MECHANICS AND MECHANICAL ENGINEERING
International Journal

Vol. 1 No. 2 1997

DYNAMICS OF A ROTATING CYLINDRICAL ISOTROPIC SHELL

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Abstract

In the paper the relative motion equations of a cylindrical isotropic shell rotating about its longitudinal axis have been derived according to the Kirchhoff – Love hipothesis. In these equations an environmental viscous suppression and material viscoelastic damping described by the nonlinear Kelvin – Voigt model have been considered. The stiffness matrix of the system has been formulated and the natural frequencies of the shell with simpy supported edges have been calculated. The results show an influence of the radial vibration amplitude on the shell natural frequencies.

Introduction

Dynamics of cylindrical ahells rotating about thier longitudinal axes has been the topic of many investigations and theoretical publications. Commonly, thin cylindrical shells are used as models of rotating cylinders or shafts in advanced gass turbines. The dynamic problems due to high speed rotation are presented in [4], [5] end [6], where critical parameters of the system and the effect of rotational speed on natural frequencies are studied. The analysis of natural frequencies of rotating isotropic shells including Coriolis component is presented in [1] among others. The problem of stability of rotating composite shafts using stochastic approach is discussed in [6].

The purpose of this paper is to obtain modal frequencies of a thin cylindrical shell of radius R and thickness h rotating with constant angular velocity Ω. In the considered case Coriolis component is included. Therefore, acceleration components are formulated taking into account rotation transportation and displacements U, V, W at any point on the shell in axial (ζ), circumferential (ξ), and radial (η) directions, respectively, given by the coordinate system which rotates with the undeformed midplane of the shell. The effect of material damping is described applying the nonlinear Kelvin – Voigt model presented in [5].

Kinematic relations

The problem of rotating shell is formulated using small deflection thin shell theory. In this case the Kirchhoff – Love asumptions are made:

1. any line perpendicular to the shell midplane before deformation remains perpendicular to the midplane.after deformation,
2. tangential displacements of midplane elements caused by small deflections are negligible.

The shell is thin i.e., the midplane radius R is much larger than the thickness h.

$$h \le (0{,}1 \div 0{,}05)R, \tag{1}$$

and

$$h \le 0{,}1L, \tag{2}$$

where L – length of the shell

Let us consider a surface of revolution S which relates to the shell midplane surfance and that its longitudinal symmetrical axis coinsides with axis of fixed coordinate system X,Y,Z. The shell equations of motion are formulated in the movable relative rotating coordinate system ξ, η, ζ fixed to the surface S. The coordinate system and displacement components U,V,W are shown in Fig. 1.

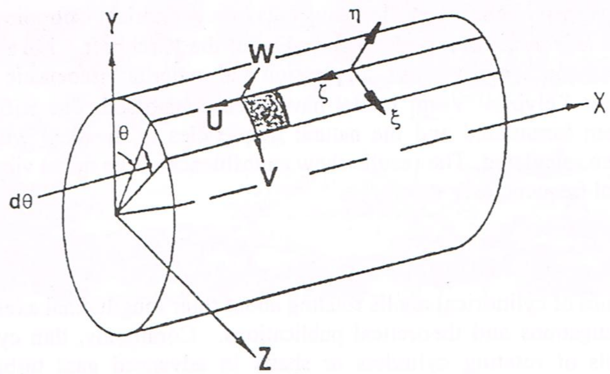

Fig. 1. Geometry of the shell and the coordinate systems

The angular velocity of the shell is given by the vector:

$$\vec{\Omega} = -\vec{k}\Omega. \tag{3}$$

In the rotating coordinates any point of the shell is determined by the position vector $\vec{r}_0$:

$$\vec{r}_0 = \vec{i}\,V + \vec{j}\,W + \vec{k}U. \tag{4}$$

After differentiation of Eq. (4), with respect to time the velocity and acceleration of the shell point in the relative motion is obtained

$$\vec{v}_r = \vec{i}\frac{\partial V}{\partial t} + \vec{j}\frac{\partial W}{\partial t} + \vec{k}\frac{\partial U}{\partial t}. \tag{5}$$

$$\vec{a}_r = \vec{i}\frac{\partial^2 V}{\partial t^2} + \vec{j}\frac{\partial^2 W}{\partial t^2} + \vec{k}\frac{\partial^2 U}{\partial t^2}. \tag{6}$$

Taking into account the transportation movement one may obtain the following formula of the acceleration vector relative to X,Y,Z system

$$\vec{a} = \vec{a}_0 + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}_0\right) + \vec{\varepsilon} \times \vec{r}_0 + 2\vec{\Omega} \times \vec{v}_r + \vec{a}_r, \tag{7}$$

where:

$\vec{a}_0 = -\vec{j}\Omega^2 R$ – acceleration of the origin the ξ,η,ζ reference (rotating system)

$\varepsilon = \dfrac{d\Omega}{dt}$ – angular acceleration of the rotation.

In the case constant rotational speed$(\varepsilon = 0)$ the acceleration components are as follows

$$a_\xi = -\Omega^2 V + 2\Omega \frac{\partial W}{\partial t}, \tag{8}$$

$$a_\eta = -\Omega^2 R - \Omega^2 W - 2\Omega \frac{\partial V}{\partial t}, \tag{9}$$

$$a_\zeta = \frac{\partial^2 U}{\partial t^2}. \tag{10}$$

Free vibration equations

Viscoelastic properties of material are described by the following constitutive relation

$$\sigma = \left[\left(\mu_0 + \mu_1\varepsilon_0 + \mu_2\varepsilon_0^2\right)\frac{d\varepsilon}{dt} + \varepsilon\right]E \tag{11}$$

where:

$\varepsilon_0 = \dfrac{W}{R}$ – strain in radial direction,

μ_0, μ_1, μ_2 – viscoelastic coefficients,

E – Young modulus.

To simplify calculations equivealent linearization of the term proportional to the strain velocity is made:

$$\gamma_{eq} = \frac{1}{2\pi\omega\varepsilon_0}\int_0^{2\pi}\frac{\mu_0 + \mu_1\varepsilon_0 + \mu_2\varepsilon_0^2}{\varepsilon_0}\sin \upsilon d\upsilon = \frac{\mu_0 + \mu_1\varepsilon_0 + \mu_2\varepsilon_0^2}{2\varepsilon_0} \tag{12}$$

Therefore the constitutive equation Eq. (11) can be reduced to the form:

$$\frac{\sigma}{\varepsilon_0} = E\left(1 + \gamma_{eq}\right) \tag{13}$$

Taking into account Eqs. (8÷12) after eliminating the axial and circumferential acceleration components and assuming external viscous damping the equations of motion refered to the rotating system (coordinate ξ,η,ζ) my be described as follows:

$$-\frac{\partial^2 U}{\partial x^2}-\frac{1-\nu}{2}\frac{\partial^2 U}{\partial y^2}-\frac{1+\nu}{2}\frac{\partial^2 V}{\partial x\partial y}-\frac{\nu}{R}\frac{\partial W}{\partial x}+2\frac{\rho h\beta}{A\left(1+\gamma_{eq}\right)}\frac{\partial U}{\partial t}=0, \tag{14}$$

$$-\frac{1+\nu}{2}\frac{\partial^2 U}{\partial x\partial y}-\frac{\partial^2 V}{\partial y^2}-\frac{1-\nu}{2}\frac{\partial^2 V}{\partial x^2}-\frac{1}{R}\frac{\partial W}{\partial y}-\left(\frac{\Omega^2}{A}V-2\frac{\Omega}{A}\frac{\partial W}{\partial t}\right)\frac{\rho h}{1+\gamma_{eq}}+$$
$$\frac{2\rho h\beta}{A\left(1+\gamma_{eq}\right)}\frac{\partial V}{\partial t}=0 \tag{15}$$

$$\frac{\nu}{R}\frac{\partial U}{\partial x}+\frac{1}{R}\frac{\partial V}{\partial y}+\frac{W}{R^2}+\frac{D}{A}\Delta^2 W+\frac{N_x}{A\left(1+\gamma_{eq}\right)}\frac{\partial^2 W}{\partial x^2}+\frac{\widetilde{N}_y}{A\left(1+\gamma_{eq}\right)}\frac{\partial^2 W}{\partial y^2}+$$
$$+\frac{\rho h}{A\left(1+\gamma_{eq}\right)}\left(\frac{\partial^2 W}{\partial t^2}-2\Omega\frac{\partial V}{\partial t}-\Omega^2 W\right)+2\frac{\rho h\beta}{A\left(1+\gamma_{eq}\right)}\frac{\partial W}{\partial t}=0 \tag{16}$$

where:

$A=\dfrac{Eh}{1-\nu^2}$ – longitudial stiffness,

$D=\dfrac{Eh^3}{12\left(1-\nu^2\right)}=\dfrac{Ah^2}{12}$ – flexural stiffness,

N_x – intensity of axial forces,

$\widetilde{N}_y=N_y-R^2\Omega^2\rho h$ – intensity of circumferential forces,

ν – Poisson ratio,

β – external damping coefficient,

ρ – mass density,

Δ – Laplace operator.

The solutions to the equations of motion (14), (15) i (16) are assumed in the form:

$$U=\sum_{m,n=1}^{\infty}\left[U_{mn}(t)\cos\beta_n y+\widetilde{U}_{mn}(t)\sin\beta_n y\right]\cos\alpha_m x, \tag{17}$$

$$V=\sum_{m,n=1}^{\infty}\left[V_{mn}(t)\sin\beta_n y+\widetilde{V}_{mn}(t)\cos\beta_n y\right]\sin\alpha_m x \tag{18}$$

$$W=\sum_{m,n=1}^{\infty}\left[W_{mn}(t)\cos\beta_n y+\widetilde{W}_{mn}(t)\sin\beta_n y\right]\sin\alpha_m x. \tag{19}$$

where:

m,n – wave numbers.

Substitution ofthe above set of displacements into equations of motion results in the ordinary differential equation system related to the time dipendant functions $U_{mn}, V_{mn}, W_{mn}, \widetilde{U}_{mn}, \widetilde{V}_{mn}, \widetilde{W}_{mn}$

$$2d\frac{dU_{mn}}{dt}+\left(\alpha_m^2+\frac{1-\nu}{2}\beta_n^2\right)U_{mn}-\frac{1+\nu}{2}\alpha_m\beta_n V_{mn}-\frac{\nu}{R}\alpha_m W_{mn}=0, \tag{20}$$

$$2d\frac{dV_{mn}}{dt}-\frac{1+\nu}{2}\alpha_m\beta_n U_{mn}+\left(\beta_n^2+\frac{1-\nu}{2}\alpha_m^2-c\Omega\right)V_{mn}+\frac{\beta_n}{R}W_{mn}+2c\frac{d\widetilde{W}_{mn}}{dt}=0, \tag{21}$$

$$\frac{\rho h}{A\left(1+\gamma_{eq}\right)}\frac{d^2W_{mn}}{dt^2}+2d\frac{dW_{mn}}{dt}-\frac{\nu}{R}\alpha_m U_{mn}+\frac{\beta_n}{R}V_{mn}+$$

$$+\left[\frac{1}{R^2}+\frac{\nu\left(\alpha_m^2+\beta_n^2\right)^2}{A\left(1+\gamma_{eq}\right)}-\frac{N_x\alpha_m^2+\hat{N}_y^2}{A\left(1+\gamma_{eq}\right)}-c\Omega\right]W_{mn}+2c\frac{d\widetilde{V}_{mn}}{dt}=0, \tag{22}$$

$$2d\frac{d\hat{U}_{mn}}{dt}+\left(\alpha_m^2+\frac{1-\nu}{2}\beta_n^2\right)\hat{U}_{mn}-\frac{1+\nu}{2}\alpha_m\beta_n\widetilde{V}_{mn}-\frac{\nu}{R}\alpha_m\widetilde{W}_{mn}=0, \tag{23}$$

$$2d\frac{d\hat{U}_{mn}}{dt}-\frac{1+\nu}{2}\alpha_m\beta_n\hat{U}_{mn}+\left(\beta_n^2+\frac{1-\nu}{2}\alpha_m^2-c\Omega\right)\widetilde{V}_{mn}+\frac{\beta_n}{R}\widetilde{W}_{mn}-2c\frac{dW_{mn}}{dt}=0, \tag{24}$$

$$\frac{\rho h}{A\left(1+\gamma_{eq}\right)}\frac{d^2\widetilde{W}_{mn}}{dt^2}+2d\frac{d\widetilde{W}_{mn}}{dt}-\frac{\nu}{R}\alpha_m\widetilde{U}_{mn}+\frac{\beta_n}{R}\widetilde{V}_{mn}+$$

$$+\left[\frac{1}{R^2}+\frac{\nu\left(\alpha_m^2+\beta_n^2\right)^2}{A\left(1+\gamma_{eq}\right)}-\frac{N_x\alpha_m^2+\widetilde{N}_y^2}{A\left(1+\gamma_{eq}\right)}-c\Omega\right]\widetilde{W}_{mn}+2c\frac{dV_{mn}}{dt}=0, \tag{25}$$

where:

$a=\dfrac{\rho h}{A}$ – constant depending on material parameters,

$d=\dfrac{2\rho h\beta}{A}=2a\beta$ – external viscous factor,

$c=\dfrac{2\rho h\Omega}{A}=2a\Omega$ – rotational speed factor.

It can be noticed that the equations (20) ÷(25) are coupled because of rotation and terms representing the elastic forces.

Using symbols L_{ij} to designate the elements of the stiffness matrix, the following differential system of equations is obtained:

$$2d\frac{dU_{mn}}{dt} + L_{11}U_{mn} + L_{12}V_{mn} + L_{13}W_{mn} = 0, \tag{26}$$

$$2d\frac{dV_{mn}}{dt} + 2c\frac{d\widetilde{W}_{mn}}{dt} + L_{21}U_{mn} + L_{22}V_{mn} + L_{23}W_{mn} = 0, \tag{27}$$

$$\frac{\rho h}{A\left(1+\gamma_{eq}\right)}\frac{d^2W_{mn}}{dt^2} + 2d\frac{dW_{mn}}{dt} + 2c\frac{d\widetilde{V}_{mn}}{dt} + L_{31}U_{mn} + L_{32}V_{mn} + L_{33}W_{mn} = 0, \tag{28}$$

$$2d\frac{d\widetilde{U}_{mn}}{dt} + L_{11}\widetilde{U}_{mn} + L_{12}\widetilde{V}_{mn} + L_{13}\widetilde{W}_{mn} = 0, \tag{29}$$

$$2d\frac{d\widetilde{V}_{mn}}{dt} - 2c\frac{d\widetilde{W}_{mn}}{dt} + L_{12}\widetilde{U}_{mn} + L_{22}\widetilde{V}_{mn} + L_{23}\widetilde{W}_{mn} = 0, \tag{30}$$

$$\frac{\rho h}{A\left(1+\gamma_{eq}\right)}\frac{d^2\widetilde{W}_{mn}}{dt^2} + 2d\frac{d\widetilde{W}_{mn}}{dt} - 2c\frac{dV_{mn}}{dt} + L_{31}\widetilde{U}_{mn} + L_{32}\widetilde{V}_{mn} + L_{33}\widetilde{W}_{mn} = 0. \tag{31}$$

where:

$$L_{11} = \alpha_m^2 + \frac{1-\nu}{2}\beta_n^2, \quad L_{12} = -\frac{1+\nu}{2}\alpha_m\beta_n, \quad L_{13} = -\frac{\nu}{R}\alpha_m$$

$$L_{21} = L_{12}, \quad L_{22} = \beta_n^2 + \frac{1-\nu}{2}\alpha_m^2, \quad L_{23} = \frac{\beta_n}{R}$$

$$L_{31} = L_{13}, \quad L_{23} = L_{32}, \quad L_{33} = \frac{1}{R} + \frac{D}{A}\left(\alpha_m^2 + \beta_n^2\right)$$

Let us assume that solutions of the above system satisfy the following relation:

$$\begin{bmatrix} U \\ V \\ W \\ \widetilde{U} \\ \widetilde{V} \\ \widetilde{W} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} e^{kt}. \tag{32}$$

Substituting Eq (32), into the governing equations one can obtain the following system of algebraic homogeneous equations:

$$\left(2dk + L_{11}\right)C_1 + L_{12}C_2 + L_{13}C_3 + 0 + 0 + 0 = 0, \tag{33}$$

$$L_{12}C_1 + \left(L_{22} + 2dk\right)C_2 + L_{23}C_3 + 0 + 0 + 2ckC_6 = 0, \tag{34}$$

$$L_{13}C_1 + L_{23}C_2 + \left(L_{33} + \frac{\rho h}{A\left(1+\gamma_{eq}\right)}k^2 + 2dk\right)C_3 + 0 + 2ckC_5 + 0 = 0, \tag{35}$$

$$0 + 0 + 0 + \left(2dk + L_{11}\right)C_4 + L_{12}C_5 + L_{13}C_6 = 0, \tag{36}$$

$$0 + 0 - 2ckC_3 + L_{12}C_4 + \left(L_{22} + 2dk\right)C_5 + L_{23}C_6 = 0, \tag{37}$$

$$0 - 2ckC_2 + 0 + L_{13}C_4 + L_{23}C_5 + \left(L_{33} + \frac{\rho h}{A\left(1+\gamma_{eq}\right)}k^2 + 2dk\right)C_6 = 0, \tag{38}$$

In order to obtain nontrivial solutions for the amplitudes C_1, C_2, C_3, C_4, C_5, C_6, the determinant of the coefficient matrix is set equal to zero:

$$\Delta = \begin{vmatrix} L_{11}+dk & L_{12} & L_{13} & 0 & 0 & 0 \\ L_{12} & L_{22}+dk & L_{23} & 0 & 0 & 2ck \\ L_{13} & L_{23} & L_{33}+ak^2+dk & 0 & 2ck & 0 \\ 0 & 0 & 0 & L_{11}+dk & L_{12} & L_{13} \\ 0 & 0 & -2ck & L_{12} & L_{22}+dk & L_{23} \\ 0 & -2ck & 0 & L_{13} & L_{23} & L_{33}+ak^2+dk \end{vmatrix} \tag{39}$$

The characteristic equation of the form Eq.(39) has been numerically solved to calculate natural frequencies of the sheel

Results

Calculations have been performed neglecting the external damping for the cylindrical sheel of dimension R=0,4 [m], h=0,005 [m], L=1,0 [m]. The shell is made of steel of the Young modulus E=2,1 10^{11} Mpa and the following viscoelast coefficients : $\mu_0 = 3{,}81\ 10^{-4}$, $\mu_1 = 350\ 10^{-2}$ oraz $\mu_2 = 1{,}27\ 10^{3}$.

The results of calculations are given in Tables 1 ÷ 3 and graphically ilustrated in Figs. 2 ÷ 4. The analysis of the results let us notice that the use of the nonlinear Kelvin-Voigt model of viscoelastic material causes an increase of the natural frequencies comparing with the purely elastic model. The similar effect is obtained by increasing the relative strain $\varepsilon_0 = \dfrac{W}{R}$. It can be observed that the minimal value of natural frequency exists for the wave numbers n = 2 and m equal or greater than 4 (m ≥ 4). The wave namber n influes much more significantly on the shell natural frequencies than the wave number m.

$\epsilon_0 = 0$	n						
m	1	2	3	4	5	6	7
1	5330,59	19524,57	43820,03	77841,64	121584,3	175047,5	238231,4
2	9258,47	19850,24	44058,91	78073,44	121814,9	175277,9	238461,7
3	17020,94	20615,03	44478,90	78463,80	122200,4	175662,3	238845,7
4	26781,99	22050,73	45109,11	79018,37	122747,3	176201,1	239383,6
5	37221,40	24313,93	45983,62	79744,36	123442,6	176895,1	240075,6
6	47481,72	27421,26	47135,98	80649,94	124303,8	177745,2	340922,2
7	57061,83	31266,72	48593,98	81743,49	125328,4	178752,3	241923,8
8	65725,36	35683,16	50375,57	83032,99	126519,3	179917,5	243080,7
9	73410,36	40495,67	52486,67	84525,32	127879,3	181241,9	244393,6
10	80156,17	45549,63	54921,06	86225,81	129410,8	182726,7	245862,9

Table 1. The shell natural frequencies for a purely elastic material

$\epsilon_0 = 0,0005$	n						
m	1	2	3	4	5	6	7
1	10941,71	40076,65	89946,14	159779,8	159779,8	359307,2	489000,1
2	19004,20	40745,13	90436,49	160255,6	250040,5	359780,1	489472,8
3	34937,63	42314,95	91298,55	161056,9	250831,8	360569,0	490261,0
4	54973,42	45261,90	92592,14	162195,2	251944,1	361675,0	491675,0
5	76401,62	49907,42	94387,18	163685,4	253381,6	363099,5	492785,6
6	97462,22	56285,61	96752,55	165544,2	255149,2	364844,4	494523,4
7	171126,6	64178,89	99745,28	167788,8	257252,4	366911,6	496579,2
8	134909,6	73244,20	103402,2	103402,2	259696,9	369303,3	498954,0
9	150684,1	83122,47	107735,5	173498,9	262488,3	372021,9	501648,8
10	164530,7	93496,37	112732,4	176989,4	265632,0	375069,5	504664,6

Table 2. The shell natural frequencies for a viscoelastic material (ε_0 = 0,0005)

$\epsilon_0 = 0,00125$	n						
m	1	2	3	4	5	6	7
1	15949,89	58420,29	131115,8	232913,3	363797,4	523767,1	712822,3
2	27702,68	59394,75	131830,5	233606,9	364487,6	524456,5	713511,4
3	931651,9	1178878	178878	50929,07	61683,09	133087,2	234774,9
4	80135,53	65978,91	65978,91	236434,2	467262,5	527218,7	716269,8
5	111371,7	72750,75	137589,5	238606,5	369357,9	529295,3	718340,5
6	142072,0	82048,32	141037,6	241316,1	371934,5	371934,5	720873,7
7	170737,1	93554,47	145400,1	244588,2	375000,4	534852,2	723870,4
8	196659,6	106769,1	150730,9	248446,5	378563,8	538338,7	727332,2
9	219654,3	121168,8	157047,6	157047,6	382632,9	542301,5	731260,5
10	239838,7	136291,0	164331,6	257999,9	387215,5	387215,5	735656,7

Table3. The shell natural frequencies for a viscoelastic material (ε_0 = 0,00125)

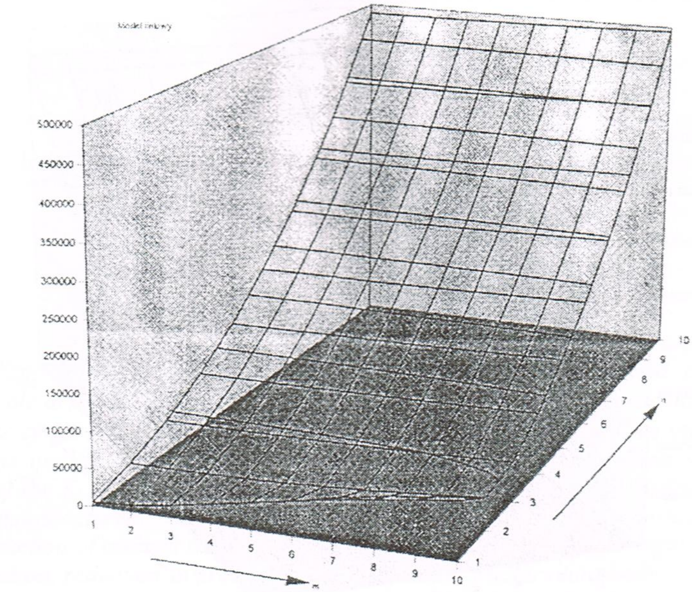

Fig.2. Plot of the natural frequencies in terms of wave numbers for a purely elastic material

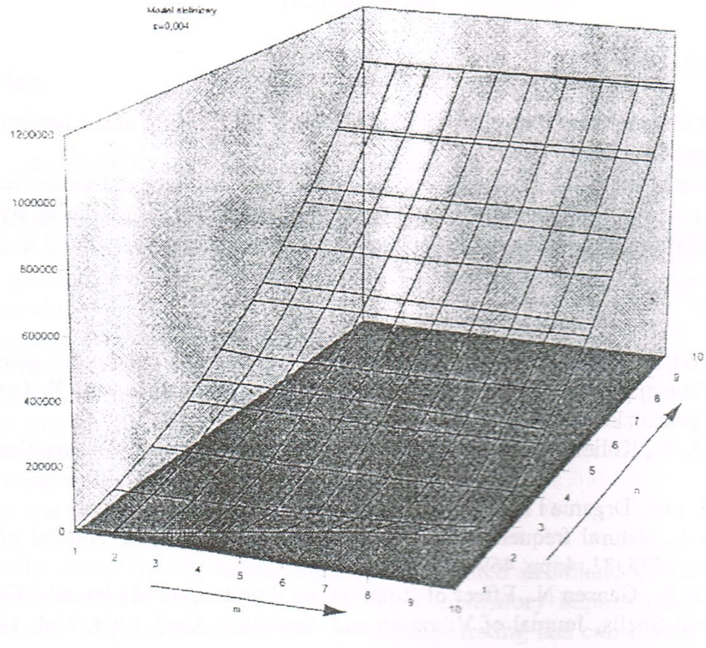

Fig.3. Plot of the natural frequencies in terms of wave numbers for a viscoelastic material ($\varepsilon_0 = 0{,}0005$)

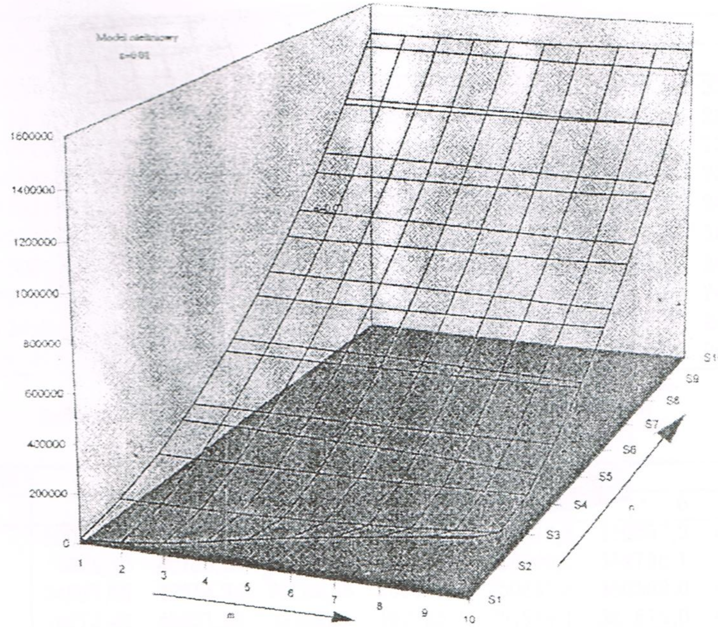

Fig.4. Plot of the natural frequencies in terms of wave numbers for a viscoelastic material ($\varepsilon_0 = 0{,}00125$)

CONCLUSIONS

1. The use of the nonlinear Kelvin-Voigt model results in increasing of the natural frequencies of the shell.
2. The increase of the radial strain ε_0 gives the greater values of natural frequencies.
3. The wave number n influes strongly on the natural frequencies in comparing with m.
4. The natural frequency reaches a minimal value for wave numbers n = 2 and m ≥ 4.

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