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# INFLUENCE OF TORQUE ON DYNAMIC STABILITY OF COMPOSITE THIN-WALLED SHAFTS WITH BRAZIER'S EFFECT

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## Abstract

It is an original result of this paper to show that for thin-walled laminated shafts an important feature is related to the Brazier's effect, which consists in considerable deformations of thin-walled cross-section contour during bending. The rotating angle-ply symmetrically laminated circular cylindrical shell is treated as a beam-like structure. The shaft is subjected to combined loading: a constant torque and an axial time-dependent (stochastic) force which can be described by the Wiener process. The uniform stochastic stability criteria involving a damping coefficient, a rotation speed and geometrical and material parameters are derived using Liapunov's direct method. Formulas determining dynamic stability regions are written explicitly.

## Introduction

Recently, composite materials find more and more applications for high-performance rotating shafts [1], [2]. Thin-walled, usually angle-ply laminated tubes relatively easy meet requirements of torsional strength and stiffness but are more flexible to bending and have specific elastic and damping properties which depend on the system geometry, physical properties of plies and on the laminate arrangement. Such systems are also sensitive to a lateral buckling.

Thin-walled shalls reveal a considerable deformation of cross-section contour during bending. The dynamic stability of rotating composite shaft described by partial differential equations [3] was investigated using the direct Liapunov method. The ovalizing phenomenon implying a degresive elestic behaviour is named Brazier's effect (see eg. [4]). This results in a specific degressive geometric nonlinearity. It can be shown that the Brazier's effect can be much stronger than the effect of curvature nonlinearity treated as dominating for isotropic axially not restrained shafts [5].

Another important problem of this paper is the description of the global damping of a where according to [9] the different engineering constants $E_{11}$, $E_{22}$, $G_{12}$ have different viscoelastic properties described $\beta_{11}$, $\beta_{22}$, $\beta_{12}$, respectively.

By analogy we can write the reduced in-plane stiffnesses in the form of operators

$$\bar{Q}^{*}_{ij} = \bar{Q}^{e}_{ij} + \bar{Q}^{v}_{ij} s, \tag{7}$$

where $\bar{Q}^{e}_{ij}$ and $\bar{Q}^{v}_{ij}$ are given as follows

$$Q^{e}_{11} = E_{11},\; Q^{e}_{22} = E_{22},\; Q^{e}_{66} = G_{12},\; Q^{v}_{11} = \beta_{11}E_{11},\; Q^{v}_{22} = \beta_{22}E_{22},\; Q^{v}_{66} = \beta_{12}G_{12}.$$

Finally the constitutive equation in the first order approach has the form

$$\sigma = E_0(\varepsilon + \beta_1\dot{\varepsilon}), \tag{8}$$

where $E_0 = \bar{Q}^{e}_{11} - (\bar{Q}^{e}_{12})^2/\bar{Q}^{e}_{22}$, $\beta_1 = (\bar{Q}^{e}_{11}\bar{Q}^{v}_{22} + \bar{Q}^{e}_{22}\bar{Q}^{v}_{11} - 2\bar{Q}^{e}_{12}\bar{Q}^{v}_{12})/(\bar{Q}^{e}_{11}\bar{Q}^{e}_{22} - (\bar{Q}^{e}_{12})^2)$.

### Geometric relation

The shaft transverse displacements $u$ and $v$ in the immovable coordinate system $(y,\ z)$ can be expressed by the polar displacement $w$ as follows

$$u = w\cos\alpha, \qquad v = w\sin\alpha \tag{9}$$

Let's consider the differential area element $dA$ of the ring-shaped cross-section. The distance from the elastic neutral stress axis$\xi$ to the element (parallel to $w$) is equal to

$$\xi = Y\cos\alpha + Z\sin\alpha \tag{10}$$

Neglecting the curvature nonlinearity, what is rigorously justified for lamination angle $0 \le \theta \le \Pi/6$ [5] we can write the strain in the form

$$\varepsilon = -\xi w_{,xx}(1 - \gamma w^2_{,xx}) \tag{11}$$

Two times differentiating Eq. (9) with respect to $x$, using Eq. (10) and substituting into Eq. (11) yield

$$\varepsilon = -(Yu_{,xx} + Zv_{,xx})\left(1 - \gamma w_{,xx}^2\right) \tag{12}$$

Due to the rotation with the constant speed $\omega$ the following formulae are valid

$$\dot{Y} = -\omega Z, \qquad \dot{Z} = \omega Y \tag{13}$$

Differentiating Eq. (12) with respect to time and and substituting Eqs. (13) the strain velocity in a linear approach is equal to

$$\dot{\varepsilon} = -Y(u_{,xxt} + \omega v_{,xx}) - Z(v_{,xxt} - \omega u_{,xx}) \tag{14}$$

**Equations of motion**

Equations of motion of the center shaft line in the $(u,\ v)$ coordinates have the form

$$\rho A(u_{,tt} + h_1 u_{,t}) + F(t)u_{,xx} = M_{Z,xx} \tag{15}$$
$$\rho A(v_{,tt} + h_1 v_{,t}) + F(t)v_{,xx} = M_{Y,xx} \tag{16}$$

where $\rho$ is the averaged density of the shaft, $h_1$ is the external damping coefficient, and $F(t)$ represents stochastic axial force, which can destabilize the smooth rotation motion. The first components (due to a pure bending) of bending moments present in Eqs. (15) and (16) are calculated by the integration of elementary moments over the shell crossection in the following way

$$(M_Y,\ M_Z) = \int_A (Z,\ Y)\sigma dA \tag{17}$$

where the inner stress $\sigma$ is obtained substituting Eqs. (12) and (14) into the constitutive Eq. (8)

$$\sigma = E_0\left\{-(Yu_{,xx} + Zv_{,xx})(1 - \gamma w_{,xx}^2) - \beta_1\left(Y(u_{,xxt} + \omega v_{,xx}) + Z(v_{,xxt} - \omega u_{,xx})\right)\right\} \tag{18}$$

After integration we have

$$M_Z = -E_0J\left\{u_{,xx}\left[1-\gamma\left(u^2_{,xx}+v^2_{,xx}\right)\right]+\beta_1 e u_{,xxt}+\beta_1 e\omega v_{,xx}\right\} \tag{19}$$

$$M_Y = -E_0J\left\{v_{,xx}\left[1-\gamma\left(u^2_{,xx}+v^2_{,xx}\right)\right]+\beta_1 e v_{,xxt}-\beta_1 e\omega u_{,xx}\right\} \tag{20}$$

The second components of bending moments are produced by a torque $M_s$ due to rotations $v_{,x}$, $u_{,x}$ of the shaft axis

$$M^*_Z = M_s v_{,x} \tag{21}$$

$$M^*_Y = -M_s u_{,x} \tag{22}$$

After substituting Eqs. (19), (20), (21) and (22) into equations of motion (15) and (16) and dividing by $\rho A$ we have the final form of dynamic equations of rotating shafts

$$u_{,tt}+h_1u_{,t}+e\left[u_{,xx}(1-\gamma w^2_{,xx})\right]_{,xx}+\beta_1 e(u_{,xxxxt}+\omega v_{,xxxx})+S(t)u_{,xx}+Lv_{,xxx}=0 \tag{23}$$

$$v_{,tt}+h_1v_{,t}+e\left[v_{,xx}(1-\gamma w^2_{,xx})\right]_{,xx}+\beta_1 e(v_{,xxxxt}-\omega u_{,xxxx})+S(t)v_{,xx}-Lu_{,xxx}=0 \tag{24}$$

where $e = E_0J/\rho A$, $\quad S(t) = F(t)/\rho A$, $\quad L = M_s/\rho A$.

Assuming that the stochastic axial force $S$ has the mean value $S_0$ and the time dependent wide-band Gaussian part with the intensity $\varsigma$, which can be expressed as the formal time-derivative of the Wiener process $\mathcal{W}$,

$$S(t) = S_0 + \varsigma\frac{d\mathcal{W}}{dt} \tag{25}$$

the dynamic equations (23) and (24) can be rewritten in the Itô form

$$du = u_{,t}dt \tag{26}$$

$$du_{,t} = -\left\{h_1 u_{,t} + e\left[u_{,xx}(1-\gamma w_{,xx}^2)\right]_{,xx} + \beta_1 e(u_{,xxxxt} + \omega v_{,xxxx}) + \right. \\ \left. + S_0 u_{,xx} + Lv_{,xxx}\right\} dt - \varsigma u_{,xx} dW \tag{27}$$

$$dv = v_{,t}dt \tag{28}$$

$$dv_{,t} = -\left\{h_1 v_{,t} + e\left[v_{,xx}(1-\gamma w_{,xx}^2)\right]_{,xx} + \beta_1 e(v_{,xxxxt} - \omega u_{,xxxx}) + \right. \\ \left. + S_0 v_{,xx} - Lu_{,xxx}\right\} dt - \varsigma v_{,xx} dW \tag{29}$$

The shaft is assumed to be simply supported at its ends. As the torque $M_s$ are acting at the shaft ends it is necessary to remind Eqs. (21) and (22). When the torque is pure tangential to the deformed shaft axis in supports the bending moments vanish. It means that the transverse displacements and the bending moments are equal to zero

$$u(0,t) = u(\ell,t) = v(0,t) = v(\ell,t) = 0 \tag{30}$$

$$u_{,xx}(0,t) = u_{,xx}(\ell,t) = v_{,xx}(0,t) = v_{,xx}(\ell,t) = 0 \tag{31}$$

## Uniform stochastic stability analysis

In order to investigate the stability of trivial solution $u = v = 0$ corresponding to the smooth shaft motion it is necessary to introduce a precise stability definition. The trivial solution is uniformly staochastically stable if the following logic sentence is true

$$\bigwedge_{\epsilon\geq 0}\bigwedge_{\delta\geq 0}\bigvee_{r\geq 0} \|u(.,0), v(.,t)\|_{.} \leq r \Rightarrow P(\sup_{t\geq 0}\|u(.,t), v(.,t)\| \geq \epsilon) \leq \delta$$

where $\|u(.,t), v(.,t)\|$ denotes a measure of distance of solutions with nontrivial initial conditions from the trivial one.

We choose the Liapunov functional in the energy-like form [10]

$$V = \frac{1}{2}\int_0^\ell \left\{u_{,t}^2 + (u_{,t} + h_1 u + \beta_1 e u_{,xxxx})^2 + v_{,t}^2 + (v_{,t} + h_1 v + \beta_1 e v_{,xxxx})^2 + \right. \\ \left. +2e(u_{,xx}^2 + v_{,xx}^2)\left[1 - \frac{\gamma}{2}(u_{,xx}^2 + v_{,xx}^2)\right] - 2S_o(u_{,x}^2 + v_{,x}^2)\right\}dx \leq V_* \tag{32}$$

where the functional $V_*$ corresponds to the linearized problem $\gamma = 0$.

For the sufficiently small curvature $w^2_{,xx}$ the functional of the nonlinear problem is positive. Therefore, the functional $V$ is locally positive-definite and we can introduce the measures of distance as the square root of functional (32)

$$\|u,v\| = V^{1/2} \tag{33}$$

The measure of distance can be upperbounded in the following way, by the nonlinearity parameter $\xi$, satisfying inequality $\xi \le 1 - \gamma w^2_{,xx}/2$ ,

$$\|u,v\| \le \sqrt{\frac{\ell e}{\gamma}(1-\xi)\xi}, \qquad \xi \in \left[\frac{1}{2}, 1\right] \tag{34}$$

It is easy to notice that,

$$1 - \gamma w^2_{,xx} \ge 2\xi - 1 \tag{35}$$

In order to calculate the differential $dV$ along the trajectory of eqs. (26)-(29) we use the appropriate generalized Itô lemma [12]

$$dV = \int_0^\ell \Big\{ u_{,t} du_{,t} + \big(u_{,t} + h_1 u + \beta_1 e u_{,xxxx}\big)\big(du_{,t} + h_1 du + \beta_1 e du_{,xxxx}\big) + v_{,t} dv_{,t} +$$

$$+\big(v_{,t} + h_1 v + \beta_1 e v_{,xxxx}\big)\big(dv_{,t} + h_1 dv + \beta_1 e dv_{,xxxx}\big) + 2e\big(u_{,xx} du_{,xx} + v_{,xx} dv_{,xx}\big)\Big[1 - \frac{\gamma}{2} w^2_{,xx}\Big] +$$

$$-e\gamma\big(u^2_{,xx} + v^2_{,xx}\big)\big(u_{,xx} du_{,xx} + v_{,xx} dv_{,xx}\big) - 2S_o\big(u_{,x} du_{,x} + v_{,x} dv_{,x}\big)\Big\} dx + \int_0^\ell \varsigma^2 \big(u^2_{,xx} + v^2_{,xx}\big) dx dt \tag{36}$$

Eliminating $du$, $dv$, $du_{,t}$, $dv_{,t}$ by means of Eqs. (26)-(29) and integrating from $t = s$ to $\tau_\delta(t)$, where $\tau_\delta(t)$ is the first random time of a trajectory exit from the domain $V^{1/2} = \delta$, conditionally averaging ($\mathcal{E}$), and taking into account inequality (35) we have

$$\mathcal{E}V(\tau_\delta(t)) = V(s) - \mathcal{E}\int_s^{\tau_\delta(t)} \mathcal{F}(t) dt \tag{37}$$

where

$$\mathcal{F} = \int_0^\ell \{h_1 \left[u_{,t}^2 + v_{,t}^2\right] + \beta_1 e \left[u_{,xxt} + \omega v_{,xx}\right]^2 + \beta_1 e \left[v_{,xxt} - \omega u_{,xx}\right]^2 +$$

$$+\beta_1 e^2 \left[u_{,xxxx}^2 + v_{,xxxx}^2\right](2\xi - 1) - \beta_1 e S_o \left[u_{,xxx}^2 + v_{,xxx}^2\right] + h_1 e \left[u_{,xx}^2 + v_{,xx}^2\right](2\xi - 1) +$$

$$-(\omega^2 \beta_1 e + \varsigma^2)\left[u_{,xx}^2 + v_{,xx}^2\right] - h_1 S_o \left[u_{,x}^2 + v_{,x}^2\right] + \beta_1 e L \left[u_{,xxxx} v_{,xxx} - v_{,xxxx} u_{,xxx}\right]$$

$$+ h_1 L \left[u v_{,xxx} - v u_{,xxx}\right] + 2L \left[u_{,t} v_{,xxx} - v_{,t} u_{,xxx}\right]\} \, dx \qquad (38)$$

Rearrangimg the first and the last terms of integrand in Eq. (38) we have

$$\mathcal{F} = \int_0^\ell \left\{ h_1 \left[ \left( u_{,t} - \frac{L}{h_1} v_{,xxx} \right)^2 + \left( v_{,t} + \frac{L}{h_1} u_{,xxx} \right)^2 \right] + \beta_1 e \left[u_{,xxt} + \omega v_{,xx}\right]^2 + \right.$$

$$+\beta_1 e \left[v_{,xxt} - \omega u_{,xx}\right]^2 + \beta_1 e^2 \left[u_{,xxxx}^2 + v_{,xxxx}^2\right](2\xi - 1) - \beta_1 e S_o \left[u_{,xxx}^2 + v_{,xxx}^2\right] +$$

$$+h_1 e \left[u_{,xx}^2 + v_{,xx}^2\right](2\xi - 1) - (\omega^2 \beta_1 e + \varsigma^2)\left[u_{,xx}^2 + v_{,xx}^2\right] - h_1 S_o \left[u_{,x}^2 + v_{,x}^2\right] +$$

$$\left. -\frac{L^2}{h_1}\left[u_{,xxx}^2 + v_{,xxx}^2\right] + h_1 L \left[u v_{,xxx} - v u_{,xxx}\right] + \beta_1 e L \left[u_{,xxxx} v_{,xxx} - v_{,xxxx} u_{,xxx}\right] \right\} dx \qquad (39)$$

Neglecting the first four positive terms of integrand and using the elementary inequality for arbitrary $\eta \in (0, 1)$

$$\pm\, ab = \pm \eta ab/\eta \leq \frac{1}{2}\left(a^2\eta^2 + b^2/\eta^2\right) \qquad (40)$$

we calculate the lowerbound of the functional $\mathcal{F}$.

$$\mathcal{F} \geq \int_0^\ell \Big\{\beta_1 e^2(2\xi - 1 - \eta^2/2)\left[u^2_{,xxxx} + v^2_{,xxxx}\right] - \left(\beta_1 e S_o + \frac{\beta_1 L^2}{2\eta^2} + \frac{L^2}{h_1}\right)\left[u^2_{,xxx} + v^2_{,xxx}\right] +$$

$$+ \left[h_1 e(2\xi - 1) - \omega^2\beta_1 e - \varsigma^2\right]\left[u^2_{,xx} + v^2_{,xx}\right] - h_1 S_o\left[u^2_{,x} + v^2_{,x}\right] - \frac{h_1^2}{2\beta_1}(u^2 + v^2)\Big\}\, dx \qquad (41)$$

Using the supermartingale property and proceeding similarly to the proof of Chebyshev's inequality we find that the trivial solution of Eqs. (26)-(29) is uniformly stochastically stable if the functional $\mathcal{F}$ is positive-definite. It is equivalent to the following algebraic inequality

$$\Bigg\{\Bigg[\Big\{e\beta_1\left(2\xi - 1 - \frac{\eta^2}{2}\right)\left[e\left(\frac{\pi}{l}\right)^2 - S_o\right] - L^2\left(\frac{1}{h_1} + \frac{\beta_1}{2\eta^2}\right)\Big\}\left(\frac{\pi}{l}\right)^2 +$$

$$+ h_1 e(2\xi - 1) - \omega^2\beta_1 e - \varsigma^2\Bigg]\left(\frac{\pi}{l}\right)^2 - h_1 S_o\Bigg\}\left(\frac{\pi}{l}\right)^2 - \frac{h_1^2}{2\beta_1} > 0 \qquad (42)$$

The critical angular velocity can be obtained maximizing over admissible $\xi$ and $\eta$

$$\omega^2 = \max_{0.5<\xi<1,\ 0<\eta<1}\Bigg\{\Bigg\{\Bigg[\Big\{e\beta_1\left(2\xi - 1 - \frac{\eta^2}{2}\right)\left[e\left(\frac{\pi}{l}\right)^2 - S_o\right] - L^2\left(\frac{1}{h_1} + \frac{\beta_1}{2\eta^2}\right)\Big\}\left(\frac{\pi}{l}\right)^2 +$$

$$+ h_1 e(2\xi - 1) - \varsigma^2\Bigg]\left(\frac{\pi}{l}\right)^2 - h_1 S_o\Bigg\}\left(\frac{\pi}{l}\right)^2 - \frac{h_1^2}{2\beta_1}\Bigg\} \Big/ \beta_1 e\left(\frac{\pi}{l}\right)^4 \qquad (43)$$

Increasing of $\xi$ enlarges the stability domain in system parameter space but decreases the stability domain in the state space described by the norm $\|.\|$. It means that the region of initial disturbances (initial conditions) described by the norm becomes smaller. For the constant axial force $\varsigma = 0$ and the stability domain in variables $\omega^2$, $\xi$, $S_0$ is defined by the following inequality

$$\omega^2 < \left[e\left(\frac{\pi}{l}\right)^2(2\xi - 1) - S_o\right]\left[\left(e\beta_1\left(\frac{\pi}{l}\right)^2 + h_1\right] \Big/ \beta_1 e\left(\frac{\pi}{l}\right)^2 \qquad (44)$$

The constant component of axial force decreases the critical rotation speed $\omega$. As for $\xi \to 1$ the domain of the positive-definiteness tends to zero (cf. Eq. (35)), the maximum value of $\omega$ is available for $\xi \approx \frac{3}{4}$. Similarly, the increase of noise intensity $\varsigma$ decreases the admissible rotation speed.

## Conclusions

A method has been presented for analysing the stability of rotating angle-ply composite shafts subjected to a torque and an axial stochastic force. The main conclusion is, that due to the ovalization of composite shafts and weakening geometrical nonlinearity the derived stability criteria have the local character. The critical parameters (e.g. rotation speed) depend on the nonlinearity parameter bounding the measure of disturbed solutions. The increase of the constant component compressive force, the intensity of stochastic force component and the torque destabilize steady-state shaft vibrations.

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