VIBRATIONS OF A MECHANICAL SYSTEM WITH 1 DEGREE OF FREEDOM

International Faculty of Engineering - BASIC MECHANICS – Exercise 11 *

1 Exercise Objective

The aim of this experiment is to analytically determine the natural frequency (period) of a single-degree-of-freedom system and experimentally verify these values. The experimental setup allows for the observation of natural vibration phenomena, as well as the generation of forced vibrations of a linear mechanical system. Measuring the frequency (period) of forced vibrations in a resonant state allows for understanding the possible consequences of such a state. A non-contact optical displacement sensor and an oscilloscope are used in the measurements.

2 Introduction

Vibrations of a mechanical system are defined as movements around the stable equilibrium position of that system. If a restoring force proportional to the displacement appears, the vibrations are called harmonic. In particular, any elastic system obeying Hooke's law (for example, a mass suspended on a spring, a flexible beam, or a stretchable wire) can perform such movement. The ability to analyze vibrations of various systems is of great practical importance. This exercise concerns the study of vibrations of a simple linear mechanical system consisting of a body suspended on a cylindrical helical spring, whose mass is small compared to the mass of the body. Both free vibrations and those forced by a variable force will be studied. Measurements of quantities characterizing the vibrations are made electrically. Measurement results will be collected in the Table 1 of the report. The natural frequency and period values obtained from them will be compared there with results obtained from the theoretical model.

3 Theoretical Description of the Phenomenon

3.1 Free Vibrations

Figure 1 shows a system consisting of a body of mass m suspended on a weightless spring with a stiffness coefficient k. Only vertical displacements of the body are considered, which means the analyzed system has only one degree of freedom.

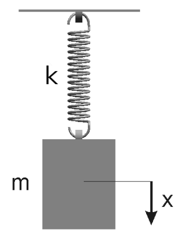


Figure 1: Body suspended on a spring

If x denotes the displacement of the body measured from its stable equilibrium position, the equation of motion will take the form:

$$m\frac{d^2x}{dt^2} = -kx, \tag{1}$$

or after transformation:

$$\frac{d^2x}{dt^2} + \alpha^2 x = 0, \tag{2}$$

where:

$$\alpha = \sqrt{\frac{k}{m}}. \tag{3}$$

*Author K. Januszkiewicz

Such a linear homogeneous differential equation has two independent solutions of the type $\sin \alpha t$ and $\cos \alpha t$; therefore, its general solution takes the form:

$$x = C_1 \sin \alpha t + C_2 \cos \alpha t. \tag{4}$$

Another form of expression (4) can be as follows:

$$x = a \sin(\alpha t + \varphi), \tag{5}$$

where the following relationships hold between the constants:

$$a = \sqrt{C_1^2 + C_2^2}, \qquad \operatorname{tg} \varphi = \frac{C_2}{C_1}. \tag{6}$$

We are thus dealing with harmonic motion, whose (angular) frequency α, called also α, is defined by formula (3). Since there is a relationship between the frequency of vibrations and their period:

$$\alpha = \frac{2\pi}{T} = \alpha, \tag{7}$$

the period of these vibrations is given by the formula:

$$T = 2\pi \sqrt{\frac{m}{c}}. \tag{8}$$

In the case of a linear mechanical system, the vibration frequency α (and thus the vibration period T) are determined solely by the structural parameters of the system (inertia and stiffness) and do not depend on the initial conditions. The coefficient a is called the vibration amplitude, while the argument $(\alpha t + \varphi)$ of the sine function is their phase. The initial phase value φ and the amplitude a depend on the starting conditions of the motion, i.e., the initial displacement $x_0 = x(0)$ and the initial velocity $\dot{x}_0 = \dot{x}(0)$. A non-zero value of at least one of these quantities is necessary for this type of vibration to occur. For a given initial displacement x_0 and initial velocity $\dot{x}_0$, the displacement of the body over time is described by the following relationship:

$$x = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\alpha}\right)^2} \sin(\alpha t + \varphi), \tag{9}$$

where:

$$\tan \varphi = \frac{\dot{x}_0}{\alpha x_0}. \tag{10}$$

3.2 Forced Vibrations and the Resonance Phenomenon

In the case where the system is constantly acted upon by some disturbance in the form of a variable force or a prescribed motion of a selected point in the system (e.g., the attachment point of the spring), we are dealing with forced vibrations. The vibrations of the system occur at the excitation frequency, which can have any value independent of its natural frequency. We consider the vibrations of the system from Fig. 1 under the influence of a harmonic forcing force with frequency ν and amplitude P_0. The currently analyzed system is shown in Fig. 2.

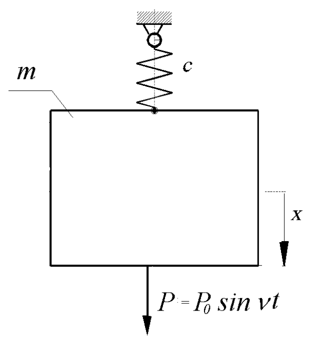


Figure 2: System with one degree of freedom excited by a harmonic force

The equation of motion of the body with mass m takes the following form:

$$m\frac{d^2x}{dt^2} = -kx + P_0 \sin \nu t, \tag{11}$$

or after transformations:

$$\frac{d^2x}{dt^2} + \alpha^2 x = \frac{P_0}{m} \sin \nu t. \tag{12}$$

This is a linear non-homogeneous differential equation, whose solution is the sum of the general solution of the homogeneous equation (belonging to free vibrations) and the particular integral of the non-homogeneous equation (describing steady-state forced vibrations). Assuming the particular integral in the form:

$$x = A \sin \nu t, \tag{13}$$

where A denotes a constant to be determined, the solution to equation (11) takes the form:

$$x = a \sin(\alpha t + \varphi) + A \sin \nu t. \tag{14}$$

Substituting (13) into differential equation (11) yields the coefficient A:

$$A = \frac{P_0}{m(\alpha^2 - \nu^2)}, \tag{15}$$

or using relationship (3):

$$A = \frac{P_0}{k - m\nu^2}. \tag{16}$$

The constants a and φ depend on the initial conditions of the motion.

Solution (14) ceases to be valid for frequency values of the forcing force where the denominator in formula (16) equals zero. In that case, the solution takes a different form due to the indeterminacy of the constant A. Such a state is called resonance; the vibration amplitude increases linearly with time (the vibrations theoretically may increase to infinity) and occurs when the forcing force frequency equals the system's natural frequency. The condition for the denominator to vanish is identical to relationship (3):

$$\nu_r = \alpha = \sqrt{\frac{k}{m}},$$

where α – natural frequency of the system shown in Figure 1, ν_r – resonance frequency of the forcing force for the system in Figure 2. Assuming that the amplitude of the forcing force $P_0 = S\nu^2$ – where S is a coefficient depending on the design parameters of the vibrating device – then the displacement amplitude of steady-state forced vibrations (absolute value of constant A) is as follows:

$$|A| = \left| \frac{S\nu^2}{m(\alpha^2 - \nu^2)} \right| = \frac{S}{m} W, \tag{17}$$

where W – amplitude magnification factor:

$$W = \left| \frac{\nu^2/\alpha^2}{1 - (\nu/\alpha)^2} \right|. \tag{18}$$

Figure 3 shows the dependence of the magnification factor w on the ratio of the forcing force frequency to the system's natural frequency, i.e., ν/α. We see that the magnification factor is infinitely large when the excitation frequency equals the natural frequency of the system.

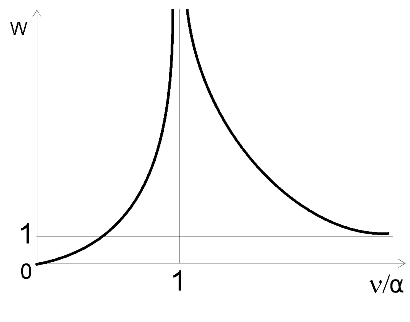


Figure 3: Resonance plot for inertial forced vibrations

The body's motion is in phase with the excitation force for frequencies ν lower than α. After exceeding the resonance frequency, the displacement of the body occurs opposite to the direction of the forcing force.

In previous considerations, the absence of vibration damping was assumed. In the case of even slight damping (in practice, mechanical energy dissipation always occurs through friction or other resistances), the amplitude of free vibrations with frequency α decreases fairly quickly with time, leaving only vibrations with frequency ν, the so-called steady-state forced vibrations. The maximum amplitude of these vibrations corresponds to a forcing force frequency slightly different from the natural vibration frequency. If the damping is small compared to its critical value, damping can be neglected in natural frequency calculations. Furthermore, across the entire range of forcing frequencies, the body's displacement lags by a certain phase angle relative to the forcing force.

4 Description of the Test Stand

A view of the test stand is shown in Fig. 4.

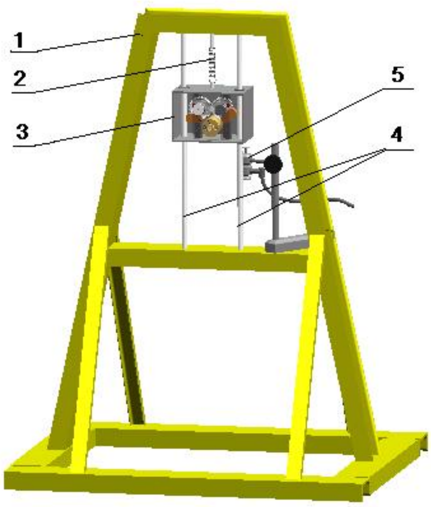


Figure 4: Test Stand

The stand consists of a frame (1), a spring (2), a body suspended on it (3) (inertial vibrator), two guides (4) ensuring the vertical movement of the vibrator, and a non-contact displacement sensor (5) recording the vibrations of the body. The body (3) can be set into vertical vibrating motion in two ways:

- by displacing the body from its static equilibrium position and suddenly releasing it,
- as a result of a harmonically variable vertical forcing force (obtained after turning on the vibrator motor, whose view is shown in Fig. 5).

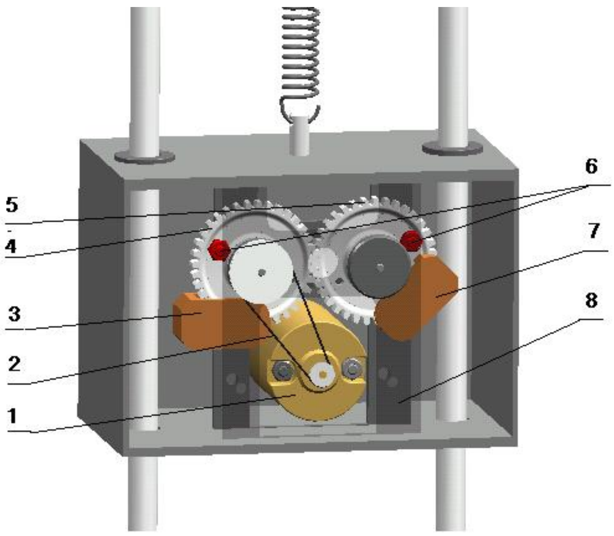


Figure 5: View of the inertial vibrator

A DC motor (1) with variable speed control drives, via a belt transmission (2), a light gear (4) cooperating with an identical gear (5). On both gears, which have 35 teeth each, identical weights (6) with mass $m_c = 9$ grams—very small compared to the mass $m = 2.6$ kg of the entire vibrator—are fixed identically (symmetrically relative to the middle plane of the vibrator). The contribution of mass m_c in the system's vibration analysis is neglected in the sense that it is assigned only the meaning of a source of centrifugal force. The weights and the spring attachment point lie in the vertical plane defined by the guide axes, in which the center of gravity of the entire vibrator is also located. As a result, the primary motion of the system is its vertical displacement. The counter-rotating gears generate two identical centrifugal forces.

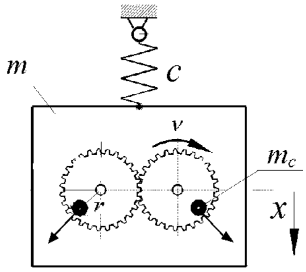


Figure 6: Harmonic force with amplitude proportional to the square of the frequency

The horizontal components of the centrifugal forces cancel each other out. The sum of the vertical components of both centrifugal forces is described by the following relationship:

$$P = 2(m_c r \nu^2 \sin \nu t) = P_0 \sin \nu t, \tag{19}$$

- $P_0 = 2m_c r\nu^2 = S\nu^2$ – amplitude of the forcing force,
- m_c – mass of the weight fixed eccentrically on the gear,
- r – distance of the weight from the gear's axis of rotation,
- ν – (steady) angular velocity of the gear.

The forcing force has a constant vertical direction and is a harmonic function of time.

4.1 Measuring Instruments and Measurement Methods

Measurements of characteristic quantities for the phenomenon under study are made using a laser displacement sensor and an oscilloscope, on which the sensor signal being proportional to the current displacement value is visualized and measured.

5 Measurement Procedure

Determining the natural frequency (period) of the system is carried out first:

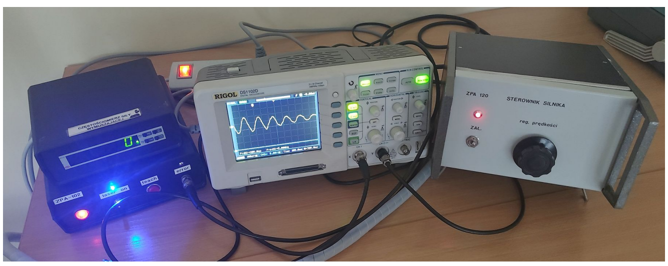
- measurement during free vibrations by introducing a disturbance to static equilibrium through displacement of the body and its sudden release, and then
- measurement during forced vibrations using the resonance method, i.e., by finding a forcing force frequency for which the largest vibration amplitude occurs.

Measurement results should be recorded in the appropriate columns of Table 1.

5.1 Measuring the Vibration Period

Measuring the natural vibration period.

1. Turn on the measurement equipment (laser sensor, oscilloscope).



2. Adjust the appropriate channel gain and oscilloscope trigger level.
3. Set the time base to single trigger.

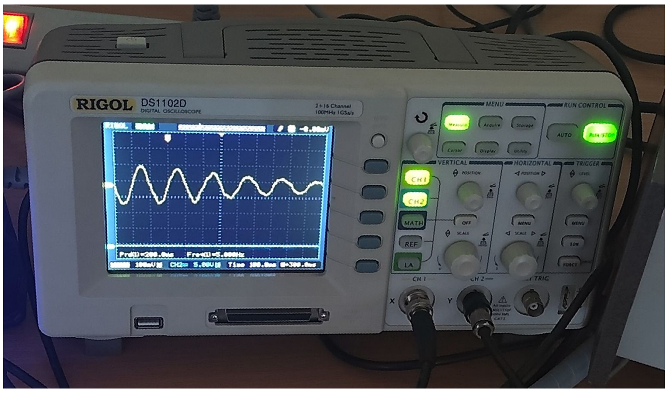
4. By pressing on the upper cover of the vibrator, cause it to displace downward by about 1 cm. Quickly release the pressure.

5. On the recorded vibration trace, determine their period using cursors.
6. Repeat the measurement three times.

Measuring the resonance frequency of forced vibrations

1. Additionally, turn on the vibrator motor speed controller, frequency meter, and the power supply for the optical rotation speed and phase sensors.
2. Connect the signal from the phase sensor to the second channel of the oscilloscope.
3. Turn on automatic time base triggering.
4. Slowly increase the motor rotation speed (i.e., the vibration forcing frequency) using the speed controller knob to find resonance. Simultaneously observe the motion of the system, the plot on the oscilloscope screen, and the frequency meter readings. The fact that the system is in resonance will be indicated by the position of the phase sensor pulse and the maximum height of the vibration sensor signal.
5. Under these conditions, measure the frequency of the forcing force. **Note! Do not keep the system in resonance for too long.**
6. Repeat the above measurement several times. To do this, turn the motor speed controller knob slightly (right or left) to detune the system (exit resonance) and then retune it.
7. Stop the motor (using the potentiometer knob) and turn off all measuring instruments and power supplies.

5.2 Description of DS 1102D Oscilloscope Operation



1. In the VERTICAL section on the front panel of the oscilloscope, turn on both channels with the CH1 and CH2 buttons (they should be backlit). Set DC coupling for the input signals. After pressing the CH1 button, set the coupling for the vibration sensor signal: CH1 → Coupling → DC. Similarly for the optical sensor signal: CH2 → Coupling → DC. To remove the drop-down menu on the screen, use the MENU ON/OFF button.

2. In the VERTICAL section, set the vertical sensitivity (in volts per division) with the SCALE knob and the vertical position of the traces with the POSITION knob:
   CH1 → SCALE → 500mV → POSITION → -1.00V
   CH2 → SCALE → 10.0V → POSITION → -40V
3. In the HORIZONTAL section, set the time base speed with the SCALE knob, displayed on the status bar. Initially, you can set Time 50.00 ms (i.e., 50ms/division). With the POSITION knob in this section, set the trigger start in the left part of the screen (for this time base speed, around 250.0ms). This is shown by the ↓ indicator at the top of the screen, and the value, e.g., T → 250.0ms, is displayed in the bottom right corner. After changing the time base speed, similarly set the trigger start in the left part of the screen.
4. In the TRIGGER section, set the source, type, and mode (Single) for triggering the time base with the signal from the optical sensor (channel 2) using the drop-down menu buttons and the multi-function knob.
   MENU → Mode → Edge (edge triggering)
   Source → CH2
   Slope → ↑ (rising edge)
   Sweep → Single (single acquisition cycle after the trigger pulse appears)
   Use the LEVEL knob to set the trigger level to about 4V: TRIG LVL= 4.00V. The trigger level value is displayed in the top right corner of the screen.
5. Measuring the period, frequency, or amplitude using cursors. After recording the traces:
   - press the Cursor button in the MENU section on the front panel of the oscilloscope.
   - set the manual cursor measurement mode (Manual) by pressing the Mode button.
   - use the Type button to set time parameters (X) or voltage parameters (Y) cursors.
   - turn the multi-function knob to set cursors A and B at the appropriate points on the trace. Change cursors A, B, or AB by repeatedly pressing the multi-function knob. At the top of the screen, you can read the time interval (ΔX) and frequency (1/ΔX), or the voltage difference (ΔY).
   - Automatic frequency measurement – press the Measure button in the MENU section.
   - set the source of the measured signal: Measure → Source → CH1.
   - select frequency measurement: Time → Freq.
   - the measurement result is immediately displayed on the screen.

   7. In case of difficulty obtaining a stable image on the screen, press the AUTO button in the RUN CONTROL section and repeat steps 1 to 5 (6).

6 Processing Measurement Results and Report

6.1 Auxiliary Calculations

After finishing the measurements, proceed to perform the calculations necessary to fill in all the sections of Table 1 and determine the measurement uncertainties. The mass m and spring constant k values needed for calculations are available in the test stand documentation. All calculation results should be rounded considering the imperfection of the experiment. Remember that there is no need to be extremely precise when describing your own imprecision. In particular, uncertainties and percentage differences should be calculated with an accuracy of one or at most two significant digits. The final result should be rounded so that the order of its last significant digit is the same as the order of the uncertainty. (Uncertainty cannot be determined with greater accuracy than the quantity itself).

Review Questions

1. Describe simple harmonic motion: give examples, explain its definition.
2. Vibrations – explain the phenomenon; list the basic quantities characterizing them.
3. Explain the terms: free vibrations, forced vibrations, damped vibrations.
4. Give practical reasons why knowledge of a system's natural frequencies is important.
5. Explain the resonance phenomenon.

Exercise 11

VIBRATIONS OF A SYSTEM WITH 1 DEGREE OF FREEDOM

Group: ____________________

Team: ____________________

Date ____________________

Student names:

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________

Measurement and Calculation Results

Mass $m =$ .... kg, Spring constant $k =$ .... N/m

Table 1: Results

Measurement	**Natural Vibrations**		**Forced Vibrations**
	Single Period	**Avg. of 10 periods**	**Single Period**
	T [s]	T [s]	T [s]
1			
2			
3			
4			
5			
6			
Average period value, measured [1–6] T_{avg}			
Avg. frequency, measured $\alpha_{avg} = \frac{2\pi}{T_{avg}}$ [rd/s]			
PERIOD, theoretical $T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$ [rd/s]			X
PERIOD, experimental			
Natural frequency, theoretical $\alpha = \sqrt{\frac{k}{m}}$ [rd/s]			X
Frequency at resonance, measured	X		
% difference in frequency $\Delta = \frac{\alpha - \alpha_{avg}}{\alpha} \cdot 100\%$			

Conclusions: