

DETERMINATION OF CRITICAL SPEEDS OF SHAFTS

Vibrations and Machine Dynamics – Exercise 4*

1 Objective of the Exercise

Determination and measurement of three consecutive critical speeds, observation of the corresponding deflection modes of the shaft, and comparison of the analytically calculated critical speeds with the measured values.

2 Theoretical Introduction

Critical speed is a concept associated with rotating machinery used to denote the angular velocity of a shaft at which the deflection amplitudes of the rotor and the associated vibrations of the machine housing increase excessively.

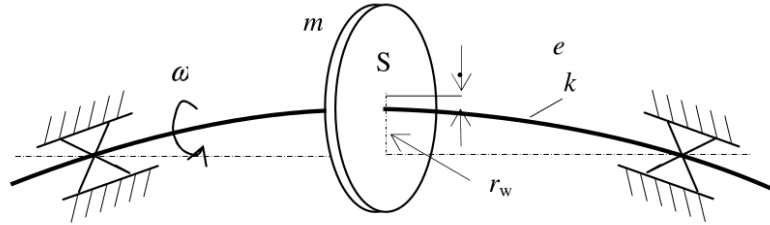


Figure 1: Diagram of a rotor with a flexible, massless shaft and a heavy disk

Consider the simplest rotor system, which consists of a disk of mass m attached at the midpoint of a span of a massless shaft of stiffness k (Fig. 1). We assume that the center of mass of the disk S does not coincide with the shaft axis and is offset from it by a distance e , which is called eccentricity or unbalance. This eccentricity is caused by inhomogeneity of the disk or its non-axial mounting on the shaft. We assume that the plane of the disk is perpendicular to the shaft axis. If the rotor rotates at a constant angular speed ω , the shaft deflects and whirls in a deflected shape. When the shaft rotates at any arbitrary rotational speed, the forces acting on the disk must be in equilibrium. The elastic restoring force of the shaft balances the centrifugal force, namely:

$$kr_w = (r_w + e)m\omega^2. \quad (1)$$

Rearranging this relationship yields the equation

$$r_w = e \frac{\omega^2}{\frac{k}{m} - \omega^2}, \quad (2)$$

from which it follows that the shaft radius r_w will increase and theoretically becomes infinitely large when the denominator of the right-hand side of Equation (2) becomes zero. This can occur when the angular speed of the shaft reaches the value:

$$\omega = \omega_{kr} = \sqrt{\frac{k}{m}}. \quad (3)$$

The velocity ω_{kr} is called the critical speed because it is dangerous for machine operation. When damping is considered, the shaft deflections r_w will be bounded; however, their magnitude may still be large enough to cause plastic deformation of the rotor or lead to rubbing of the rotor against, for example, the stator windings of an electric motor.

It is easy to notice that the expression obtained for the critical speed is identical to the formula defining the natural angular frequency of a single-degree-of-freedom system. In the discussed case, the critical speed of the rotating shaft with a mounted disk is equal to the natural frequency of transverse vibrations of the disk-rotor system.

Systems with multiple degrees of freedom (e.g., a shaft carrying several disks or a shaft with continuously distributed mass) possess a higher number of natural frequencies and multiple critical speeds.

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2.1 Rotor as a Single-Degree-of-Freedom System

If the gyroscopic effect is neglected in the system under consideration, the values of consecutive critical speeds are equal to the values of consecutive transverse natural frequencies. Calculating the critical speeds thus reduces, in most cases, to calculating the natural frequencies. One of the simplest methods to determine the fundamental natural frequency is to use the relationship defining the natural angular frequency of a single-degree-of-freedom system

$$\alpha = \sqrt{\frac{k}{m}}, \quad (4)$$

where k and m are replaced by the equivalent parameters k_z and m_z

$$\alpha_1 = \sqrt{\frac{k_z}{m_z}}. \quad (5)$$

In the case of the model rotor under consideration, i.e., a beam of constant cross-section simply supported at both ends, we have

$$k_z = \frac{48 \times EJ}{l^3}. \quad (6)$$

The equivalent mass is determined by the relationship obtained via Rayleigh's energy method

$$m_z = \frac{17}{35} m. \quad (7)$$

Substituting the above quantities into (5) yields

$$\alpha_1 = \sqrt{\frac{48 \times 35 \times EJ}{17 \times m \times l^3}}, \quad (8)$$

and expressing the mass of the beam via length l and mass per unit length μ ($m = \mu l$), the expression for the natural frequency, and thus for the (only, in this approach) critical speed, takes the form

$$\omega_{krI} = \alpha_1 = \frac{9.941}{l^2} \sqrt{\frac{EJ}{\mu}}. \quad (9)$$

2.2 Rotor as a System with a Finite Number of Degrees of Freedom. Myklestad Method

2.2.1 State Vector and Transfer Matrix

A complex or continuous system can be divided into segments, also referred to as finite elements. This is one of the methods used to approximate systems with continuous mass distribution using an equivalent discrete system. To describe the discretization process, the concepts of state vector and transfer matrix are introduced. The state vector is a column of numbers, each representing the value of a variable at a given location in the system. Each component of the state vector is called a state variable. Examples of state vectors are shown in Fig. 2.

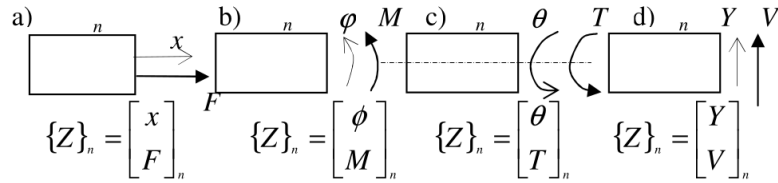


Figure 2: State vectors for various loading conditions: a) tension, b) bending, c) torsion, d) shear

The order of the components in the state vector is arbitrary. Under a combined loading state, e.g., in bending and shear, the state vector has the form

$$\{Z\}_n = \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n = [Y \quad \phi \quad M \quad V]_n^T. \quad (10)$$

The transfer matrix is used to determine the state vector at a specific location in the system using the state vector at another location. An example of partitioning a system subjected to bending into beam segments (finite elements) is shown in Fig. 3.

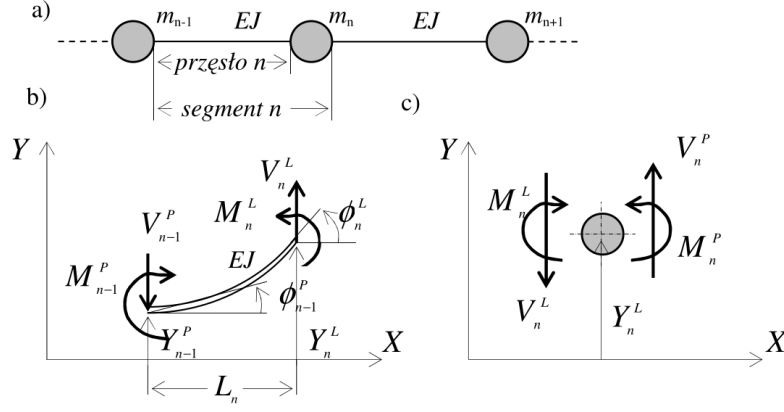


Figure 3: Partitioning of a system subjected to bending into beam segments

A single n -th segment consists of a massless span representing a spring of stiffness EJ and a lumped mass m_n . Superscripts L and P in the notation in Fig. 3 denote the left and right sides of mass m_n , respectively. Due to such segment construction, the transfer matrix of this element consists of two parts:

1. the matrix of the massless elastic part of the segment,
2. the matrix of the mass part.

The matrix of the massless elastic part relates the state vectors $\{Z\}_{n-1}^P$ and $\{Z\}_n^L$ at the beginning and end of the n -th span. Based on Fig. 3b, the equilibrium conditions give

$$V_n^L = V_{n-1}^P, \quad M_n^L = M_{n-1}^P - L_n V_{n-1}^P, \quad (11)$$

where: M – bending moment, V – shear force.

According to Fig. 3a, the change in slope angle is defined by

$$\phi_n^L = \phi_{n-1}^P + M_{n-1}^P \left(\frac{L}{EJ} \right)_n + V_{n-1}^P \left(\frac{L^2}{2EJ} \right)_n. \quad (12)$$

Substituting (11) into (12) yields

$$\phi_n^L = \phi_{n-1}^P + M_{n-1}^P \left(\frac{L}{EJ} \right)_n - V_{n-1}^P \left(\frac{L^2}{2EJ} \right)_n. \quad (13)$$

The deflection change of the span is given by

$$Y_n^L = Y_{n-1}^P + L_n \phi_{n-1}^P + M_{n-1}^P \left(\frac{L^2}{2EJ} \right)_n + V_{n-1}^P \left(\frac{L^3}{6EJ} \right)_n. \quad (14)$$

Substituting (11) into (14) yields

$$Y_n^L = Y_{n-1}^P + L_n \phi_{n-1}^P + M_{n-1}^P \left(\frac{L^2}{2EJ} \right)_n + V_{n-1}^P \left(\frac{L^3}{6EJ} \right)_n. \quad (15)$$

The matrix of the elastic part is obtained by expressing (11), (13), and (15) in matrix form

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^L = \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{L^2}{2EJ} \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P. \quad (16)$$

To determine the matrix of the mass part of the n -th beam element, mass m_n isolated from its constraints is considered (Fig. 3c). From the equations of forces and moments acting on a beam element undergoing harmonic motion with angular frequency α , one obtains

$$V_n^P = V_n^L - \alpha^2 m_n Y_n^L \quad \text{and} \quad M_n^P = M_n^L - \alpha^2 J_0 \phi_n^L. \quad (17)$$

During rigid body movement of m_n , the following equalities hold:

$$\phi_n^L = \phi_n^P, \quad Y_n^L = Y_n^P. \quad (18)$$

The matrix of the mass part is established on the basis of (17) and (18)

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\alpha^2 J_0 & 1 & 0 \\ -\alpha^2 m & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^L. \quad (19)$$

The transfer matrix of the n -th beam segment is obtained by inserting vector (16) into (19)

$$\begin{aligned} \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_n^P &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\alpha^2 J_0 & 1 & 0 \\ -\alpha^2 m & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{2EJ}{L^2} \\ 0 & 0 & 1 & -\frac{2EJ}{L} \\ 0 & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P = \\ &= \begin{bmatrix} 1 & L & \frac{L^2}{2EJ} & -\frac{L^3}{6EJ} \\ 0 & 1 & \frac{L}{EJ} & -\frac{2EJ}{L^2} \\ 0 & -\alpha^2 J_0 & 1 - \alpha^2 J_0 \frac{L}{EJ} & -L + \alpha^2 J_0 \frac{L^2}{2EJ} \\ -\alpha^2 m & -\alpha^2 mL & -\alpha^2 m \frac{L^2}{2EJ} & 1 + \alpha^2 m \frac{L^3}{6EJ} \end{bmatrix}_n \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_{n-1}^P. \end{aligned} \quad (20)$$

In abbreviated notation, Equation (20) takes the following form:

$$\{Z\}_n^P = H_n \{Z\}_{n-1}^P, \quad (21)$$

where H_n is the transfer matrix of the n -th element. The state vector of the n -th element is related to the state vector $\{Z\}_0^P$ at the boundary of the system by:

$$\{Z\}_n^P = \{H_n H_{n-1} \dots H_2 H_1\} \{Z\}_0^P. \quad (22)$$

2.2.2 Myklestad Method for Determining System Natural Frequencies

Consider a beam structure partitioned into a finite number of segments. It is assumed that the masses of the segments are concentrated at their ends.

Equation (22), relating the state vector of the n -th element to the state vector at the boundary of the system, illustrates the calculation formula for the beam system under consideration. The boundary conditions for the most common support types in beam systems are presented in Table 1.

Table 1:

	Y	ϕ	M	V
Simply supported	0	ϕ	0	V
Free end	Y	ϕ	0	0
Clamped	0	0	M	V

It follows that at the initial point (location 0) of the beam, there are two non-zero initial conditions depending on the support type. A similar situation applies to the other end of the beam. The procedure for calculating natural frequencies using the Myklestad method involves assuming a value of α and carrying out calculations using transfer matrices. The value of frequency α that simultaneously satisfies the boundary conditions at both ends of the system is the desired natural frequency.

Example. Using the Myklestad method, determine the flexural natural frequencies of the cantilever beam shown in Fig. 4a.

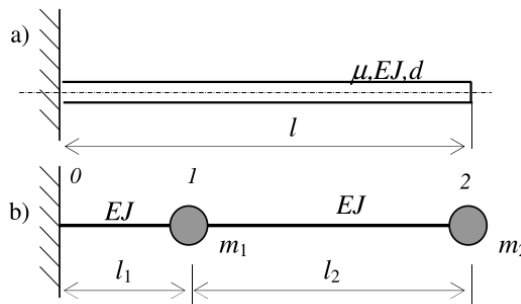


Figure 4: Cantilever beam and vibration model

Data:

- $m = 0.222$ kg (beam mass);
- $l = 1$ m (beam length);
- $d = 0.006$ m (beam diameter);
- $J = \pi d^4/64 = 63.6 \times 10^{-12}$ m⁴;
- $E = 2.1 \times 10^{11}$ N/m²;
- $\mu = m/l = 0.222$ kg/m.

Solution

The beam is divided into two segments of length $l_1 = 0.25$ m and $l_2 = 0.75$ m, respectively (Fig. 4b). The mass of the 1st segment is concentrated at its right end, while the mass of the 2nd segment is divided into two equal parts placed at both ends. The lumped masses determined in this way are $m_1 = 0.13875$ kg and $m_2 = 0.08325$ kg. Next, the transfer matrices of the segments are established, followed by utilizing relationship (22) linking the state vector of the 2nd segment to the state vector at the system boundary.

$$\{Z\}_2^P = H_2 H_1 \{Z\}_0^P, \quad (a)$$

where:

$$\{Z\}_0^P = [Y \quad \phi \quad M \quad V]_0^T = [0 \quad 0 \quad M_0 \quad V_0]^T. \quad (b)$$

Substituting numerical values into (20) determines the transfer matrices for both segments

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_1^P = \begin{bmatrix} 1 & 0.25 & 2.34 \times 10^{-3} & -1.95 \times 10^{-4} \\ 0 & 1 & 1.87 \times 10^{-2} & -2.34 \times 10^{-3} \\ 0 & 0 & 1 & -0.25 \\ -0.138\alpha^2 & -3.47 \times 10^{-2}\alpha^2 & -3.245 \times 10^{-4}\alpha^2 & 1 + 2.704 \times 10^{-5}\alpha^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_0 \\ V_0 \end{bmatrix}, \quad (c)$$

$$\begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_2^P = \begin{bmatrix} 1 & 0.75 & 2.1 \times 10^{-2} & -5.26 \times 10^{-3} \\ 0 & 1 & 5.61 \times 10^{-2} & -2.1 \times 10^{-2} \\ 0 & 0 & 1 & -0.75 \\ -8.32 \times 10^{-2}\alpha^2 & -6.24 \times 10^{-2}\alpha^2 & -1.75 \times 10^{-3}\alpha^2 & 1 + 4.38 \times 10^{-5}\alpha^2 \end{bmatrix} \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_1^P. \quad (d)$$

Taking into account (c) and (d) in (a) using general notation yields

$$\{Z\}_2^P = H_2 H_1 \{Z\}_0^P = [H] \{Z\}_0^P \Rightarrow \begin{bmatrix} Y \\ \phi \\ M \\ V \end{bmatrix}_2^P = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_0 \\ V_0 \end{bmatrix}. \quad (e)$$

The bending moment and shear force at the free end must vanish (Table 1). Using the notation applied in (e), this condition can be written as

$$\begin{aligned} M_2^P &= H_{33}M_0 + H_{34}V_0 = 0, \\ V_2^P &= H_{43}M_0 + H_{44}V_0 = 0. \end{aligned} \quad (f)$$

The condition for non-trivial solutions to exist for the homogeneous system of equations (f) is the vanishing of its characteristic determinant, hence

$$\begin{vmatrix} H_{33} & H_{34} \\ H_{43} & H_{44} \end{vmatrix} = 0. \quad (g)$$

This equation allows for determining the target natural frequencies. Substituting the numerical data into Equation (g) yields

$$\begin{vmatrix} 1 + 2.434 \times 10^{-4}\alpha^2 & -1 - 2.028 \times 10^{-5}\alpha^2 \\ -3.44 \times 10^{-5}\alpha^2 - 1.42 \times 10^{-7}\alpha^4 & 1 + 1.06 \times 10^{-7}\alpha^2 + 1.18 \times 10^{-8}\alpha^4 \end{vmatrix} = 0. \quad (h)$$

In expanded form, Equation (h) can be written as

$$\alpha^4 - 35.97 \times 10^3 \alpha^2 + 1.69 \times 10^7 = 0. \quad (i)$$

Solving Equation (i) enables the calculation of the natural frequencies:

$$\alpha_1 = 21.8 \text{ [rad/s]} \Rightarrow f_1 = 3.47 \text{ [Hz]}, \quad (j)$$

$$\alpha_2 = 188.4 \text{ [rad/s]} \Rightarrow f_2 = 30 \text{ [Hz]}. \quad (23)$$

2.3 Formulation of Equations for Structural Strength Quantities in Bending Beams

Let us begin by recalling fundamental principles from mechanics of materials. Figure 5a shows a cantilever beam of stiffness EI and length l , loaded at its free end by a transverse force F . We can write the differential equation governing the relation between the second derivative of the beam deflection and the bending moment:

$$EIy''(x) = F(l - x) \quad (a)$$

The linear dependence of y'' on coordinate x is shown in Figure 5c. Integrating Equation (a) yields the function defining the slope angle of the beam y' :

$$EIy'(x) = F \left(lx - \frac{x^2}{2} \right). \quad (b)$$

The parabolic dependence of y' on coordinate x is shown in Figure 5b. Integrating Equation (b) yields the function governing the deflection of the beam y :

$$EIy(x) = F \left(\frac{lx^2}{2} - \frac{x^3}{6} \right). \quad (c)$$

The relation (cubic curve) between y and coordinate x is depicted in Figure 5a. These are basic concepts known from mechanics of materials. Returning to Equation (a), differentiating Equation (a) yields the function defining the shear force in the beam:

$$EIy'''(x) = -F. \quad (d)$$

The shear force distribution is shown in Figure 5d.

It should be noted that writing equations (a)–(d) was possible because the beam loading is known, and in particular, because it is independent of the beam deflection $y(x)$.

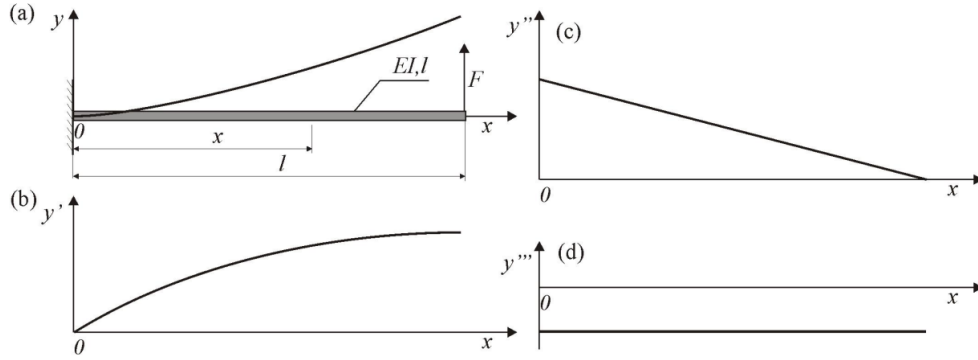


Figure 5: Cantilever beam bent by a concentrated force

Another example:

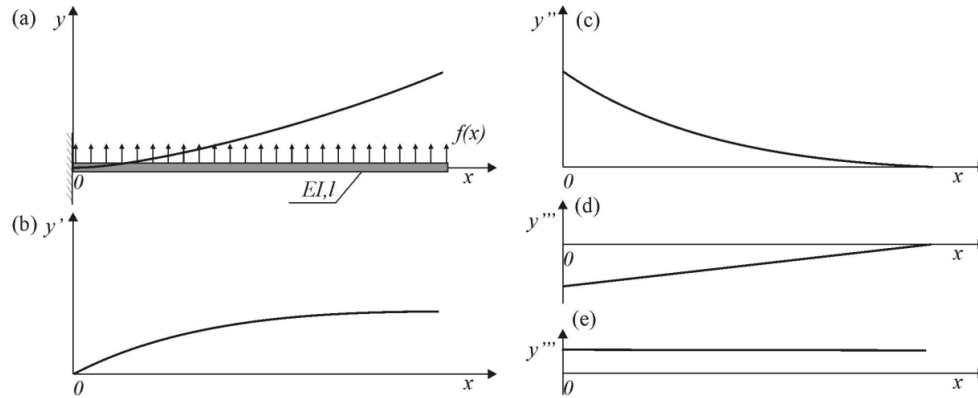


Figure 6: Cantilever beam bent by a continuously distributed load

Figure 6a shows a cantilever beam of flexural rigidity EI and length l , subjected along its entire length to a constant distributed load of intensity $f(x)$ (N/m). We can write the differential equation describing the relation between the second derivative of beam deflection and bending moment:

$$EIy''(x) = f(x) \frac{(l - x)^2}{2} \quad (e)$$

The parabolic dependence of y'' on coordinate x is shown in Figure 6c. Integrating Equation (e) gives the expression defining the slope angle y' :

$$EIy'(x) = f(x) \left(\frac{l^2x}{2} - \frac{lx^2}{2} + \frac{x^3}{6} \right). \quad (f)$$

The dependence of y' on coordinate x is shown in Figure 6b. Integrating Equation (f) yields the function describing the beam deflection y :

$$EIy(x) = f(x) \left(\frac{l^2 x^2}{4} - \frac{lx^3}{6} + \frac{x^4}{24} \right). \quad (g)$$

The relationship (quartic curve) between y and coordinate x is shown in Figure 6a. These are also fundamental concepts known from strength of materials.

Returning to Equation (e), differentiating it yields the function defining the shear force along the beam:

$$EIy'''(x) = -f(x)(l - x) \quad (h)$$

The shear force is shown in Figure 6d.

Finally, differentiating Equation (h) yields the function determining the intensity of the continuous load:

$$EIy''''(x) = f(x) \quad (i)$$

It should be emphasized that writing equations (e)–(h) was again possible solely because the load on the beam is known, and specifically independent of the beam deflection $y(x)$.

The system under consideration in this exercise is a rotating single-span beam shown in Fig. 7.



Figure 7: Beam model corresponding to a shaft in bending

Due to the motion of the beam, the function y is a function of two variables: the position of the beam section x and time t . Furthermore, the continuously distributed load consists of the centrifugal inertia force, whose magnitude depends on the acceleration of individual cross-sections of the beam. Not knowing the function $y(x, t)$ prevents formulation of an equation analogous to Equation (e). However, it is possible to formulate an equation analogous to (i) of the form:

$$EIy''''(x, t) = -\frac{m}{l}\ddot{y}(x, t) \quad (j)$$

where: m – mass of the beam, l – length of the beam.

The dependence of load intensity on the acceleration of beam cross-sections prevents direct integration of this equation. It is solved as follows, after writing the equation in the form

$$\ddot{y}(x, t) + q^2 y''''(x, t) = 0, \quad q^2 = \frac{EI}{m} \quad (k)$$

Its solution is assumed in the separable form

$$y(x, t) = Y(x)T(t), \quad T(t) = a \cos \alpha t + b \sin \alpha t. \quad (l)$$

Thus, it is assumed that vibrations are harmonic or represent a linear combination of harmonic vibrations (when multiple values of α exist for which function (l) satisfies Equation (k)). From (l) it follows that

$$\begin{aligned} \ddot{y}(x, t) &= Y(x)\ddot{T}(t) = -\alpha^2 Y(x)T(t) \\ y''''(x, t) &= Y''''(x)T(t) \end{aligned} \quad (m)$$

Substituting (m) into (k) yields

$$(-\alpha^2 Y(x) + q^2 Y''''(x))T(t) = 0 \quad (n)$$

from which, given $T(t) \neq 0$,

$$Y''''(x) = \eta^4 Y(x), \quad \eta^4 = \frac{\alpha^2}{q^2} = \frac{\alpha^2 m}{EI} \quad (o)$$

Equation (15) is satisfied by the following particular solutions:

$$\begin{aligned} Y_1(x) &= A \cos \eta x, \\ Y_2(x) &= B \sin \eta x, \\ Y_3(x) &= C \cosh \eta x, \\ Y_4(x) &= D \sinh \eta x. \end{aligned} \quad (p)$$

Subject to the boundary conditions

$$Y(0) = 0, \quad Y''(0) = 0, \quad Y(l) = 0, \quad Y''(l) = 0, \quad (q)$$

one obtains $A = C = D = 0$ and $B \sin \eta l = 0$. Thus, given the obvious condition $B \neq 0$, it follows that

$$\sin \eta l = 0 \quad (r)$$

and consequently:

$$\eta l = i\pi, \quad i = 1, 2, 3, \dots \quad \eta^4 l^4 = i^4 \pi^4 = \frac{\alpha^2 m l^4}{EI} \quad (s)$$

From the last equation, one obtains the formula determining the consecutive natural angular frequencies of the free vibrations of a single-span beam subjected to centrifugal inertia forces:

$$\alpha_i = \frac{i^2 \pi^2}{l^2} \sqrt{\frac{EI}{m}}, \quad i = 1, 2, 3, \dots \quad (t)$$

The harmonic vibrations occurring at individual angular frequencies take the form of the function:

$$y(x, t) = B \sin \left(\frac{i\pi}{l} x \right) (a \cos(\alpha_i t) + b \sin(\alpha_i t)) \quad (u)$$

2.4 Rotor as a Continuous Mass System

To determine the precise value of the first natural frequency as well as higher natural frequencies for a beam of uniform cross-section, the classical Euler-Bernoulli equation for transverse beam vibrations can be used:

$$\frac{\partial^4 y}{\partial x^4} + a^2 \frac{\partial^2 y}{\partial t^2} = 0, \quad (24)$$

where:

$$a^2 = \frac{\mu}{EJ}. \quad (25)$$

A solution to Equation (23) is sought as a product of a spatial function x and a time-dependent function t

$$y(x, t) = U(x)T(t). \quad (26)$$

Substituting (25) into (23) yields

$$U^{IV}(x)T(t) + a^2 U(x)\ddot{T}(t) = 0, \quad (27)$$

hence

$$\frac{U^{IV}(x)}{U(x)} = -a^2 \frac{\ddot{T}(t)}{T(t)}. \quad (28)$$

Satisfying Equation (27) for arbitrary values of x and t requires both sides of the equation to be equal to a constant, denoted by k^4

$$\frac{U^{IV}(x)}{U(x)} = -a^2 \frac{\ddot{T}(t)}{T(t)} = k^4. \quad (29)$$

Hence, solving the transverse beam vibration problem reduces to solving two independent ordinary differential equations:

$$U^{IV}(x) - k^4 U(x) = 0, \quad (4.29)$$

$$\ddot{T}(t) + \alpha^2 T(t) = 0, \quad (4.30)$$

where:

$$\alpha^2 = \frac{k^4}{a^2}. \quad (30)$$

The general solution to Equation (29) takes the following form:

$$U(x) = A \sin kx + B \cos kx + C \sinh kx + D \cosh kx, \quad (31)$$

where A, B, C, D are integration constants determined from boundary conditions. For pinned end supports, as in the experimental setup, the boundary conditions are

$$\begin{aligned} y(0, t) &= 0, & y(l, t) &= 0, \\ \frac{\partial^2 y(x, t)}{\partial x^2} \Big|_{x=0} &= 0, & \frac{\partial^2 y(x, t)}{\partial x^2} \Big|_{x=l} &= 0. \end{aligned} \quad (32)$$

Substituting solution (32) into (33) ultimately yields the sequence of eigenfunctions governing the deflection mode shapes of a beam simply supported at both ends:

$$U_n(x) = \sin k_n x = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots \quad (33)$$

The solution to differential Equation (30) is given by:

$$T_n(t) = K_n \cos \alpha_n t + L_n \sin \alpha_n t, \quad (34)$$

where K_n and L_n are constants determined from initial conditions. The values of natural frequencies α_n corresponding to consecutive eigenfunctions of the beam are determined according to (31) and (34) by

$$\alpha_n = \frac{k_n^2}{a} = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EJ}{\mu}}, \quad n = 1, 2, \dots \quad (35)$$

Finally, the solution to the classical differential equation of transverse beam vibrations simply supported at both ends is expressed as:

$$y(x, t) = \sum_{n=1}^{\infty} (K_n \cos \alpha_n t + L_n \sin \alpha_n t) \sin \frac{n\pi x}{l}. \quad (36)$$

3 Experimental Setup

The schematic diagram of the test apparatus is shown in Fig. 5. It is a laboratory bench designed to observe and measure three consecutive critical speeds of a shaft.

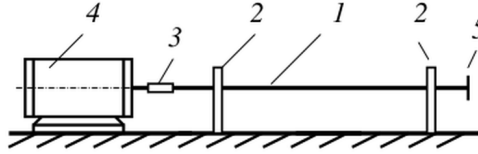


Figure 8: Schematic diagram of the experimental apparatus

Shaft 1, mounted in self-aligning ball bearings housed in bearing pedestals 2, is driven through a flexible coupling 3 by a DC motor 4. The motor and bearing housings are mounted on a common base plate. A small disk 5 is attached to the shaft end to enable rotational speed measurements. The power supply system allows adjustment of the motor speed. The shaft dimensions were selected such that within the speed range of the motor ($0 \div 6000$ rpm), three consecutive critical speeds can be observed. The shaft is formed by a steel rod with length $l = 1$ m, diameter $d = 6$ mm, and mass $m = 0.222$ kg.

4 Procedure

Using the provided geometry and mass data of the shaft mounted on the test bench, calculate its three lowest critical speeds. These calculations should be conducted treating the rotor successively as a single-degree-of-freedom system, a system with continuous mass distribution, and a discrete system with a finite number of degrees of freedom.

By treating the rotor as a single-degree-of-freedom system, the fundamental critical speed can be calculated using Equation (9) or Equation (36) for $n = 1$. Higher critical speeds (second and third) can also be evaluated using Equation (36) for subsequent values of n , treating the rotor as a continuous mass system. The calculated values should be recorded in the summary table in the report.

Next, perform critical speed calculations for the shaft mounted on the test apparatus treated as a discrete system with a finite number of degrees of freedom.

Due to the structural complexity of turbomachinery, engineering practice for critical speed evaluation usually relies on system discretization followed by computational analysis of the discretized model. The discretization approach (partitioning into elements) directly influences the accuracy of the computed results.

The Myklestad method is employed in this exercise to compute the critical speeds of the test shaft modeled as a discrete system. Using numerical software, calculate the critical speed values corresponding to different element partitions. Compare the resulting critical speeds with predictions from the other theoretical models.

In the experimental part of the exercise, observe and measure the critical speeds of the test shaft mounted on the test stand.

To observe rotor critical speeds, start the motor and gradually increase the rotational speed of the shaft. Measure the rotational speed at which maximum shaft deflections occur. Repeat this measurement three times. If deflections are excessive and lead to impact against safety limiters, limit measurements to two repetitions, performed below and above the critical speed where impacts do not occur. Record all measurements in the experiment table and determine the average value. To facilitate passing through critical speeds without excessive deflection or limiter contact, decrease motor speed prior to acceleration and traverse the resonance region with higher acceleration. Draw the observed deflection mode shapes corresponding to each critical speed.

Upon completing all measurements, compare the measured critical speeds with analytical calculations obtained via the various methods. Formulate conclusions based on the comparison of the results obtained.

5 References

1. Parszewski Z.: *Drgania i dynamika maszyn* (Vibrations and Machine Dynamics). PWN, Warszawa 1982.
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3. Tse F. S., Morse I. E., Hinkle R. T.: *Mechanical Vibrations – Theory and Applications*, Allyn and Bacon Inc., Boston London Sydney Toronto, 1978.

LABORATORY OF MECHANICAL VIBRATIONS

Exercise 4

DETERMINATION OF CRITICAL SPEEDS OF SHAFTS

Group: _____
Team: _____

Date: _____

Full Name:

1. _____
2. _____
3. _____
4. _____
5. _____
6. _____

Results

Shaft: steel rod of length $l = 1$ m, diameter $d = 6$ mm, and mass $m = 0.222$ kg.

Calculation of critical speeds of the shaft considered as a single-degree-of-freedom system [rad/s]:

$$\alpha_1 = \sqrt{\frac{48 \times 35 \times EJ}{17 \times m \times l^3}} \approx \omega_{kr} = \frac{9.941}{l^2} \sqrt{\frac{EJ}{\mu}} =$$

Calculation of critical speeds of the shaft considered as a continuous system [rad/s]:

$$\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EJ}{\mu}} = \quad , \quad \omega_2 = \frac{4\pi^2}{l^2} \sqrt{\frac{EJ}{\mu}} = \quad , \quad \omega_3 = \frac{9\pi^2}{l^2} \sqrt{\frac{EJ}{\mu}} =$$

	Measurement	Calculations					
Crit. Speed	ω_{meas} [rad/s]	ω_{calc1} [rad/s]	$\frac{\omega_{\text{calc1}}}{\omega_{\text{meas}}} [\%]$	ω_{calc2} [rad/s]	$\frac{\omega_{\text{calc2}}}{\omega_{\text{meas}}} [\%]$	ω_{calc3} [rad/s]	$\frac{\omega_{\text{calc3}}}{\omega_{\text{meas}}} [\%]$
I							
II							
III							

Table of measurements and calculation results

Sketches of deflection mode shapes corresponding to three consecutive critical speeds:

Conclusions: