

DETERMINATION OF THE MINIMUM PERIOD OF FREE OSCILLATIONS OF A PHYSICAL PENDULUM

Vibrations and Machine Dynamics – Exercise 1B*

Aim of the exercise

The aim of the exercise is to determine the position of the axis of rotation of a physical pendulum for which the period of free oscillations of this pendulum is the shortest.

Description of the experimental stand

The object under consideration is a physical pendulum in the form of a homogeneous rod of length $l = 1.03$ m, capable of performing oscillatory motion about the axis of rotation A. The pendulum is shown in Fig. 1.

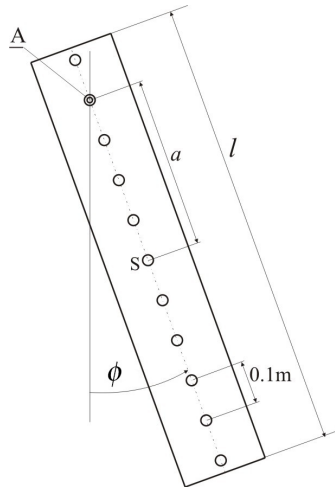


Figure 1

On the basis of the dynamic equation of rotational motion of a rigid body (the equation of equilibrium of moments of forces about the axis of rotation, taking into account the moment of the inertia force), the following equation of motion of the pendulum can be formulated:

$$B_A \ddot{\phi} + mga\phi = 0 \quad (1)$$

where:

- B_A – moment of inertia with respect to the axis of rotation,
- m – mass,
- g – gravitational acceleration,

*In: LABORATORY OF MECHANICAL VIBRATIONS, K. Marynowski (ed.), Łódź 2002

- a – distance between the axis passing through the centre of mass S and the axis of rotation,
- ϕ – angle of rotation.

The period of free oscillations of such a pendulum is:

$$T_A = 2\pi \sqrt{\frac{B_A}{mga}} \quad (2)$$

Do you know how to solve equation (1) and derive formula (2)?

Theoretical analysis

According to Steiner's theorem on moments of inertia of a body with respect to parallel axes

$$B_A = B_S + ma^2 \quad (3)$$

where B_S – moment of inertia with respect to the axis passing through the centre of mass.

The moment B_S is constant and, in the case of the analysed pendulum in the form of a homogeneous rod, is expressed by the formula

$$B_S = \frac{ml^2}{12} \quad (4)$$

Substituting equation (3) into (2) yields the following formula for the period of oscillations of the pendulum:

$$T_A(a) = 2\pi \sqrt{\frac{B_S + ma^2}{mga}} = 2\pi \sqrt{\frac{\frac{l^2}{12} + a^2}{ga}} \quad (5)$$

As indicated, the period of oscillations can be given the form of a function whose argument is the distance a between the axis of rotation and the axis passing through the centre of mass.

It is readily seen that when the distance a tends to zero (the axis of rotation approaches the centre of mass), the period of oscillations tends to infinity: the numerator of the expression under the square root tends to the constant value B_S while the denominator tends to zero.

It is similarly easy to observe that when the distance a tends to infinity (the axis of rotation moves away from the centre of mass to an unlimited distance many times greater than l), the period of oscillations also tends to infinity: the derivative of the numerator of the expression under the square root ($2ma$) tends to infinity while the derivative of the denominator is constant (mg). The question arises for what value of the argument a the function (5) attains an extremal value and what that value is. To find the answer, equation (5) must be differentiated with respect to the variable a , yielding:

$$\frac{dT_A(a)}{da} = \frac{2\pi}{2\sqrt{\frac{B_S + ma^2}{mga}}} \frac{2m^2a^2g - (B_S + ma^2)mg}{m^2g^2a^2} = \frac{\pi \left(a^2 - \frac{l^2}{12}\right)}{\sqrt{\frac{gl^2a^3}{12} + ga^5}} \quad (6)$$

It is readily noticed that this function attains the value zero for

$$\frac{dT_A(a)}{da} = 0 \quad \rightarrow \quad a = \frac{l}{\sqrt{12}} \quad (7)$$

and this is the only extremum – logic suggests that it is a minimum. The period of oscillations T_A is the shortest for the distance a equal to the length of the pendulum divided by the square root of 12. Substituting this value into (5), one can calculate

$$T_{Amin} = 2\pi \sqrt{\frac{ml^2 + \frac{l^2}{12}}{mg \frac{l}{\sqrt{12}}}} = 2\pi \sqrt{\frac{\sqrt{12}l}{6g}} \quad (8)$$

It is necessary to verify whether, for the value of a expressed by (7), the function (5) indeed attains a minimum rather than a maximum. To this end, by differentiating (6) we determine the second derivative of the period T_A with respect to the distance a :

$$\frac{d^2 T_A(a)}{da^2} = \pi \frac{2a \sqrt{\frac{gl^2 a^3}{12} + ga^5} - \left(a^2 - \frac{l^2}{12}\right) \frac{5ga^4 + \frac{3gl^2 a^2}{12}}{2\sqrt{\frac{gl^2 a^3}{12} + ga^5}}}{\frac{gl^2 a^3}{12} + ga^5} \quad (9)$$

The formula looks TERRIFYING – everyone can see that; but: for $a = \frac{l}{\sqrt{12}}$ the expression in parentheses in the numerator is equal to zero, so the second term of the numerator vanishes. The first term is positive, the denominator is positive, hence for

$$a = \frac{l}{\sqrt{12}} = 0 \quad \rightarrow \quad \frac{d^2 T_A(a)}{da^2} > 0. \quad (10)$$

thus the period is indeed minimal for this value of a .

Procedure

- Determine the duration of 20 periods of motion of the pendulum suspended successively by means of five holes located at distances of 0.50, 0.40, 0.30, 0.20 and 0.10 metre from the centre of mass. Dividing this time by 20, determine the periods of oscillations T_A as a function of a .
- Using formula (5), calculate T_A for the various values of a given in the table.
- Draw the theoretical graph of $T_A(a)$. Plot the measurement results on it.

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Exercise 1B

DETERMINATION OF THE MINIMUM PERIOD OF FREE OSCILLATIONS OF A PHYSICAL PENDULUM

Group: _____

Team: _____

date _____

Name and surname:

1. _____

2. _____

3. _____

4. _____

5. _____

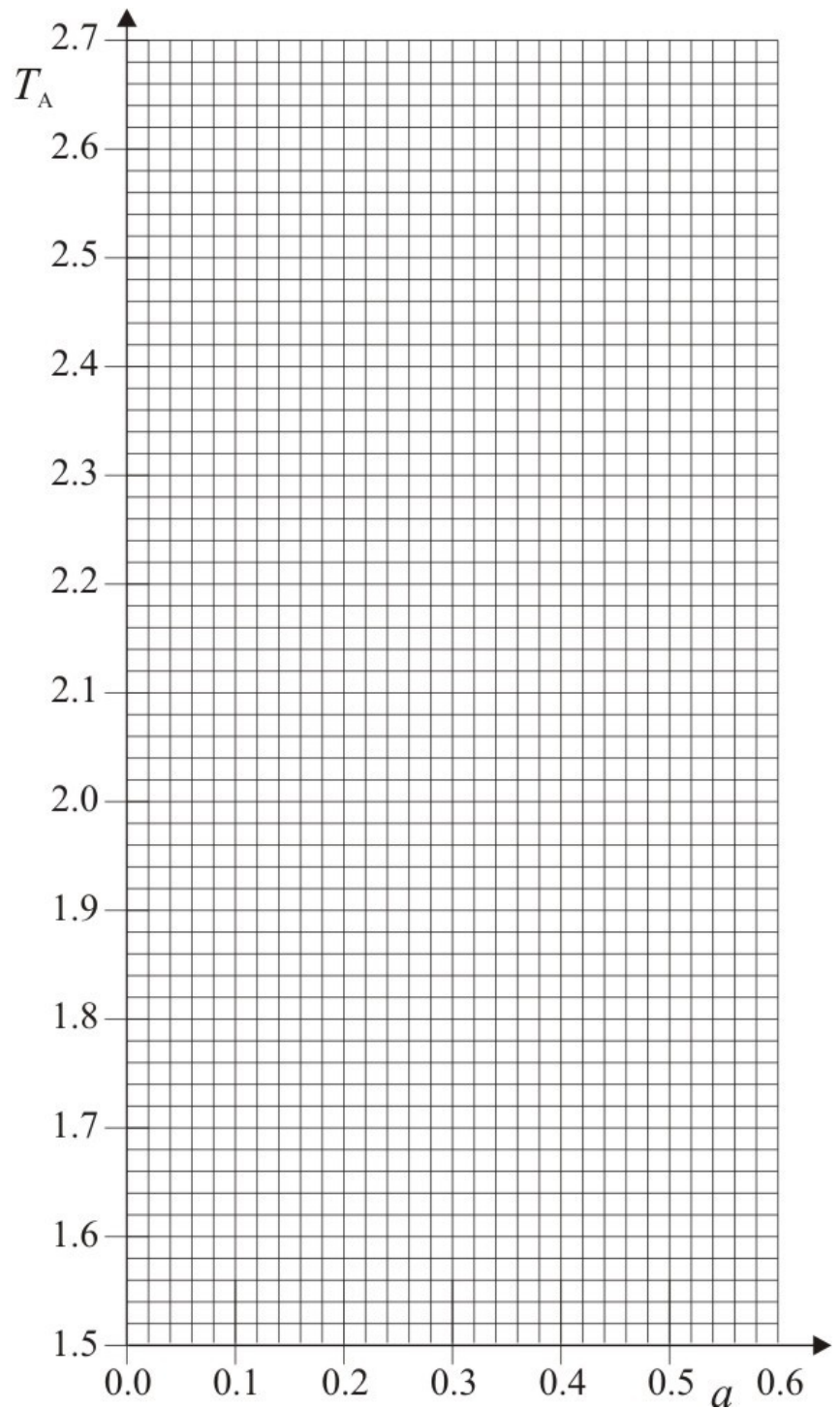
6. _____

Measurement results:

a [m]	$20T_A$ [s]	T_A [s]
0.10		
0.20		
0.30		
0.40		
0.50		

Calculation results:

a [m]	T_A [s]
0.05	
0.10	
0.15	
0.20	
0.25	
0.30	
0.35	
0.40	
0.45	
0.50	
0.55	
0.60	



Formula for calculating the period of oscillations:

$$T_A(a) = 2\pi \sqrt{\frac{l^2}{12} + a^2}, \quad l = 1.03 \text{ m}, \quad g = 9.81 \frac{\text{m}}{\text{s}^2} \quad (11)$$