

# DETERMINATION OF RESONANT FREQUENCIES OF TORSIONAL VIBRATIONS OF A SHAFT

Vibrations and Machine Dynamics – Exercise 5\*

## 1 Aim of the exercise

Measurement of the resonant frequencies of torsional vibrations of a shaft with three discs and observation of the corresponding modes of shaft torsion, as well as comparison of the resonant-frequency values obtained from measurements with the natural frequencies calculated analytically by Holzer's method.

## 2 Theoretical introduction

Holzer's method is a classical iterative numerical-tabular method used to determine the natural frequencies and the corresponding mode shapes (amplitudes) of discrete multi-degree-of-freedom systems, particularly with regard to torsional vibrations. It is most frequently applied to the analysis of complex rotor systems, drive shafts, internal-combustion engines or turbomachinery that can be modeled as an equivalent system of masses and elastic shafts. A continuous rotating system of shafts and masses (e.g. pistons, a flywheel, a propeller) is reduced to a discrete system consisting of masses (discs of moments of inertia  $B_i$ ) connected by shaft segments of torsional stiffness  $k_i$ .

Holzer's method is a version of the trial-and-error method. The calculations begin by assuming a value of the circular natural frequency  $f$  (or the frequency  $\alpha$ ). Then, in a tabular manner (easy to program even in a spreadsheet), a unit initial amplitude (angle of twist) is assumed for the first element and the sought quantities are calculated successively for each subsequent segment of the system. The calculations proceed from one end of the system to the other. The residual value obtained at the end (of moment or angle of twist) indicates whether the assumed frequency  $f$  was correct. If the equilibrium condition at the free or fixed end is not satisfied (the residual is not zero), the value of  $f$  is modified and the iteration is repeated until convergence is achieved.

The method is applied to the model of the experimental stand for the vibrating system under consideration shown in Fig. 1, which is a three-degree-of-freedom system. Free torsional vibrations of the system are considered when the shaft is fixed at one of its ends. Three identical discs of moment of inertia  $B$  are mounted on a shaft of stiffnesses  $k_1$  and  $k$ .

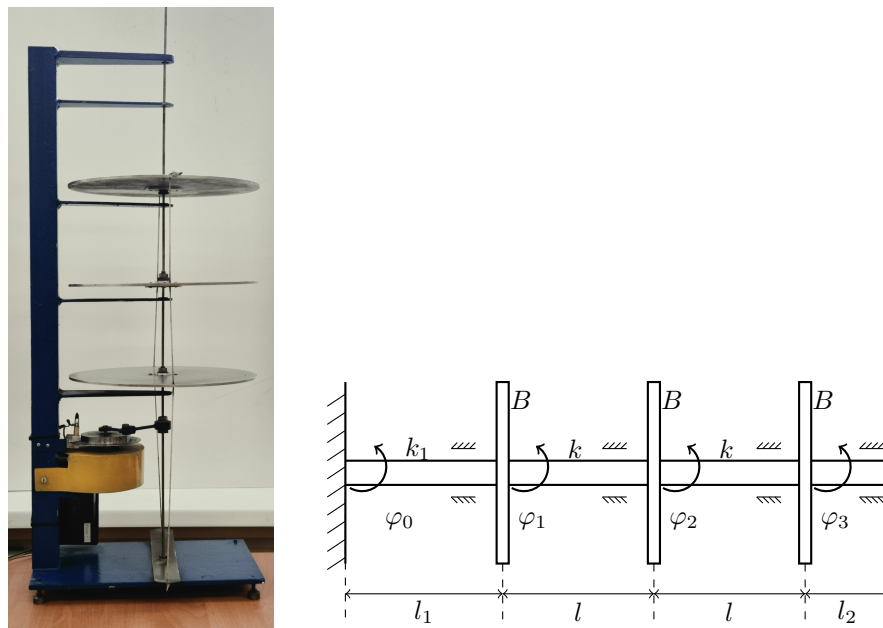


Figure 1: Experimental stand on the left and model of the vibrating system on the right

\*Author: J. Wawrzeki, in: LABORATORY OF MECHANICAL VIBRATIONS, K. Marynowski (ed.), Łódź 2002. Modifications: B. Jagiełło, J. Wojewoda 2026

The differential equations of motion have the following form:

$$\begin{aligned} B\ddot{\varphi}_1 + k(\varphi_1 - \varphi_2) + k_1(\varphi_1 - \varphi_0) &= 0 \\ B\ddot{\varphi}_2 + k(\varphi_2 - \varphi_3) - k(\varphi_1 - \varphi_2) &= 0 \\ B\ddot{\varphi}_3 - k(\varphi_2 - \varphi_3) &= 0 \end{aligned} \quad (1)$$

In order to determine the natural frequencies of torsional vibration of the system, Holzer's method has been applied. Particular solutions of the system of equations (1) are assumed in the form of harmonic functions:

$$\varphi_1 = \Phi_1 \sin \alpha t; \quad \varphi_2 = \Phi_2 \sin \alpha t; \quad \varphi_3 = \Phi_3 \sin \alpha t, \quad (2)$$

where:  $\alpha$  – the sought natural frequency of free vibration,  $\Phi_1, \Phi_2, \Phi_3$  – amplitudes of torsional vibration of the individual discs. After substitution of the solutions (2) into the system of differential equations (1) one obtains the system of algebraic equations:

$$B\Phi_1\alpha^2 - k(\Phi_1 - \Phi_2) - k_1(\Phi_1 - \Phi_0) = 0; \quad (3)$$

$$B\Phi_2\alpha^2 - k(\Phi_2 - \Phi_3) + k(\Phi_1 - \Phi_2) = 0; \quad (4)$$

$$B\Phi_3\alpha^2 + k(\Phi_2 - \Phi_3) = 0. \quad (5)$$

The system of equations (3)–(5) constitutes the basis of the program for calculating the natural frequencies of torsional vibration of the system under investigation. An arbitrary value of the amplitude of vibration of the third disc  $\Phi_3$  is assumed, e.g.  $\Phi_3 = 1$  rad, and, inserting successive values of  $\alpha$  from the expected frequency range, the amplitude  $\Phi_2$  is determined from (5):

$$\Phi_2 = \Phi_3 - \frac{B\Phi_3\alpha^2}{k} \quad (6)$$

The value of  $\Phi_2$  thus obtained is taken into account in equation (4) and the amplitude  $\Phi_1$  is calculated:

$$\Phi_1 = \Phi_2 - \frac{B\Phi_2\alpha^2 + B\Phi_3\alpha^2}{k} \quad (7)$$

The value of  $\Phi_1$  thus obtained is taken into account in equation (3) and the amplitude  $\Phi_0$  is calculated:

$$\Phi_0 = \Phi_1 - \frac{B\Phi_1\alpha^2 + B\Phi_2\alpha^2 + B\Phi_3\alpha^2}{k_1} \quad (8)$$

If  $\alpha$  is a natural frequency of the system under consideration, the boundary condition  $\Phi_0 = 0$  must be satisfied. The foregoing description shows that **Holzer's method applied to the system under consideration consists in searching for the zeros of the function (8)**. This is done by evaluating the function for successively assumed values of  $\alpha$ . If the assumed value of  $\alpha$  is not a natural frequency of the system, the boundary condition of vanishing of the amplitude  $\Phi_0$  will likewise not be satisfied. The assumed values of  $\alpha$  are refined in the vicinity of a zero (a change of sign of  $\Phi_0$ ) until a value of  $\alpha$  (natural frequency) that makes the residual zero is obtained with the required accuracy.

### 3 Experimental stand

The schematic diagram of the experimental stand is shown in Fig. 2. It consists of a vertical column 1 with a base and a shaft 2 of length  $l = 0.8$  m and diameter  $d = 0.004$  m with three discs mounted on it. The lower end of the shaft is fixed, while the upper end is supported in a manner that permits free rotation. Torsional vibrations of the system are excited kinematically by means of a lever mechanism 3. The value of the adjustable excitation frequency is read on the meter 4. Threads 5 connected to the discs display the corresponding modes of the angle of twist of the shaft.

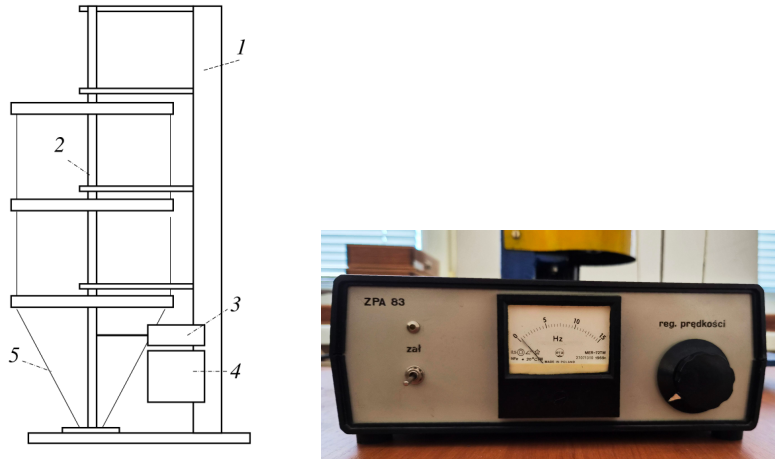


Figure 2: Schematic diagram of the experimental stand on the left, control-and-measuring device ZPA 83 on the right

The device ZPA 83 permits regulation of the rotational speed – the excitation frequency of the vibrations – in the range from 1 to 15 Hz, where three critical speeds can be identified.

## 4 Procedure

1. **Using Holzer's method, calculate the values of the natural frequencies of torsional vibration of the system by means of the computational table given below.** At the beginning of the calculations successive extreme values of  $\alpha$  from the expected frequency range (here 1–15 Hz) should be assumed, within which the search is to be conducted. The residual values of the amplitude  $\Phi_0$  obtained in this way facilitate determination of the exact value of  $\alpha$  after the search region has been narrowed to those frequency values at which a change of sign of  $\Phi_0$  occurs.

These calculations can be performed with the aid of a spreadsheet in which a table should be constructed according to a scheme containing the formulae that evaluate the successive angles  $\Phi_i$ ,  $i = 3, 2, 1, 0$  (formulae (6)–(8)), starting from the assumed value of the angle  $\Phi_3 = 1$  rad. The excitation range  $f$  is from 1 to 15 Hz, as in Fig. 3.

|   | A   | B        | C        | D                                             | E                                                               | F                                                                                   |
|---|-----|----------|----------|-----------------------------------------------|-----------------------------------------------------------------|-------------------------------------------------------------------------------------|
| 1 | $f$ | $\alpha$ | $\Phi_3$ | $\Phi_2 = \Phi_3 - \frac{B\Phi_3\alpha^2}{k}$ | $\Phi_1 = \Phi_2 - \frac{B\Phi_2\alpha^2 + B\Phi_3\alpha^2}{k}$ | $\Phi_0 = \Phi_1 - \frac{B\Phi_1\alpha^2 + B\Phi_2\alpha^2 + B\Phi_3\alpha^2}{k_1}$ |
| 2 | Hz  | rad/s    | rad      | rad                                           | rad                                                             | rad                                                                                 |
| 3 | 1   | 6.28     | 1        | 0.9745                                        | 0.9241                                                          | 0.7762                                                                              |
| 4 | 2   | 12.56    | 1        | 0.8978                                        | 0.7040                                                          | 0.1724                                                                              |

Figure 3: Computational table in a spreadsheet, scheme of the vibrating system according to Fig. 1

After the table has been constructed a graph of the values of  $\Phi_0(f)$  should be plotted; this will allow the zeros of the relation to be located approximately. They occur, of course, between the values of  $f$  that correspond to a change of sign of  $\Phi_0$  in the last column of the table. This can be done more accurately by using the built-in Goal Seek function of the spreadsheet: TOOLS – GOAL SEEK. The cell containing  $\Phi_0$  in the vicinity of its change of sign should be selected as the formula cell. The target value is of course 0, and the cell to be changed in the search process is the frequency  $f$  in the same row. The result – the eigenvalue – is the value of  $f$  modified by the Goal Seek procedure.

| Hz | rad/s  | rad | rad    | rad    | rad     |
|----|--------|-----|--------|--------|---------|
| 1  | 6.2832 | 1   | 0.9745 | 0.9240 | 0.7760  |
| 2  | 12.566 | 1   | 0.8978 | 0.7040 | 0.1724  |
| 3  | 18.85  | 1   | 0.7702 | 0.3633 | -0.6174 |
| 4  | 25.133 | 1   | 0.591  | 0.0000 | -1.3113 |
| 5  | 31.416 | 1   | 0.361  | -0.634 | -1.5980 |
| 6  | 37.699 | 1   | 0.080  | -1.068 | -1.2214 |
| 7  | 43.982 | 1   | -0.251 | -1.553 | -0.0882 |
| 8  | 50.265 | 1   | -0.634 | -2.090 | 1.6000  |
| 9  | 56.549 | 1   | -1.068 | -2.677 | 3.1909  |
| 10 | 62.832 | 1   | -1.553 | -3.291 | 3.4017  |
| 11 | 69.115 | 1   | -2.090 | -3.849 | 0.1149  |
| 12 | 75.398 | 1   | -2.677 | -4.491 | -9.8491 |

Figure 4: Dialogue box of the GOAL SEEK function

Numerical data:  $B = 0.0085 \text{ kgm}^2$ ;  $k = 13.14 \text{ Nm/rad}$ ;  $k_1 = 6.57 \text{ Nm/rad}$ ;

2. Start the system that excites torsional vibrations of the discs and measure the three successive resonant frequencies at which local maxima of the amplitude of torsional vibration of the discs are observed. The search should be performed in the vicinity of the values of  $f$  identified numerically above. The angles of twist of the discs (mode shapes) are observed by means of the deflection lines of the threads, which, when the positions of the discs change at the resonant frequencies, form shapes corresponding to the principal modes of vibration.
3. Enter the resonant-frequency values read from the meter in the measurement table.
4. Draw the observed mode shapes.
5. Compare the results of the calculations with the results of the experimental investigations and formulate your own conclusions.

## 5 Literature

1. Kapitaniak T.: *Introduction to Vibration Theory*, Publishing House of the Technical University of Łódź, Łódź 1992.
2. Parszewski Z.: *Dynamics and Vibrations of Machines*. WNT, Warsaw 1982.
3. Osiński Z.: *Theory of Vibrations*, PWN, Warsaw 1978.

**LABORATORY OF  
MECHANICAL  
VIBRATIONS**

**Exercise 5**

**DETERMINATION OF RESONANT FREQUENCIES OF  
TORSIONAL VIBRATIONS OF A SHAFT**

**Group:** \_\_\_\_\_  
**Team:** \_\_\_\_\_

**date** \_\_\_\_\_

**Name and surname:**

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
5. \_\_\_\_\_
6. \_\_\_\_\_

**RESULTS**

|                                                        |              |   |   |    |              |       |                     |
|--------------------------------------------------------|--------------|---|---|----|--------------|-------|---------------------|
| Natural<br>frequencies<br>of<br>torsional<br>vibration | Measurements |   |   |    | Calculations |       | Relative difference |
|                                                        | 1            | 2 | 3 | Hz | Hz           | rad/s | %                   |
| I                                                      |              |   |   |    |              |       |                     |
| II                                                     |              |   |   |    |              |       |                     |
| III                                                    |              |   |   |    |              |       |                     |

**Drawings of the observed mode shapes**

**Conclusions**