omega_1} \quad \Rightarrow \quad \left|G_1(j\omega)\right|_{dB}=20\log\omega-20\log\omega_1\,\cdot \qquad (9.12)$$

Hence, the high frequency asymptote plots a straight line with the positive slope of 20 dB/dec that passes through the 0 dB level at $\omega = \omega_1$ (Fig. 9.2). The intersection of these two asymptotes occurs on the 0 dB level at $\omega = \omega_1$, which is called the corner frequency of the first-order factor. Using Eq. (9.10), the phase angle ϕ_1 increases continuously from 0^o at $\omega = 0$ to 45^o when $\omega = \omega_1$ and to 90^o as $\omega \to \infty$. Since $(Ae^{j\phi})^{-1} = (1/A)\,e^{-j\phi}$ and $20\log(1/A) = -20\log A$, the amplitude and phase Bode graphs of the first-order pole factor $(Ts+1)^{-1}$ plot as mirror images of the corresponding graphs of the first-order zero factor $(Ts+1)$.

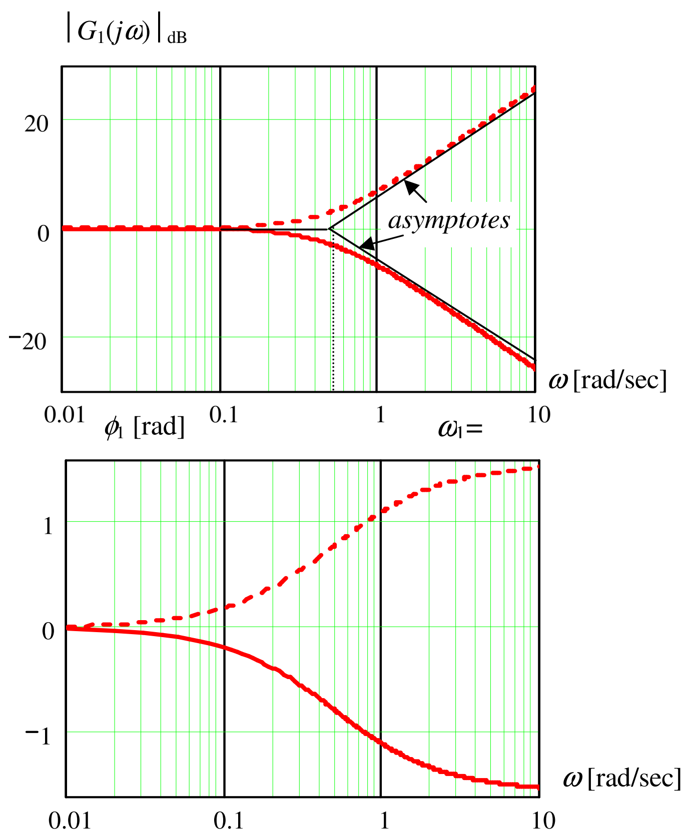

Fig. 9.2. Actual and asymptotic Bode graphs of the zero (dashed line) and pole (solid line) first order factor (ω_1 = 0.5 rad/sec)

The second-order factor in the corner frequency ω_2 factored form is expressed as:

$$(T^2s^2+2T\xi s+1)=\left(\frac{s^2}{\omega_2^2}+\frac{2\xi s}{\omega_2}+1\right)\cdot \qquad (9.13)$$

The corresponding sinusoidal transfer function in the exponential form is as follows:

$$G_2(j\omega) = |G_2(j\omega)| e^{j\phi_1} = \sqrt{\left(1-\frac{\omega^2}{\omega_2^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_2^2}}\, e^{j\left(\tan^{-1}\frac{-2\xi\frac{\omega}{\omega_2}}{1-\frac{\omega^2}{\omega_2^2}}\right)}. \quad (9.14)$$

The amplitude can be written as:

$$|G_2(j\omega)|_{dB} = 20\log\sqrt{\left(1-\frac{\omega^2}{\omega_2^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_2^2}}. \quad (9.15)$$

The low frequency asymptote (when $\omega << \omega_2$) takes the form:

$$|G_2(j\omega)| = 1 \quad \Rightarrow \quad |G_2(j\omega)|_{dB} = 20\log 1 = 0\ dB. \quad (9.16)$$

Thus, the low frequency asymptote of the second-order factor plots a horizontal line at 0 dB (Fig. 9.3). The high frequency asymptote (when $\omega >> \omega_1$) is then:

$$|G_2(j\omega)| = \frac{\omega^2}{\omega_2^2} \quad \Rightarrow \quad |G_2(j\omega)|_{dB} = 40\log\omega - 40\log\omega_2. \quad (9.17)$$

The high frequency asymptote plots a straight line with the positive slope of 40 dB/dec that passes through the 0 dB level at $\omega = \omega_2$ (Fig. 9.3). The intersection of these two asymptotes occurs on the 0 dB line at $\omega = \omega_2$. In the light of Eq. (9.14), the phase angle ϕ_2 increases continuously from 0^o at $\omega = 0$ to $\pi/2$ when $\omega = \omega_2$ and to π as $\omega \to \infty$. Similarly as in the case of the first-order factor, the amplitude and phase Bode graphs of the second-order pole factor plot as mirror images of the corresponding graphs of the second-order zero factor.

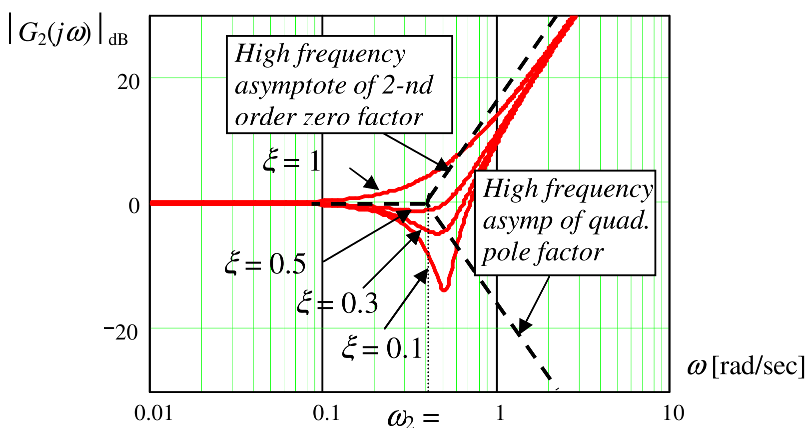

Fig. 9.3. Actual (solid lines) and asymptotic (dashed lines) amplitude Bode graphs of the second-order factor (ω_1 = 0.5 rad/sec)

2.3. Example of the determination of the transfer function on the basis of actual Bode diagrams

In this example, one should determine the transfer function of an unknown minimum phase dynamic system on the basis of its Bode diagrams (Fig. 9.4), which are obtained by experimental frequency response measurements.

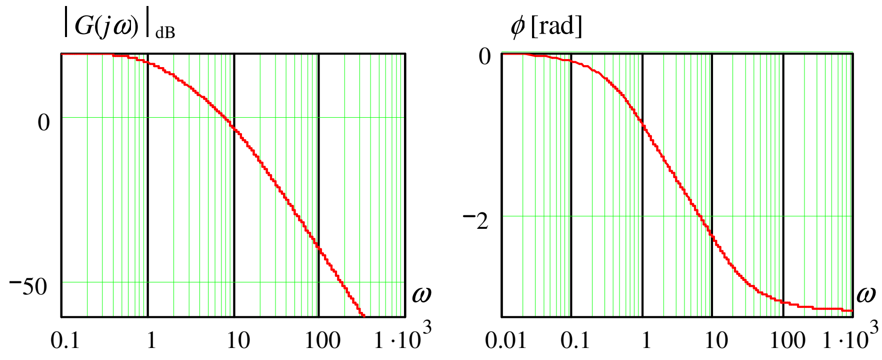

Fig. 9.4. Actual Bode graphs of an unknown system

The solution is presented in the graphic form in Fig. 9.5.

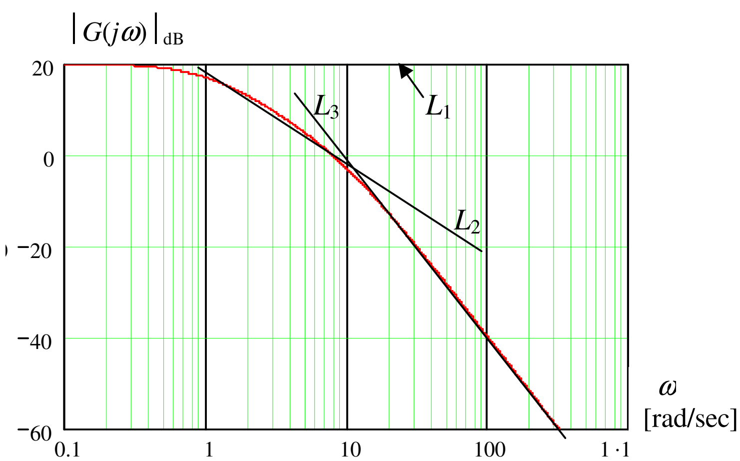

Fig. 9.5. Amplitude Bode graph approximation

At first, one should approximate the actual amplitude Bode graph by an asymptotic amplitude plot. Figure 9.5 shows this approximation with three straight lines: L_1 (initial low frequency asymptote), L_2 and L_3. From the location of the initial low frequency asymptote, the gain factor value k and the integer n, representing the number of system zeros, can be determined. The intersection of two consecutive asymptotes indicates the corner frequency. From the change in the slope of two consecutive asymptotes at the corner frequency, one can identify the factor type. The dimensionless damping coefficient ξ for each quadratic factor can be assigned from the error between the actual and asymptotic amplitude values at the corner frequency. Table 9.2 shows the investigation results.

Table 9.2. Investigation results

Asymptote	*Corner freq. [rad/sec]*	*New slope [dB/dec]*	*Slope change [dB/dec]*	*Factor type*	*Transfer function*
L_1	0	0	no change	$n = 0$	$k = 10$
L_2	1	-20	-20	first-order pole	$(s+1)$
L_3	10	-40	-20	first-order pole	$(0.1s+1)$

Finally, the transfer function of the system has the following form:

$$G(s) = 10\frac{1}{(s+1)\left(\frac{s}{10}+1\right)} = \frac{100}{s^2+11s+10} \tag{9.18}$$

3. Course of the exercise

Using the MATLAB software package and the Bode diagram identification procedure, determine the transfer function of an unknown, minimum phase dynamic system. The system is characterized by the Bode diagrams, which have been obtained by experimental frequency response measurements. The experimental investigation results will be presented during the exercise.

4. Laboratory report should contain:

1. Aim of the exercise.
2. Experimental and computational results of the frequency response data.
3. Actual Bode graphs.
4. Asymptotic amplitude plot in the graphic form.
5. Investigation results recorded in the table.
6. Transfer function of the system.
7. Phase angle Bode diagram in the graphic form.
8. Conclusions and remarks.

References

1. Wolovich W.A.: *Automatic Control Systems, Basic Analysis and Design,* Harcour Brace College Publishers, 1994.
2. Ogata K.: *Modern Control Engineering,* IV-th Edition, Prentice Hall, 2002.
3. Friendland B.: *Control System Design,* McGraw-Hill Book Company, 1987.