MOMENTUM CONSERVATION LAW

International Faculty of Engineering – BASIC MECHANICS – Exercise 20 *

1 Theoretical introduction

Momentum (p) is one of the fundamental notions in mechanics. For a single particle (point mass), it is defined as the product of the particle's mass and velocity; thus, it is a vector quantity.

$$\mathbf{p} = m\mathbf{v} \tag{1}$$

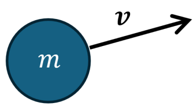

Figure 1: Momentum of a particle as the product of its mass and velocity

Components of this vector are connected with the corresponding components of velocity: $\mathbf{p} = (p_x, p_y, p_z) = (mv_x, mv_y, mv_z)$. In many applications (including the present exercise), it is sufficient to observe just a single component. Using this quantity, Newton's second law can be reformulated:

$$\sum \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}} \tag{2}$$

which means that **the time derivative of the particle's momentum equals the resultant force applied to it**. In other words, the resultant force decides on the rate of change of the particle's momentum.

Eq. (2) can be distributed into subsequent components. Consequently, for any selected axis x, we can write the corresponding scalar equation (3). Of course, analogous equations are available for axes y and z.

$$\sum F_x = ma_x = m\dot{v}_x = \dot{p}_x \tag{3}$$

The importance of the notion of momentum can be clearly noticed when **the center of mass** – C for multiple point masses is investigated. For a set of particles with masses $m_1, m_2, \ldots$ whose positions are indicated by vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots$ the location of the mass center is defined as follows:

$$R = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m_i\mathbf{r}_i}{\sum m_i} \tag{4}$$

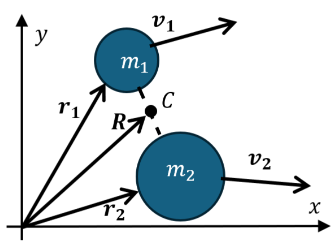

Figure 2: The center of mass , indicated by the position vector $\mathbf{R}$, for a set of two particles

From Eq. (4) it can be noticed that components x_c, y_c z_c of vector $\mathbf{R} = [x_c,\, y_c\, z_c]$, being coordinates of the center of mass, can be calculated using Eq. (5) (or its analogs for axes y and z). The x-coordinate of the center of mass is:

$$x_C = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m_ix_i}{\sum m_i} \tag{5}$$

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If the total mass is $M = \sum m_i$, then:

$$M\mathbf{R} = [Mx_C my_C Mz_C] = m_1 r\mathbf{r}_1 + m_2 r\mathbf{r}_2 + \ldots = \sum m_i \mathbf{r}_i \tag{6}$$

with its component x_C expressed by

$$Mx_C = m_1 x_1 + m_2 x_2 + \ldots = \sum m_i x_i \tag{7}$$

Since the velocity vector is defined as the time derivative of the position vector with respect to time, $v = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$, the differentiation of Eq. (6) gives the total momentum $\mathbf{p}$:

$$\mathbf{p} = M\dot{\mathbf{R}} = m_1 r\dot{\mathbf{r}}_1 + m_2 r\dot{\mathbf{r}}_2 + \ldots = m_1 r\mathbf{v}_1 + m_2 r\mathbf{v}_2 + \ldots = \mathbf{p}_1 + \mathbf{p}_2 + \ldots \tag{8}$$

In scalar form, which refers to a selected direction x, it is expressed as

$$p_x = M\dot{x}_C = Mv_{Cx} = m_1 v_{1x} + m_2 v_{2x} + \ldots = p_{1x} + p_{2x} + \ldots \tag{9}$$

where $v_{Cx} = \dot{x}_C$ is the component of the mass center's velocity along the x axis, v_{ix} is the component of velocity of the i-th particle and p_x is the component of momentum of the i-th particle.

Equation (8) shows that the total momentum of the set of particles m_1, m_2, ... gives the momentum of the center of mass of the system. In other words, the momentum $p = M\dot{\mathbf{M}}$, associated with the center of mass of the system, is equal to the sum of all momenta $p_i = m_i \mathbf{v}_i$ of the particles that constitute the system under investigation.

An even more important fact follows when the third Newton's law is taken into account. In a set of particles, the forces acting upon them can originate: a) from outside the system (external forces) and b) from one particle on another (internal forces).

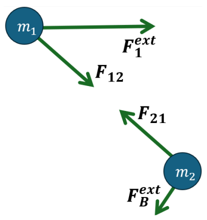

Figure 3: A set of two particles with external forces F_1^{ext}, F_2^{ext} and internal ones F_{12}, F_{21}

Here, the force F_{12} is the force exerted on particle m_1 by particle m_2, and for F_{21} – vice versa. From the third Newton's law we know that $F_{12} = F_{21}$. Further, for each particle separately, the second Newton's law can be formulated.

$$\begin{aligned} \Sigma\mathbf{F}_1 &= \mathbf{F}_1^{ext} + \mathbf{F}_{12} = m_1\mathbf{a}_1 = m_1\dot{\mathbf{v}}_1 = \dot{\mathbf{p}}_1 \\ \Sigma\mathbf{F}_2 &= \mathbf{F}_2^{ext} + \mathbf{F}_{21} = m_2\mathbf{a}_2 = m_2\dot{\mathbf{v}}_2 = \dot{\mathbf{p}}_2 \end{aligned} \tag{10}$$

Obviously, the internal forces affect the motion of each particle separately. However, if the time derivative of the whole system's momentum is investigated:

$$\dot{\mathbf{p}} = M\ddot{\mathbf{R}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{F}_1^{ext} + \mathbf{F}_{12} + \mathbf{F}_2^{ext} + \mathbf{F}_{21} = \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} \tag{11a}$$

$$\dot{\mathbf{p}} = M\dot{\mathbf{R}} = \Sigma\mathbf{F}^{ext} \tag{11b}$$

The last transformation in Eq. (11a) is valid due to the third Newton's law, which implies $F_{12} = F_{21}$, or $F_{12}+F_{21} = 0$. Please note that the internal forces F_{12}, F_{21} do not yield a zero resultant in Eq. (10), since they are applied to different particles; however, their influence vanishes when the momentum of the whole system is investigated. Moreover, note that although Eq. (11a) has been – for simplicity – derived for just two particles, the more general result (11b) is valid for any set of particles in which the third Newton's law holds.

Again, the presented result can be expressed in terms of components. In particular, for the x direction, eq. (11) yields the following result

$$\dot{\mathbf{p}}_x = M\ddot{\mathbf{x}}_c = M\dot{\mathbf{v}}_{Cx} = \sum \mathbf{F}_x^{ext} \tag{12}$$

which states that the time derivative of the x component of momentum of a set of particles is equal to the sum of all external forces acting in the x direction on any particle in the system. Furthermore, in the special case of $\sum \mathbf{F}^{ext} = 0$, Eq. (11b) implies the following result

$$\sum \mathbf{F}^{ext} = 0 \rightarrow \dot{\mathbf{p}} = 0 \rightarrow \mathbf{p} = M\dot{\mathbf{R}} = const \tag{13}$$

with the single-component, scalar counterpart

$$\sum F_x^{ext} = 0 \rightarrow \dot{p}_x = 0 \rightarrow p_x = Mv_{Cx} = M\dot{x}_C = const \tag{14}$$

Equations (13) and (14) constitute the law of momentum conservation: if no external forces act on the system, its total momentum is preserved. Consequently, the velocity of the center of mass of the system remains constant.

Application of Eqs. (11-14) are exemplified by the following examples.

Example 1: Slippery Ice

Why can't a man walk on ideally slippery ice, with a zero static friction coefficient?

Solution

Due to Eqs. (13) and (14), in the absence of an external horizontal force (i.e., friction), the time derivative of the horizontal momentum of the center of mass of the human is zero. Consequently, the horizontal component of the velocity of the center of mass is constant, and if it was zero initially, it will remain zero until he receives a helping hand.

Example 2: Plane Crash

Two planes, flying horizontally at the height of 2000m above the sea level, crash into one another and start to fall together. What will be the falling time?

Solution

Since the initial momentum of both planes together had a horizontal direction, and until the crash the weights of the planes were balanced by lift forces, the total vertical momentum was constant and equal to 0. This remains true during the crash, apart from the fact that the lift force disappears. Consequently, according to Eq. (12), the center of mass of the wreckage of both planes undergoes free fall with zero initial vertical velocity. Thus, the falling time is $t = \sqrt{2h/g}$.

2 Description of the experiment

The experiment tests the conservation of momentum using carts on an air track shown in Fig. 4.

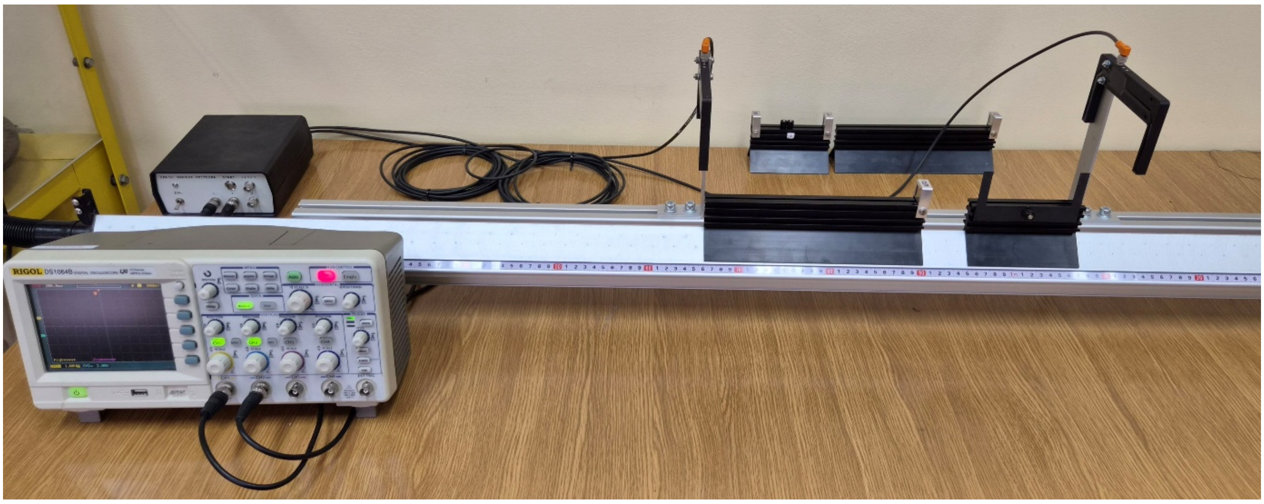

Figure 4: Photograph of the air track

The validity of the conservation of momentum law is going to be tested by investigating the impacts between the carts on the track. The setup allows you to observe the results of direct collisions, as well as collisions with a spring, magnets, or even modelling clay between the contacting carts. For each medium used in the collision, the results may be observed for both: a moving cart impacting a stationary one or for the impact of two carts, both in motion with respect to the track.

2.1 Procedure

Each measurement is conducted in the following manner.

1. Determine the masses M and m of the carts.
2. Measure the distance between the metal sheets on the carts used as markers for timing.

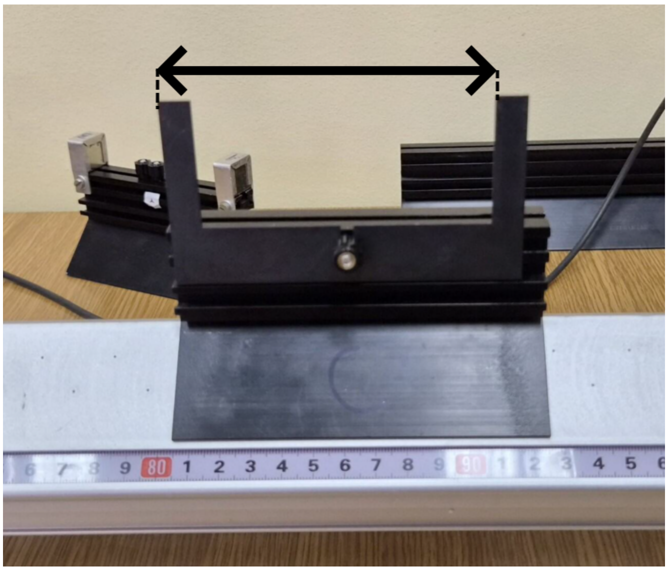

Figure 5: The distance between the metal sheets to be measured

Start with the larger cart being stationary: place it between the sensors and push the smaller one into it, thus causing the collision. Use the scope to measure the times necessary for each cart to move through the sensors (Fig. 6). With such data, knowing the masses and the distances between the metal sheets, calculate the total momentum and the total energy before and after the collision.

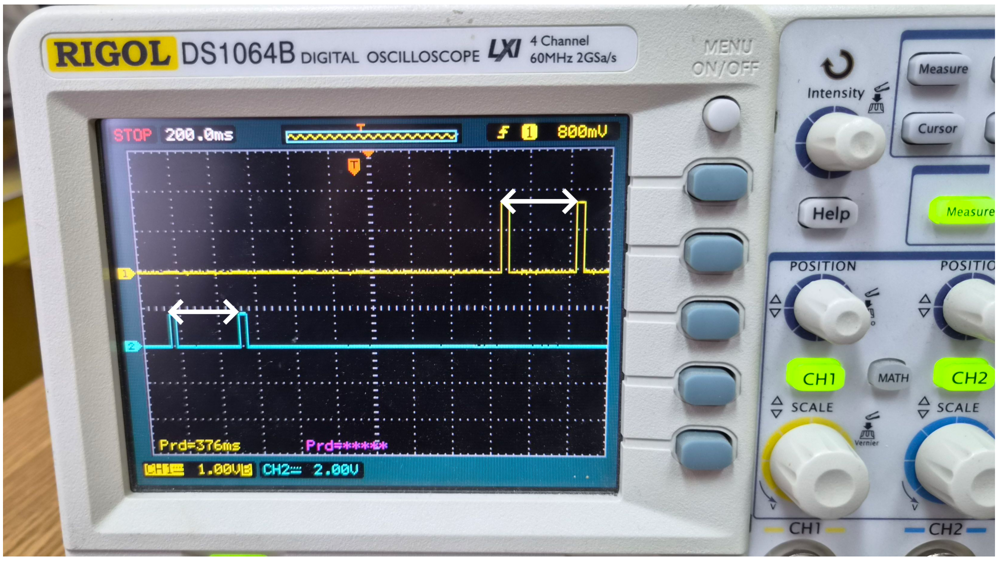

Figure 6: Use an oscilloscope to measure the travel times through sensors to calculate velocities

3. Perform collisions starting with one stationary cart, then with both in motion.
4. Calculate total momentum and energy before and after each collision.

Repeat the procedure three times for different impact media (springs, magnets, clay) to check for conservation of momentum and kinetic energy.

LABORATORY

OF

TECHNICAL MECHANICS

Exercise 20

MOMENTUM CONSERVATION LAW

Group: ______________________

Team: ______________________

date ______________________

Name and surname:

1. ______________________
2. ______________________
3. ______________________
4. ______________________
5. ______________________
6. ______________________

Observations and Conclusions

Case 1: Direct Impact

Mass of the smaller cart: ________________ Measurement length of the smaller cart: ________________
Mass of the larger cart: ________________ Measurement length of the larger cart: ________________
Note: SC = small cart; LC = large cart.

Case 1a – Larger cart stationary

Before the impact				After the impact							
SC time [s]	SC vel. [m/s]	Init. Mom. [kg·m/s]	Init. En. [J]	SC time [s]	SC vel. [m/s]	LC time [s]	LC vel. [m/s]	Final Mom. [kg·m/s]	Final En. [J]	Mom. loss	En. loss

Case 1b – Larger cart in motion

Before the impact						After the impact							
SC time	SC vel.	LC time	LC vel.	Init. Mom.	Init. En.	SC time	SC vel.	LC time	LC vel.	Final Mom.	Final En.	Mom. loss	En. loss

Case 2: Impact with Magnets

Mass of the smaller cart: ________________ Measurement length of the smaller cart: ________________

Mass of the larger cart: ________________ Measurement length of the larger cart: ________________

Note: SC = small cart; LC = large cart.

Case 2a – Larger cart stationary

Before the impact				After the impact							
SC time [s]	SC vel. [m/s]	Init. Mom. [kg·m/s]	Init. En. [J]	SC time [s]	SC vel. [m/s]	LC time [s]	LC vel. [m/s]	Final Mom. [kg·m/s]	Final En. [J]	Mom. loss	En. loss

Case 2b – Larger cart in motion

Before the impact						After the impact							
SC time	SC vel.	LC time	LC vel.	Init. Mom.	Init. En.	SC time	SC vel.	LC time	LC vel.	Final Mom.	Final En.	Mom. loss	En. loss

Case 3: Impact through Spring

Mass of the smaller cart: ____________ Measurement length of the smaller cart: ____________

Mass of the larger cart: ____________ Measurement length of the larger cart: ____________

Note: SC = small cart; LC = large cart.

Case 3a – Larger cart stationary

Before the impact				After the impact							
SC time [s]	SC vel. [m/s]	Init. Mom. [kg·m/s]	Init. En. [J]	SC time [s]	SC vel. [m/s]	LC time [s]	LC vel. [m/s]	Final Mom. [kg·m/s]	Final En. [J]	Mom. loss	En. loss

Case 3b – Larger cart in motion

Before the impact						After the impact							
SC time	SC vel.	LC time	LC vel.	Init. Mom.	Init. En.	SC time	SC vel.	LC time	LC vel.	Final Mom.	Final En.	Mom. loss	En. loss

Case 4: Impact through Modelling Clay

Mass of the smaller cart: ____________ Measurement length of the smaller cart: ____________

Mass of the larger cart: ____________ Measurement length of the larger cart: ____________

Note: SC = small cart; LC = large cart.

Case 4a – Larger cart stationary

Before the impact				After the impact							
SC time [s]	SC vel. [m/s]	Init. Mom. [kg·m/s]	Init. En. [J]	SC time [s]	SC vel. [m/s]	LC time [s]	LC vel. [m/s]	Final Mom. [kg·m/s]	Final En. [J]	Mom. loss	En. loss

Case 4b – Larger cart in motion

Before the impact						After the impact							
SC time	SC vel.	LC time	LC vel.	Init. Mom.	Init. En.	SC time	SC vel.	LC time	LC vel.	Final Mom.	Final En.	Mom. loss	En. loss