DETERMINATION OF THE PERIOD OF SMALL AND LARGE OSCILLATIONS OF A PHYSICAL PENDULUM

International Faculty of Engineering – BASIC MECHANICS – Exercise 5 *

1 Exercise Objective

The objective of the exercise is to measure the oscillation period of a physical pendulum and compare the results obtained from the experiment with the results of theoretical calculations. The calculations are performed analytically — based on the solution of the equation describing arbitrarily large movements of the pendulum (using elliptic integrals) and as using the approximate equation of motion describing small oscillations. These results are compared with measurements at the experimental stand.

2 Introduction

A physical pendulum is defined as a rigid body rotating around a fixed horizontal axis in a uniform gravitational field[1]. The axis of rotation cannot pass through the center of gravity of the pendulum. The analysis of physical pendulum motion is a classic task in mechanics. The pendulum can serve as an example from the field of dynamics of a body moving in rotational motion. It is also used as an example of harmonic motion and harmonic vibrations.

Considering the dynamic equations of motion of a body, one can — using the pendulum as an example — quickly show how the equations of motion of a system are derived using Lagrange's equations, and what the linearization of a mathematical model consists of[2]. Including motion resistance (friction in bearings, air resistance) complicates the equations of motion but allows for the illustration of a number of other phenomena (oscillation decay, quasi-periodic motion). Measuring the oscillation period of a body around a specific axis allows for the experimental determination of the body's moment of inertia relative to that axis (this is a simple and accurate method for determining the moment of inertia). In machine dynamics analysis, the physical pendulum is used to model the movement of a suspended load (e.g., in crane calculations).

3 Basic Theoretical Relationships

The pendulum model (Fig. 1) rotating around the y axis is considered. Its position is described by the angle φ. The following symbols were used to denote the body's parameters:

- J_y – mass moment of inertia relative to the axis of oscillation (y),
- m – total mass of the pendulum,
- s – distance of the center of gravity (point C) of the pendulum from the axis of oscillation.

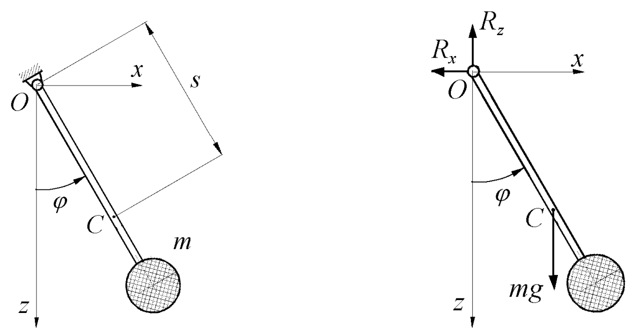


Figure 1: Physical pendulum

*Author: B. Mianowski, in – K. Januszkiewicz, J. Grabski: Ćwiczenia Laboratoryjne z Mechaniki, Łódź 2008

[1] A rigid body on which geometric constraints are imposed allowing for the rotation of the body around a point is called a physical pendulum. Various other types of systems called pendulums are also interesting, like the Foucault one (the Coriolis acceleration resulting from the Earth's rotational motion is included in its equations of motion), the elliptical pendulum (one whose suspension point moves in harmonic motion), or sympathetic pendulums (two identical mathematical pendulums connected by a spring).

[2] This refers to replacing non-linear equations of motion with linear equations.

3.1 Equations of Motion of a Physical Pendulum

The equation of motion of a physical pendulum (Fig. 5.1) has the form

$$J_y \ddot{\varphi} = -mgs \sin \varphi. \tag{1}$$

or

$$\ddot{\varphi} + \frac{mgs}{J_y} \sin \varphi = 0. \tag{2}$$

Equation (2) is a *non-linear differential equation.* The analytical solution of such an equation can be determined using elliptic integrals. This method of solution is discussed in section 3.3. This equation can also be solved through numerical integration. Assuming that for small angles φ the function $\sin \varphi$ can be replaced by the angle φ expressed in radians ($\sin \varphi \approx \varphi$), the equation of motion of the pendulum (2) can be presented in the form

$$\ddot{\varphi} + \frac{mgs}{J_y} \varphi = 0. \tag{3}$$

This form is a *linear differential equation*, which describes the movement of the pendulum at small displacements (meaning not exceeding a few degrees) from the equilibrium position. The general solution to this equation is a harmonic function of the form

$$\varphi = a \sin(\omega_0 t + \alpha_0). \tag{4}$$

3.2 Period of Small Oscillations of a Physical Pendulum

Function (4) is a periodic function with a period $T = \frac{2\pi}{\omega_0}$, where

$$\omega_0 = \sqrt{\frac{mgs}{J_y}}. \tag{5}$$

The period of physical pendulum oscillations for small displacement angles is therefore

$$T = 2\pi \sqrt{\frac{J_y}{mgs}} = 2\pi \sqrt{\frac{l_{\text{red}}}{g}}, \tag{6}$$

where: l_{red} – is called the reduced length of the pendulum ($l_{\text{red}} = \frac{J_y}{ms}$). The reduced length for a physical pendulum can be presented in the form

$$l_{\text{red}} = \frac{J_y}{ms} = \frac{J_{yC} + ms^2}{ms} = \frac{J_{yC}}{ms} + s, \tag{7}$$

where J_{yC} denotes the moment of inertia of the body relative to the central axis (passing through the center of gravity C). An example plot of the function $l_{\text{red}}(s)$ is presented in Fig. 2 (for given values of J_{yC} and m). This function has a local minimum, so for every physical pendulum, such a relative position of the axis of rotation and the center of gravity can be determined for which the oscillation period T reaches a minimum.

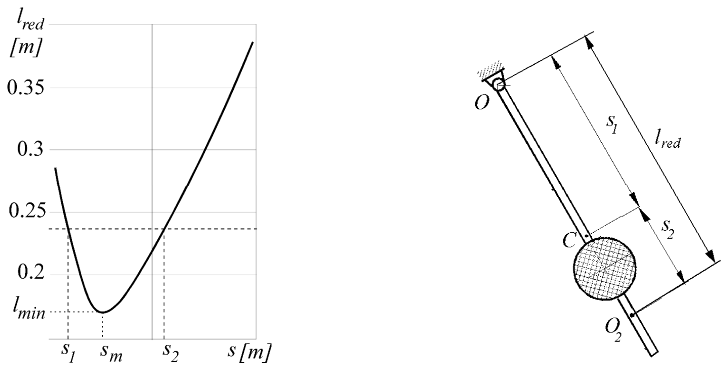


Figure 2: Plot of changes in the reduced length of the pendulum (l_{red}) as a function of the distance of the pendulum's center of gravity from the axis of rotation (s)

From the analysis of the function $l_{\text{red}} = f(s)$, the following conclusions arise:

- The minimum period of physical pendulum oscillations is defined by the relationship

$$T_{\text{min}} = 2\pi \sqrt{\frac{2J_{yC}}{mg}}.$$

- The minimum reduced length of the pendulum corresponds to the minimum oscillation period

$$l_{\text{red min}} = 2\sqrt{\frac{J_{yC}}{m}}.$$

- The minimum oscillation period of the pendulum occurs when the distance of the body's center of gravity from the rotation axis is

$$s_m = \frac{1}{2} l_{\text{red min}} = \sqrt{\frac{J_{yC}}{m}}.$$

If the distance s is different from s_m ($s \neq s_m$), then for each reduced length ($l_{\text{red}} > l_{\text{red min}}$) there are two values of distance s ($s = s_1$ and $s = s_2$, where $s_1 + s_2 = l_{\text{red}}$) for which the oscillation period is identical. Thus, a body suspended on an axis passing through point O_2 has the same oscillation period as in the case of an axis passing through point O and is called an inverted or reversible pendulum.

3.3 Period of Large Oscillations of a Physical Pendulum – Analytical Solution

The equation of motion of the pendulum at arbitrarily large displacements has the form given by equation (1), meaning with the non-linear sine function it has the form

$$J_y \frac{d^2\varphi}{dt^2} = -mgs \sin\varphi. \tag{8}$$

Introducing the reduced length of the physical pendulum $l_{\text{red}} = \frac{J_y}{ms}$, the equation of motion can be presented in the form

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l_{\text{red}}} \sin\varphi. \tag{9}$$

Utilizing the following relationship

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\varphi}\frac{d\varphi}{dt} = \frac{d\omega}{d\varphi}\omega = \frac{1}{2}\frac{d}{d\varphi}(\omega^2), \tag{10}$$

equation (9) can be presented as

$$\frac{d}{d\varphi}(\omega^2) = -\frac{2g}{l_{\text{red}}} \sin\varphi. \tag{11}$$

Separating the variables

$$d(\omega^2) = -\frac{2g}{l_{\text{red}}} \sin\varphi \, d\varphi \tag{12}$$

and integrating both sides of the equation yields

$$\omega^2 = \frac{2g}{l_{\text{red}}} \cos\varphi + 2h, \tag{13}$$

where the integration constant has been denoted as $2h$ ($2h = \text{const}$). Assuming the initial conditions (for $t = 0$) of the form: $(\omega)_{t=0} = 0$, $(\varphi)_{t=0} = \varphi_0$ (Fig. 3) and determining the integration constant for these conditions yields

$$2h = -\frac{2g}{l_{\text{red}}} \cos\varphi_0. \tag{14}$$

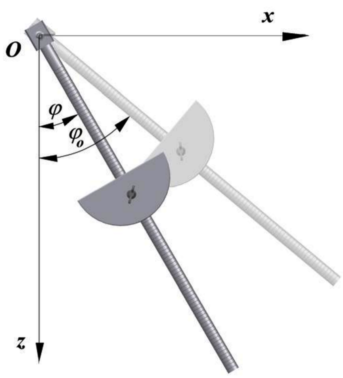


Figure 3: Initial position (φ_0) and arbitrary position (φ) of the pendulum

The equation defining the square of the pendulum's angular velocity is thus

$$\omega^2 = \left(\frac{d\varphi}{dt}\right)^2 = \frac{2g}{l_{\text{red}}}(\cos\varphi - \cos\varphi_0). \tag{15}$$

To determine the period of large oscillations of the pendulum, it is necessary to solve the equation

$$\frac{d\varphi}{dt} = \pm\sqrt{\frac{2g}{l_{\text{red}}}(\cos\varphi - \cos\varphi_0)}. \tag{16}$$

This equation is also solved using the method of separation of variables – hence

$$dt = \pm\sqrt{\frac{g}{l_{\text{red}}}}\frac{d\varphi}{(2\cos\varphi - \cos\varphi_0)}. \tag{17}$$

Integrating the left side within the limits $(0, t)$, and the right side within the limits (φ_0, φ) yields

$$t = \pm\sqrt{\frac{l_{\text{red}}}{g}}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{2(\cos\varphi - \cos\varphi_0)}}. \tag{18}$$

Since for the pendulum the angle φ changes within the limits $(-\varphi_0, +\varphi_0)$, the expression $(\cos\varphi - \cos\varphi_0)$ takes positive values $((\cos\varphi - \cos\varphi_0) > 0)$, and the angle in the first phase of motion will decrease (Fig. 5.3). Therefore, the minus sign should be placed before the root

$$t = -\sqrt{\frac{l_{\text{red}}}{2g}}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\sqrt{2(\cos\varphi - \cos\varphi_0)}}. \tag{19}$$

In formula (19), the following identities can be used:

$$\cos\varphi = 1 - 2\sin^2\frac{\varphi}{2}, \qquad \cos\varphi_0 = 1 - 2\sin^2\frac{\varphi_0}{2} \tag{20}$$

and a new variable ψ can be introduced

$$\sin\psi = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}. \tag{21}$$

Determining the total differential of expression (21) yields

$$\cos\psi\, d\psi = \frac{1}{2}\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}\, d\varphi, \tag{22}$$

hence

$$d\varphi = 2\frac{\sin\frac{\varphi_0}{2}}{\cos\frac{\varphi}{2}}\cos\psi\, d\psi. \tag{23}$$

When applying the change of variables, new integration limits must be established (based on relationship (21)):

- lower limit (after substituting $\varphi = \varphi_0$)

$$(\sin\psi)_{\varphi=\varphi_0} = \frac{\sin\frac{\varphi_0}{2}}{\cos\frac{\varphi_0}{2}} = 1, \tag{24}$$

hence

$$\psi_{\varphi=\varphi_0} = \frac{\pi}{2}, \tag{25}$$

- upper limit (after substituting $\varphi = 0$)

$$(\sin\psi)_{\varphi=0} = \frac{0}{\sin\frac{\varphi_0}{2}} = 0, \tag{26}$$

then

$$(\psi)_{\varphi=0} = 0. \tag{27}$$

Taking the above into account in relationship (19), after transformations, one obtains

$$t = -\sqrt{\frac{l_{\text{red}}}{g}}\int_{\pi/2}^{0}\frac{d\psi}{\sqrt{1 - \sin^2\frac{\varphi_0}{2}\sin^2\psi}}, \tag{28}$$

or changing integration boundaries

$$t = \sqrt{\frac{l_{\text{red}}}{g}}\int_{0}^{\pi/2}\frac{d\psi}{\sqrt{1 - \sin^2\frac{\varphi_0}{2}\sin^2\psi}}, \tag{29}$$

after introducing

$$k = \sin\frac{\varphi_0}{2}. \tag{30}$$

we get

$$t = \sqrt{\frac{l_{\text{red}}}{g}}\int_{0}^{\psi}\frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}}, \tag{31}$$

An integral of the form

$$\int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \tag{32}$$

is called an *elliptic integral of the first kind in Legendre's form.* In the case where the upper integration limit is $\pi/2$, the discussed integral is called a *complete elliptic integral of the first kind*

$$F\left(\frac{\pi}{2}\middle|k\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \equiv K\,. \tag{33}$$

It does not have any elementary solution. Its numerical values are given in tables of elliptic integrals as functions of the parameter k. The sought solution to equation (26), which is the solution to the pendulum's equation of motion (9) with respect to the variable t, can thus be presented as

$$t = \sqrt{\frac{l_{\rm red}}{g}}\,K. \tag{34}$$

The time t determined in this way for the angle φ changing within the limits $(\varphi_0, 0)$ corresponds to 1/4 of the oscillation period. Thus, **the period of large oscillations of the pendulum** τ ($\tau = 4t$) is expressed by the relationship

$$\tau = 4K\sqrt{\frac{l_{\rm red}}{g}} \quad = 4K\sqrt{\frac{J_y}{mgs}}. \tag{35}$$

In calculations, the expansion of the function $K = f(k)$ into a series with respect to parameter k can be used

$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \dots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n} + \dots\right]. \tag{36}$$

By limiting the expansion to the first few terms ($n = 3$) and using (33), the following formula for rapid practical use is obtained

$$K \approx \frac{\pi}{2}\left[1 + \frac{1}{4}\sin^2\frac{\varphi_0}{2} + \frac{9}{64}\sin^4\frac{\varphi_0}{2} + \frac{25}{256}\sin^6\frac{\varphi_0}{2}\right]. \tag{37}$$

In Figure 4, the values of K obtained from the approximate (33) and exact relationships were compared.

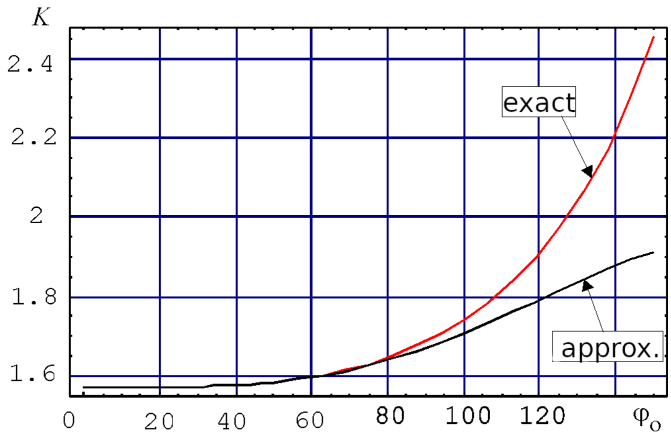


Figure 4: Exact and approximate value of the elliptic integral K

It can be seen that for a displacement angle smaller than 80° the differences between the two values are imperceptible.

4 Description of the Measurement Station

A view of the station where the pendulum oscillation period measurements are carried out is shown in Fig. 5.

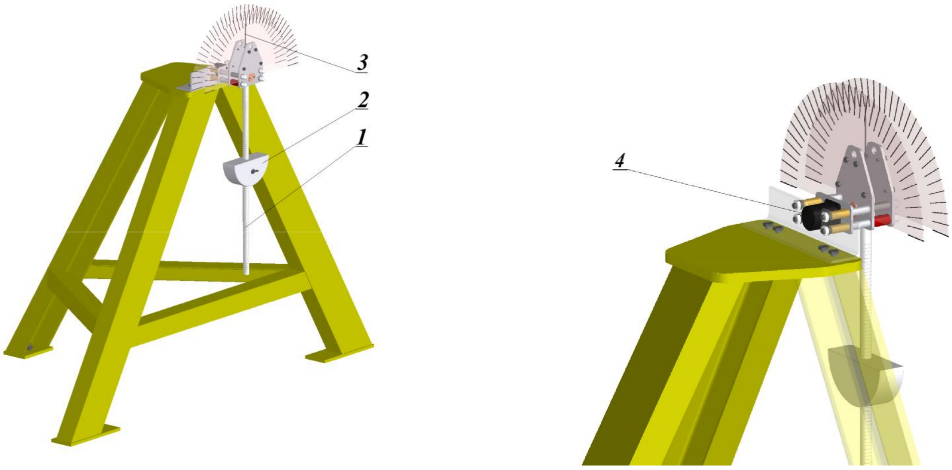


Figure 5: View of the measurement station

The main part of the station is a rod (1) placed on a horizontal axis. Additional bodies can be fixed on the rod — a semi-cylinder (2) or a cylinder (Fig. 6). A pointer (3) is used to read the pendulum's displacement angle from the vertical. An optical sensor (4) cooperating with a digital time meter is attached to the pendulum's axis, which enables the measurement of the oscillation period.

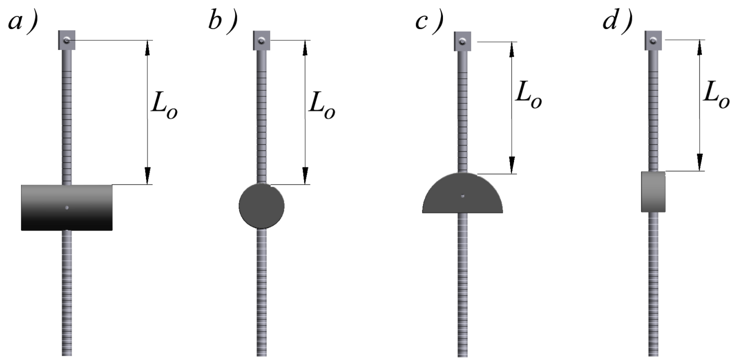


Figure 6: Pendulum configurations with different moments of inertia depending on the position of additional bodies

4.1 Moments of Inertia of the Pendulum

The period of physical pendulum oscillations depends on the distribution of the pendulum's mass and the position of the oscillation axis relative to the center of mass. Period measurements are carried out for several pendulum variants — differing in mass and moment of inertia (Fig. 6). The change in the pendulum's moment of inertia while maintaining an unchanged position of the center of gravity relative to the oscillation axis is achieved by rotating the body fixed on the pendulum rod by 90°.

4.1.1 Dimensions and Masses of Pendulum Components

Bar: $l = 0.540$ m, $s = 0.25$ m, $m = 0.922$ kg.
Cylinder: $H = 0.15$ m, $R = 0.0375$ m, $r' = 0.008$ m, $h = 2R = 0.075$ m, $m = 5.08$ kg.
Semi-cylinder: $H = 0.04$ m, $R = 0.066$ m, $r = 0.008$ m, $h = R = 0.066$ m, $m = 2.04$ kg.

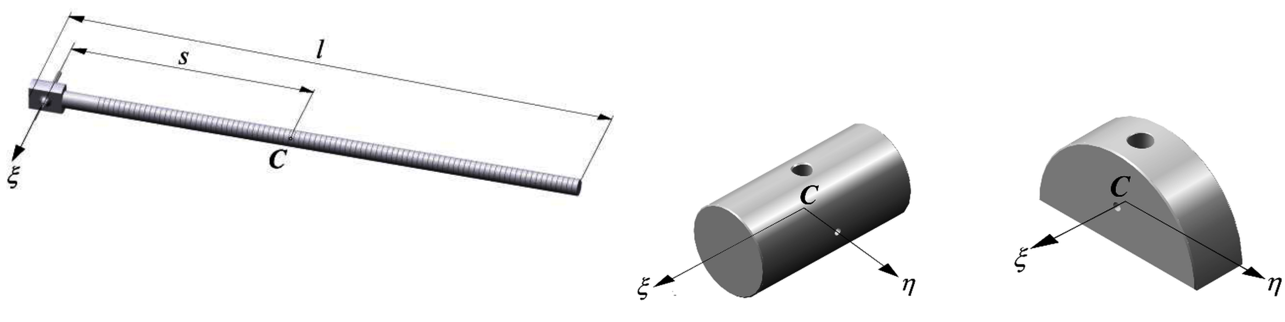


Figure 7: Pendulum components with different moments of inertia

4.1.2 Calculation of Moments of Inertia

For each of the studied pendulum variants (shown in Fig. 6), its moment of inertia relative to the oscillation axis y must be determined. The calculation of the moment of inertia relative to the oscillation axis y for a pendulum with a fixed body (cylinder or semi-cylinder) is carried out for the current position of the body (distance L_0 in Fig. 6). When determining the moments of inertia, the Steiner theorem should be used, $J = J_{CA} + mL_0^2$, where $J_{CA} = J_\xi$ or J_η. The moments of inertia of the rod, cylinder, and semi-cylinder were given in Table 1 (in the calculation of these quantities, the cylindrical cutout — the drilled hole — was taken into account).

Table 1: Summary of moment of inertia values for pendulum components

Component	m [kg]	J_ξ [kg m^2]	J_η [kg m^2]	Comments – see Fig. 7
bar	0.922	$80 \cdot 10^{-3}$	—	the ξ axis is not the central axis for the rod
cylinder	5.08	$3.60 \cdot 10^{-3}$	$11.52 \cdot 10^{-3}$	ξ, η are central axes (CA)
semi-cylinder	2.04	$2.92 \cdot 10^{-3}$	$0.87 \cdot 10^{-3}$	ξ, η are central axes (CA)

5 Measurements and Report

Before starting the measurements, the variants are be selected from those presented in Figure 8. Below are the subsequent activities to be performed during the exercise.

1. Turn on the power supply for the oscillation period sensor connected to the time meter.
2. Assemble the pendulum according to the variants given below:

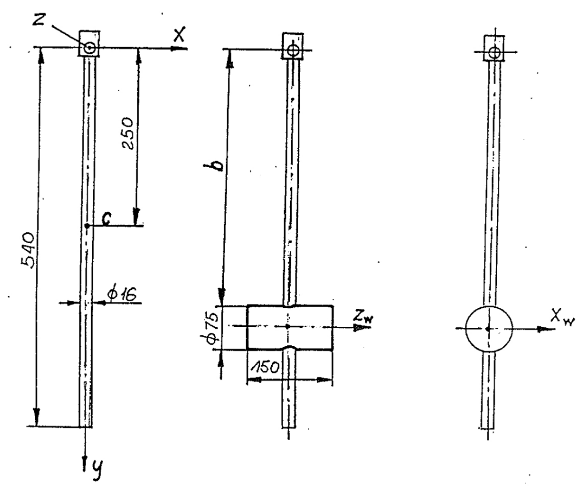


Figure 8: Pendulum configuration with different moments of inertia

   (a) Move the pendulum rod by an angle of 5° and release freely (initial velocity=0).
   (b) Read the period meter indication T_{exp} and record it in Table 2. Repeat measurements for initial displacements of 10°, 30°, and 45°.
   (c) According to the measurement plan, fix a cylinder or semi-cylinder on the pendulum rod. Record the distance L_0 (determine it based on the notches on the rod, made every 5 mm; the first notch is at a distance of 50 mm from the oscillation axis).
   (d) Perform measurements for other initial displacements and record the results.

3. Enter the required numerical data and perform calculations. Record the calculation results displayed on the screen (pendulum mass m, distance of the center of mass s, pendulum moment of inertia J_y, reduced length l_{red}, oscillation period T) respectively in appropriate columns of Table 2.
4. Using formulas (27), (31), and (33) and any spreadsheet, calculate the period values for the case of non-linear vibration theory, record in Table 2.

After finishing the exercise, remove the fixed body from the rod, turn off the sensor power supply, the meter, and the computer, and tidy up the measurement station.

5.1 Auxiliary Calculations

After finishing the measurements, calculate the oscillation periods according to linear theory, formula (6), and record them in Table 2. Next, for the selected modified pendulum variant, determine the minimum oscillation period and the corresponding distance of the oscillation axis from the center of gravity (relevant formulas in section 3.2). Compare the results obtained with those contained in report Table, relating to the given pendulum variant; mark the position of the oscillation axis and the pendulum's center of gravity on the diagram.

The report should include: a summary of measurement results in Table 2 and theoretical calculations, and conclusions resulting from the measurements and calculations carried out.

6 Review Questions

1. What is the difference between a physical pendulum and a mathematical pendulum?
2. What is the reduced length of a physical pendulum?
3. What is a reversible pendulum?
4. Derive the equation describing pendulum motion using the Newton-Euler method.
5. Why are the cases of large and small pendulum displacements distinguished?
6. Does the oscillation period depend on the pendulum's displacement angle?
7. In what case can pendulum motion be treated as harmonic motion?

LABORATORY OF MECHANICS

Exercise 5

DETERMINATION OF THE PERIOD OF SMALL AND LARGE OSCILLATIONS OF A PHYSICAL PENDULUM

Group: ______________________

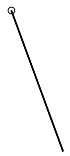
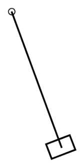
Team: ______________________

Date ______________________

Student names:

1. ______________________
2. ______________________
3. ______________________
4. ______________________
5. ______________________
6. ______________________

Determination of the period of small and large oscillations of a physical pendulum – report

Variant	mass		moment of inertia acc. to rot. axis		body pos.	l-red.	dist. O–CG	initial angle	PERIOD			
	bar	att. body	bar	att. body $J = J_{CA} + mL_0^2$	L_0	l_{red}	s	ϕ_0	theory			exp.
	m_B	m_B	J_ξ / J_η	J_y				[°]	T_{Lin}	K	T_{NonLin}	T_{exp}
	[kg]	[kg]	[kgm^2]	[kgm^2]	[m]	[m]	[m]		[s]	[–]	[s]	[s]
		X		X	X			5				
		X		X	X			10	X			
		X		X	X			30	X			
		X		X	X			45	X			
								5				
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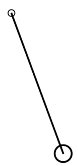								5				
								10	X			
								30	X			
								45	X			

Variant 1: pendulum rod only; Variant 2: Rod + body in the plane of pendulum motion; Variant 3: Rod + body placed transversely to the plane of pendulum motion

bar	$m = 0.922$ kg	$l = 0.540$ m	$s = 0.25$ m			$J_O = 80.0 \cdot 10^{-3}$ kgm^2		axis is not the central axis
cylinder	$m = 5.08$ kg	$H = 0.15$ m	$R = 0.0375$ m	$r' = 0.008$ m	$h = 2R = 0.075$ m	$J_\eta = 11.52 \cdot 10^{-3}$ kgm^2	$J_\xi = 3.60 \cdot 10^{-3}$ kgm^2	ξ, η axes are central axes (CA)
semi-cyl.	$m = 2.04$ kg	$H = 0.04$ m	$R = 0.066$ m	$r = 0.008$ m	$h = R = 0.066$ m	$J_\eta = 2.92 \cdot 10^{-3}$ kgm^2	$J_\xi = 0.87 \cdot 10^{-3}$ kgm^2	ξ, η axes are central axes (CA)

Conclusions: