

FREE VIBRATIONS OF A TWO-DEGREE-OF-FREEDOM SYSTEM

Vibrations and Machine Dynamics – Exercise 7 *

1 Aim of the exercise

Experimental determination of the natural frequencies of a two-degree-of-freedom system, demonstration of the mode shapes corresponding to these frequencies, and observation and analysis of the phenomenon of beating.

2 Theoretical introduction

The object of consideration is the system shown in Fig. 7.1. It consists of two identical physical pendulums connected by a spring of stiffness k . The pendulums are characterised by the fact that the mass of the body m fixed at the end of a rod of length l is approximately equal to the mass of the rod m_p . For simplicity the dimensions of the suspended weights have been neglected, treating them as bodies of concentrated masses located at points at a distance l from the axis of rotation. The spring is attached at a distance from the axis of rotation such that $l_1 = 0.5l$.

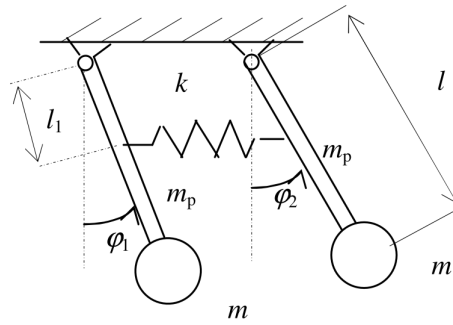


Figure 1: Model of the system

To derive the dynamic equations of motion about the equilibrium positions of the pendulums shown in Fig. 1 in linearised form, Newton's second law for rotational motion has been applied:

$$B\ddot{\phi} = \sum M, \quad (1)$$

where:

- B – mass moment of inertia of the pendulum;
- ϕ – angle of rotation of the pendulum measured from the equilibrium position (Fig. 1);
- M – moment of the external forces (gravity forces and spring deflection forces) acting on the pendulum.

The mass moment of inertia of the pendulum with respect to the axis of rotation is given by the relation:

$$B = \frac{m_p l^2}{3} + m l^2 = \frac{4}{3} m l^2. \quad (2)$$

For the first pendulum performing small oscillations about the equilibrium position, the dynamic equation of motion (1) has the following form:

$$\frac{4}{3} m l^2 \ddot{\phi}_1 = - \left(m_g \frac{l}{2} + m g l \right) \phi_1 - k \left(\frac{l}{2} \right)^2 \phi_1 + k \left(\frac{l}{2} \right)^2 \phi_2. \quad (3)$$

The analogous equation of motion of the second pendulum is of the form:

$$\frac{4}{3} m l^2 \ddot{\phi}_2 = - \left(m_g \frac{l}{2} + m g l \right) \phi_2 - k \left(\frac{l}{2} \right)^2 \phi_2 + k \left(\frac{l}{2} \right)^2 \phi_1. \quad (4)$$

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The system of coupled differential equations (3) and (4) constitutes the linearised mathematical model of the system under investigation. After rearrangements this model can be presented in the following form:

$$\begin{aligned} \frac{4}{3}ml^2\ddot{\phi}_1 + \left(\frac{3}{2}mgl + \frac{kl^2}{4}\right)\phi_1 - \frac{kl^2}{4}\phi_2 &= 0; \\ \frac{4}{3}ml^2\ddot{\phi}_2 + \left(\frac{3}{2}mgl + \frac{kl^2}{4}\right)\phi_2 - \frac{kl^2}{4}\phi_1 &= 0. \end{aligned} \quad (5)$$

By introducing initial conditions a portion of energy is supplied to the system in order to excite its free vibrations. Particular solutions of the homogeneous system of differential equations (5) are then sought in the form:

$$\begin{aligned} \phi_1 &= A_1 \sin(\alpha t + \delta); \\ \phi_2 &= A_2 \sin(\alpha t + \delta). \end{aligned} \quad (6)$$

After substitution of the solutions (6) into (5) and formulation of the condition that the solutions be satisfied at every instant of time, one obtains a homogeneous system of algebraic equations with the unknown amplitudes A_1 and A_2 and the frequency α as a parameter in the form:

$$\begin{aligned} (-b_{11}\alpha^2 + k_{11})A_1 + k_{12}A_2 &= 0; \\ k_{21}A_1 + (-b_{22}\alpha^2 + k_{22})A_2 &= 0, \end{aligned} \quad (7)$$

where: $b_{11} = b_{22} = \frac{4}{3}ml^2$; $k_{12} = k_{21} = -\frac{kl^2}{4}$; $k_{11} = k_{22} = \frac{3}{2}mgl + \frac{kl^2}{4}$.

The condition for the existence of non-zero solutions of the system of equations (7) is the vanishing of its principal determinant:

$$\begin{vmatrix} -b_{11}\alpha^2 + k_{11} & k_{12} \\ k_{21} & -b_{22}\alpha^2 + k_{22} \end{vmatrix} = 0. \quad (8)$$

On the basis of condition (8) one obtains the frequency equation in the form:

$$\alpha^4 - \frac{b_{11}k_{22} + b_{22}k_{11}}{b_{11}b_{22}}\alpha^2 + \frac{k_{11}k_{22} - k_{12}k_{21}}{b_{11}b_{22}} = 0. \quad (9)$$

Solving the biquadratic equation (9) one determines the natural frequencies of the system under investigation:

$$\begin{aligned} \alpha_1^2 &= \frac{9}{8} \frac{g}{l}; \\ \alpha_2^2 &= \frac{9}{8} \frac{g}{l} + \frac{3}{8} \frac{k}{m}. \end{aligned} \quad (10)$$

There exist, therefore, always two circular frequencies α_1 and α_2 at which harmonic vibrations of the system under consideration are possible. The mode of vibration of the system in which it performs harmonic oscillations is called a **principal mode of vibration**. Substituting α_1 in place of α into the system of algebraic equations (7), one obtains the system of equations for the amplitudes of free vibration of the first mode:

$$\begin{aligned} (-b_{11}\alpha_1^2 + k_{11})A_{11} + k_{12}A_{21} &= 0; \\ k_{21}A_{11} + (-b_{22}\alpha_1^2 + k_{22})A_{21} &= 0, \end{aligned} \quad (11)$$

where:

A_{11} – amplitude of vibration of the first pendulum corresponding to the first mode of vibration,

A_{21} – amplitude of vibration of the second pendulum corresponding to the first mode of vibration.

Equations (11) are linearly dependent, which follows from the vanishing of their principal determinant. From either of them one can determine the value of the amplitude ratio $\nu_1 = A_{21}/A_{11} = 1$, which defines the configuration of the system during harmonic vibrations of frequency α_1 .

Proceeding analogously with the frequency α_2 , one obtains the value of the ratio $\nu_2 = A_{22}/A_{12} = -1$. The ratios obtained in this way are called the **coefficients of the principal modes of vibration**.

The general solution of the system of equations (5) under consideration can be presented in the form:

$$\begin{aligned} \phi_1(t) &= A_{11} \sin(\alpha_1 t + \delta_1) + A_{12} \sin(\alpha_2 t + \delta_2); \\ \phi_2(t) &= A_{11}\nu_1 \sin(\alpha_1 t + \delta_1) + A_{12}\nu_2 \sin(\alpha_2 t + \delta_2). \end{aligned} \quad (12)$$

After taking into account the numerical values of ν_1 and ν_2 the solutions (12) can be written in the form:

$$\begin{aligned} \phi_1(t) &= A_{11} \sin(\alpha_1 t + \delta_1) + A_{12} \sin(\alpha_2 t + \delta_2); \\ \phi_2(t) &= A_{11} \sin(\alpha_1 t + \delta_1) - A_{12} \sin(\alpha_2 t + \delta_2), \end{aligned} \quad (13)$$

where: $A_{11}, A_{12}, \delta_1, \delta_2$ – constants dependent on the initial conditions.

One may pose the question: What initial conditions must be imposed in order to observe the individual principal modes of vibration? Let us adopt the following notation:

$$\begin{aligned}\phi_{10} &= \phi_1(t=0); & \phi_{20} &= \phi_2(t=0); \\ \omega_{10} &= \left. \frac{d\phi_1}{dt} \right|_{t=0}; & \omega_{20} &= \left. \frac{d\phi_2}{dt} \right|_{t=0}.\end{aligned}\quad (14)$$

Taking (14) into account in (13) one obtains:

$$\begin{aligned}\phi_{10} &= A_{11} \sin \delta_1 + A_{12} \sin \delta_2; \\ \phi_{20} &= A_{11} \sin \delta_1 - A_{12} \sin \delta_2; \\ \omega_{10} &= A_{11} \alpha_1 \cos \delta_1 + A_{12} \alpha_2 \cos \delta_2; \\ \omega_{20} &= A_{11} \alpha_1 \cos \delta_1 - A_{12} \alpha_2 \cos \delta_2.\end{aligned}\quad (15)$$

If we wish to observe vibrations of the first mode we must assume $A_{12} = 0$. Then $\phi_{10} - \phi_{20} = 0$, i.e. $\phi_{10}/\phi_{20} = 1 = \nu_1$, so the initial conditions concerning the displacements of the pendulums are proportional to the principal mode of vibration. Analogously $\omega_{10} - \omega_{20} = 0$, i.e. $\omega_{10} = \omega_{20}$ (in particular one may assume $\omega_{10} = \omega_{20} = 0$).

This means that free vibrations of the first mode can be observed by adopting the initial condition $\phi_{10} = \phi_{20}$, which means deflection of both pendulums by the same angle (with respect to its value and sign). Proceeding analogously, one can conclude that free vibrations of the second mode occur under the initial condition $\phi_{10} = -\phi_{20}$. The graphical representation of the free vibrations of the first and second modes is shown in Fig. 2.

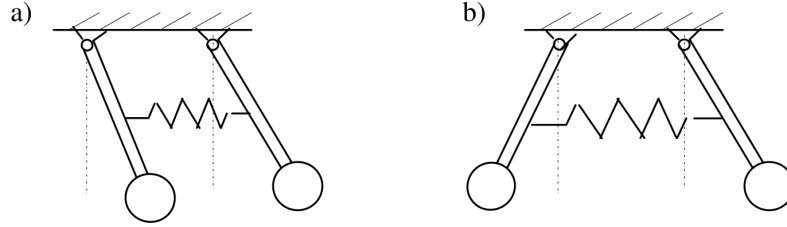


Figure 2: Principal modes of vibration: a) first, b) second

2.1 Phenomenon of beating

An interesting case can be obtained by adopting at the initial instant of motion of the pendulums: $\phi_{10} = \phi_0$ and $\phi_{20} = 0$, $\omega_{10} = \omega_{20} = 0$. On the basis of (15) one then has:

$$\begin{aligned}\phi_0 &= A_{11} \sin \delta_1 + A_{12} \sin \delta_2; \\ 0 &= A_{11} \sin \delta_1 - A_{12} \sin \delta_2; \\ 0 &= A_{11} \alpha_1 \cos \delta_1 + A_{12} \alpha_2 \cos \delta_2; \\ 0 &= A_{11} \alpha_1 \cos \delta_1 - A_{12} \alpha_2 \cos \delta_2.\end{aligned}\quad (16)$$

From the first and second equations of (16) one obtains:

$$\phi_{10} = 2A_{11} \sin \delta_1. \quad (17)$$

From the third and fourth equations of (16) and from the condition of existence of non-zero solutions for A_{11} and A_{12} one obtains $\delta_1 = \delta_2 = \pi/2$. Taking the above relations into account one obtains:

$$A_{11} = A_{12} = \frac{1}{2}\phi_0, \quad (18)$$

and the general solution in the following form:

$$\begin{aligned}\phi_1(t) &= \frac{1}{2}\phi_0 \cos \alpha_1 t + \frac{1}{2}\phi_0 \cos \alpha_2 t; \\ \phi_2(t) &= \frac{1}{2}\phi_0 \cos \alpha_1 t - \frac{1}{2}\phi_0 \cos \alpha_2 t.\end{aligned}\quad (19)$$

Making use of trigonometric identities the expressions (19) can be presented as follows:

$$\begin{aligned}\phi_1(t) &= \phi_0 \cos \left(\frac{\alpha_2 - \alpha_1}{2} t \right) \cos \left(\frac{\alpha_2 + \alpha_1}{2} t \right); \\ \phi_2(t) &= \phi_0 \sin \left(\frac{\alpha_2 - \alpha_1}{2} t \right) \sin \left(\frac{\alpha_2 + \alpha_1}{2} t \right).\end{aligned}\quad (20)$$

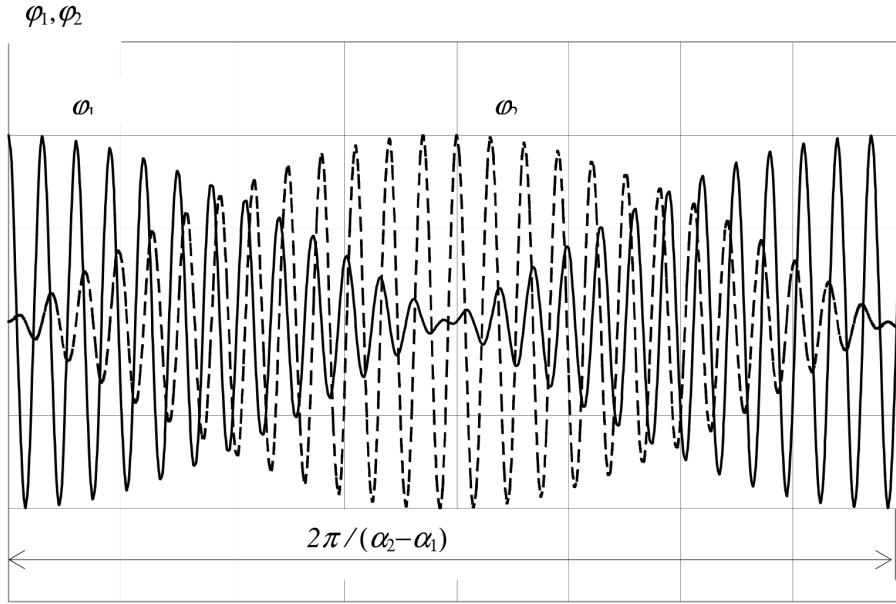


Figure 3: Solutions (20) illustrating the phenomenon of beating

The course of the solutions (20) is shown in Fig. 3. It can be seen that in the case of coupling of two identical single-degree-of-freedom vibrating systems the vibrations in the coupled system have the character of **beats**. The energy determined by the initial conditions is transferred periodically from one system to the other. This phenomenon is illustrated in Fig. 4.

At the first instant pendulum 1 performs vibrations while pendulum 2 is stationary (Fig. 4a). This motion can be interpreted as the sum of two free vibrations of the first and second modes of frequencies α_1 and α_2 . When these frequencies are sufficiently close a certain time (corresponding to several periods) is required for a phase shift to occur. At a certain instant the phase shift of the two modes is 180° , which is illustrated in Fig. 4b. Adding the two motions shown, one can observe that pendulum 1 is now stationary while pendulum 2 performs vibrations of amplitude ϕ_0 . This phenomenon repeats itself and the vibrations are transferred from one pendulum to the other.

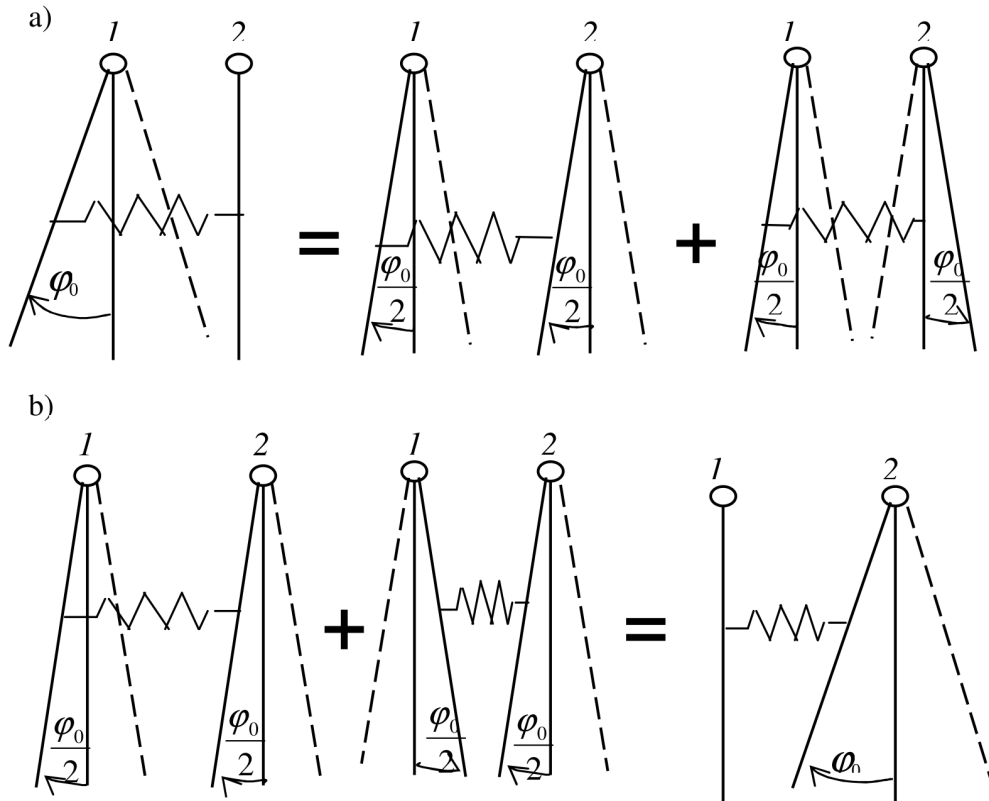


Figure 4: Principal modes of vibration: a) first, b) second

3 Experimental stand

The experimental stand is shown schematically in Fig. 5. It corresponds to the theoretical model presented in Fig. 1. The pendulums are supported in two prisms and can perform vibrations only in the vertical plane.

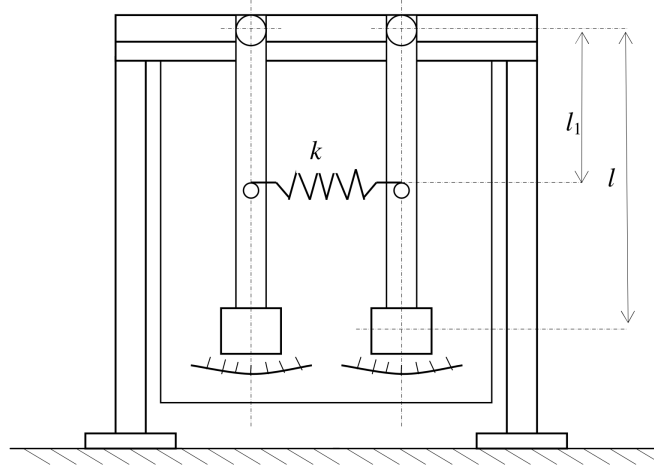


Figure 5: Schematic diagram of the experimental stand

4 Procedure

1. In order to determine the first natural frequency of the system one should:
 - (a) deflect both pendulums by the same angle in magnitude and sense,
 - (b) using a stop-watch measure the time of 20 periods of free vibration,
 - (c) repeat the measurement of the time of 20 periods of vibration twice,
 - (d) calculate the average value of the period of free vibration,
 - (e) on the basis of the calculated period determine the value of the first natural frequency of the system under investigation.
2. In order to determine the second natural frequency of the system one should:
 - (a) deflect both pendulums by equal angles in magnitude but of opposite sense,
 - (b) using a stop-watch measure the time of 20 periods of free vibration,
 - (c) repeat the measurement of the time of 20 periods of vibration twice,
 - (d) calculate the average value of the period of free vibration,
 - (e) on the basis of the calculated period determine the value of the second natural frequency of the system under investigation.
3. In order to determine the beat frequency one should:
 - (a) deflect one pendulum by a small angle ($\phi_0 \leq 5^\circ$),
 - (b) using a stop-watch measure the time of 5 periods of beating,
 - (c) repeat the measurement of the time of 5 periods of beating twice,
 - (d) calculate the average value of the beat period,
 - (e) on the basis of the calculated period determine the value of the beat frequency.
4. In order to compare the determined frequencies with their theoretical values one should:
 - (a) calculate the free-vibration frequencies α_1 and α_2 on the basis of formulae (10),
 - (b) calculate the value of the beat frequency $\alpha_d = \alpha_2 - \alpha_1$,
 - (c) perform an analysis of the theoretical and experimental results.

5 Literature

1. Kaliski S.: *Vibrations and Waves*. PWN, Warsaw 1966.
2. Kapitaniak T.: *Introduction to Vibration Theory*, Publishing House of the Technical University of Łódź, Łódź 1992.
3. Collective work: *Mechanical Vibrations – Laboratory*, Publishing House of Warsaw University of Technology, Warsaw 1999.

**LABORATORY OF
MECHANICAL
VIBRATIONS**

Exercise 7

**FREE VIBRATIONS OF A TWO-DEGREE-OF-FREEDOM
SYSTEM**

Group: _____
Team: _____

date _____

Name and surname:

1. _____
2. _____
3. _____
4. _____
5. _____
6. _____

Determination of the 1st circular natural frequency of the system

Time of 20 periods of free vibration:

$$T_{11} = \dots\dots\dots [\text{s}]; \quad T_{12} = \dots\dots\dots [\text{s}]; \quad T_{13} = \dots\dots\dots [\text{s}]$$

$$\text{Average value of the period of free vibration: } T_1 = \frac{T_{11}+T_{12}+T_{13}}{3 \times 20} = \dots\dots\dots$$

$$\text{Value of the 1st circular natural frequency: } \alpha_1 = \frac{2\pi}{T_1} = \dots\dots\dots$$

Determination of the 2nd circular natural frequency of the system

Time of 20 periods of free vibration:

$$T_{21} = \dots\dots\dots [\text{s}]; \quad T_{22} = \dots\dots\dots [\text{s}]; \quad T_{23} = \dots\dots\dots [\text{s}]$$

$$\text{Average value of the period of free vibration: } T_2 = \frac{T_{21}+T_{22}+T_{23}}{3 \times 20} = \dots\dots\dots$$

$$\text{Value of the 2nd circular natural frequency: } \alpha_2 = \frac{2\pi}{T_2} = \dots\dots\dots$$

Determination of the beat frequency

Time of 5 periods of beating:

$$T_{d1} = \dots\dots\dots [\text{s}]; \quad T_{d2} = \dots\dots\dots [\text{s}]; \quad T_{d3} = \dots\dots\dots [\text{s}]$$

$$\text{Average value of the beat period: } T_d = \frac{T_{d1}+T_{d2}+T_{d3}}{3 \times 5} = \dots\dots\dots$$

$$\text{Value of the beat frequency: } \alpha_d = \frac{2\pi}{T_d} = \dots\dots\dots$$

Measured and determined numerical data:

spring stiffness: $k = \dots\dots\dots$;

mass at the end of the rod: $m = \dots\dots\dots$;

rod length: $l = \dots\dots\dots$

Results of theoretical calculations

Calculation of the first circular natural frequency:

$$\alpha_1 = \sqrt{\frac{9}{8} \frac{g}{l}} = \dots\dots\dots$$

Calculation of the second circular natural frequency:

$$\alpha_2 = \sqrt{\frac{9}{8} \frac{g}{l} + \frac{3}{8} \frac{k}{m}} = \dots\dots\dots$$

Calculation of the beat frequency:

$$\alpha_d = \alpha_2 - \alpha_1 = \dots\dots\dots$$

Comparison of experimental and theoretical results

Conclusions