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METHOD OF THE ANALYSIS OF VARIATIONS IN THE DYNAMICAL SYSTEM POTENTIAL ENERGY AIMING AT THE STABILITY ANALYSIS OF CHARACTERISTIC POINTS

Przemysław Szumiński and Tomasz Kapitaniak
Technical University of Łódź
Division of Machines Dynamics
90-924 Łódź, Stefanowskiego 1/15, Poland
Tel. (0-42) 6312229, e-mail: przeszum@ck-sg.p.lodz.pl.

Abstract

A method of the stability analysis of critical points of dynamical systems, based on equations of the so-called potential energy of perturbations, has been proposed. This method has been called the method of potential energy perturbation analysis. Its essence consists in formulation of relations and an analysis of changes in the potential energy of a perturbation of any number of generalised co-ordinates of the dynamical system. An example of the application of this method to analyse the stability in vicinities of characteristic points of a sample dynamical system in the form of a robot manipulator has been also shown.

Introduction

Studying the literature devoted to the subject matter, one should notice that the most often used method of analysis of dynamical system behaviour in vicinities of critical points is the methods based on the analysis of eigenvalues [3, 7], the Lyapunov exponents [2, 3]. The mathematical-numerical method which consists in the analysis of changes in generalised co-ordinate perturbations of the dynamical system as time functions, phase diagrams, Poincaré maps performed for subsequent generalised co-ordinates as a function of one or more motion perturbations of the system can also be employed [8]. The method of eigenvalues makes it possible to describe generally and qualitatively dynamical properties of the system in the neighbourhood of a selected critical point. The method proposed in the present paper allows one not only to answer the question concerning the type of the dynamical system critical point, but to perform any quantitative analyses of the dynamical system behaviour in the vicinity of a given critical point with regard to the number, magnitude, and character of the motion perturbation. The method consists in formulation of relations describing the potential energy, called the potential energy of dynamical system perturbations, and in the analysis of this energy changes in the neighbourhood of the system critical point, on the basis of which it is possible to determine qualitatively and quantitatively a character of the system motion.

The paper consists of four parts. In the first part, general assumptions of the proposed method of potential energy perturbation investigations are presented. Mathematical relations used for the stability analysis and determination of types of critical points of a sample dynamical system are presented in the second part. A *MAR* robot manipulator has been assumed as an example of the mechanical system. The third section includes some sample results of numerical simulations and the conclusions drawn from these numerical simulations. The fourth section is devoted to a comparison of advantages and disadvantages of the proposed method with the methods applied so far and it presents some general remarks.

Theoretical introduction

Let us assume that a vector of spatial generalised co-ordinates of the dynamical system (for instance, of the robot manipulator) assumes the following form:

$$\overline{q} = [\, q_1 , q_2 , \dots , q_i , \dots , q_n \,]^T \tag{1}$$

where:

i - 'i'-th generalised co-ordinate of the dynamical system;

n - number of generalised co-ordinates of the dynamical system.

Let us now introduce a perturbation of the 'i'-th generalised co-ordinate in the form:

$$q_i = q_i + \varphi_i \tag{2}$$

where:

$q_i \in \overline{M}$

$\overline{M}$ - set of perturbed generalised co-ordinates of the dynamical system;

φ_i - perturbation of the 'i'-th generalised co-ordinate of the dynamical system.

Let us assume that the relation expressing the potential energy of the nominal dynamical system motion as a function of its vector of generalised co-ordinates is known. The dynamical system potential energy, taking into account the effect of a perturbation of selected generalised co-ordinates, has been expressed as follows:

$$E_p = A + Z(\varphi_i) \tag{3}$$

where:

A - potential energy terms dependent on the location of the nominal dynamical system in the system state space (e.g. on the configuration of a robot in its operating space) and the data concerning the mass and geometry of the system structure. Here the potential energy terms describing the nominal motion of the dynamical system are included;

$Z(\varphi_i)$ - terms of the potential energy equation expressing the dynamical system motion perturbations (e.g. the industrial robot motion perturbations).

In general, the potential energy of the dynamical system is expressed as:

$$E_p = E_p^n + E_p^* \tag{4}$$

where:

E_p^n - potential energy of the nominal motion of the dynamical system;

E_p^* - potential energy of the nominal motion perturbation.

As a result of the assumption of the analysis of the system dynamics in the close vicinity of the manipulator equilibrium points, the trigonometric functions of the system potential energy have been replaced by a Taylor series. The obtained relations represent changes of the potential energy of the dynamical system perturbation in vicinities of individual configurations in the system state space as a function of the perturbation magnitude of individual generalised co-ordinates and critical point parameters in the space of motion parameters of the dynamical system.

In order to carry out an analysis of the system dynamics in neighbourhoods of critical points, the stability criteria based on the following cases:

for $E_p^* = 0$ => a lack of perturbations, i.e. the perturbation potential energy equals to zero;

for $E_p^* > 0$ => an increase in the system potential energy resulting from the occurrence of motion perturbations; the equilibrium point is stable;

for $E_p^* < 0$ => a loss of the robot potential energy resulting form the occurrence of perturbations, the equilibrium point is unstable;

have been given.

As a result of numerical simulations, one can perform an analysis of changes in the potential energy with respect to the perturbation location of subsequent generalised co-ordinates of the dynamical system as a function of the perturbation magnitude and for the case when:

- perturbations of a selected generalised co-ordinate of the dynamical system under consideration occur;
- perturbations of a higher number or of all generalised co-ordinates of the system under consideration occur.

As a result, some information on types and characters of individual critical points of the dynamical system analysed can be for instance obtained.

In the next section an example of the application of the method for an analysis of types of characteristic points of a sample robot manipulator has been presented.

Analysis of characteristic points of a sample robot manipulator

Let us consider a robot manipulator with three degrees of freedom as an example. A scheme of such a manipulator is shown in Fig. 1. It is a four-link manipulator with the following kinematic chain structure: R-R-P (R - rotational kinematic pair, P - prismatic kinematic pair).

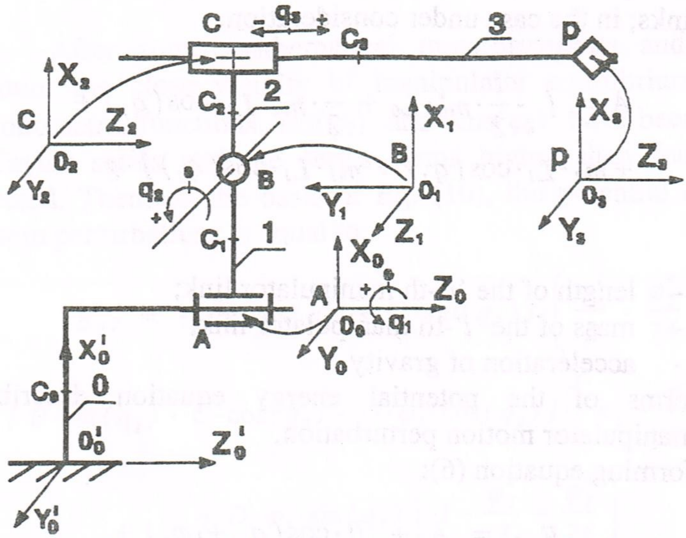

Fig. 1. Scheme of the robot manipulator *MAR*. Choice of local co-ordinate systems by Denavit-Hartenberg notation. The potential energy is described with reference to plain Y_0Z_0.

Now let us consider a relation describing the manipulator potential energy in the second stage of the motion (the motion of the second and third generalised co-ordinate, whereas the first co-ordinate remains stationary).

Let us introduce perturbations of the co-ordinates q_2 and q_3 in the following form:

$$q_2 = q_2 + \varphi_2 \quad , \qquad q_3 = q_3 + \varphi_3 \tag{5}$$

We obtain then the relation describing the manipulator potential energy, in the second stage of the motion, in the form:

$$E_{p2} = A + Z(\varphi_2, \varphi_3) \tag{6}$$

where:

A - terms of the potential energy equation describing the basic (nominal) motion; factors depending on the robot configuration and on the mass and geometrical data of its links; in the case under consideration:

$$A = [-\frac{1}{2} \cdot m_0 \cdot L_0 + \frac{1}{2} \cdot m_1 \cdot L_1 \cdot \cos(q_1) + + m_2 \cdot L_1 \cdot \cos(q_1) + m_3 \cdot L_1 \cdot \cos(q_1)] \cdot g \tag{7}$$

L_i - length of the 'i'-th manipulator link;
m_i - mass of the 'i'-th manipulator link;
g - acceleration of gravity.

Z - terms of the potential energy equation describing the manipulator motion perturbation,

or transforming equation (6):

$$E_{p2} = A + B \cdot \cos(q_2 + \varphi_2) + + C \cdot \sin(q_2 + \varphi_2) + D \cdot \varphi_3 \cdot \sin(q_2 + \varphi_2) \tag{8}$$

where:

$$B = \left(\frac{1}{2} \cdot m_2 + m_3 \right) \cdot L_2 \cdot g \cdot \cos(q_1)$$
$$C = m_3 \cdot (s_3 + q_3) \cdot g \cdot \cos(q_1) \tag{9}$$
$$D = m_3 \cdot g \cdot \cos(q_1)$$

s_3 - initial position of the centre of gravity for link *3*, with regard to position of point *C*.

Thus, after the motion perturbation occurs, we can express the manipulator potential energy as a sum of the manipulator nominal motion potential energy and of the potential energy, called the potential energy of the manipulator motion perturbation, i.e., generally in the form:

$$E_{p2} = E_{p2} + E^{*}_{p2} \tag{10}$$

After some mathematical transformations and taking into account the close vicinity of manipulator equilibrium points, the trigonometric functions 'sin(φ_2)' and 'cos(φ_2)' have been replaced by a Taylor series and the series terms higher than four have been rejected. Then, on the basis of Eq. (10), the potential energy of the system perturbations is equal to:

$$\begin{aligned} E^{*}_{p2} = & -[\,B\cdot\cos(q_2) + C\cdot\sin(q_2)\,]\cdot\left(\frac{\varphi_2^2}{2} - \frac{\varphi_2^4}{24}\right) + \\ & -[\,B\cdot\sin(q_2) - C\cdot\cos(q_2) - D\cdot\varphi_3\cdot\cos(q_2)\,]\cdot\left(\varphi_2 - \frac{\varphi_2^3}{6} + \frac{\varphi_2^5}{120}\right) + \\ & + D\cdot\varphi_3\cdot\sin(q_2)\cdot\left(1 - \frac{\varphi_2^2}{2} + \frac{\varphi_2^4}{24}\right) \end{aligned} \tag{11}$$

If we assume the values of the parameters of individual equilibrium points, then on the basis of Eq. (9), we obtain the values of the coefficients *B*, *C*, *D* of Eq. (11) in the neighbourhood of manipulator individual equilibrium points.

A collection of the obtained results has been presented in Table 1.

Table 1

Manipulator configuration	Parameters of critical point in kinematic pairs	Coefficient B	Coefficient C	Coefficient D
High	$q_1 = 0$ $q_2 = 0$ $q_3 = -s_3$	$(1/2*m_2+m_3)*L_2*g$	0	m_3*g
High	$q_1 = 0$ $q_2 = \pi$ $q_3 = -s_3$	$(1/2*m_2+m_3)*L_2*g$	0	m_3*g
Low	$q_1 = \pi$ $q_2 = 0$ $q_3 = -s_3$	$-(1/2*m_2+m_3)*L_2*g$	0	$-m_3*g$
Low	$q_1 = \pi$ $q_2 = \pi$ $q_3 = -s_3$	$-(1/2*m_2+m_3)*L_2*g$	0	$-m_3*g$

Employing the data included in Table 1, Eq. (11) assumes the form:

$$
\begin{aligned}
E^*_{p2} = & -\frac{1}{120}\cdot B\cdot\sin(q_2)\cdot\varphi_2^5 + \frac{1}{24}\cdot B\cdot\cos(q_2)\cdot\varphi_2^4 + \frac{1}{6}\cdot B\cdot\sin(q_2)\cdot\varphi_2^3 + \\
& -\frac{1}{2}\cdot B\cdot\cos(q_2)\cdot\varphi_2^2 - B\cdot\sin(q_2)\cdot\varphi_2 + D\cdot\sin(q_2)\cdot\varphi_3 + \\
& +\frac{1}{120}\cdot D\cdot\cos(q_2)\cdot\varphi_2^5\cdot\varphi_3 + \frac{1}{24}\cdot D\cdot\sin(q_2)\cdot\varphi_2^4\cdot\varphi_3 + \\
& -\frac{1}{6}\cdot D\cdot\cos(q_2)\cdot\varphi_2^3\cdot\varphi_3 - \frac{1}{2}\cdot D\cdot\sin(q_2)\cdot\varphi_2^2\cdot\varphi_3 + \\
& + D\cdot\cos(q_2)\cdot\varphi_2\cdot\varphi_3
\end{aligned}
\tag{12}
$$

Equation (12) allows for an analysis of changes of the manipulator perturbation potential energy as a function of the perturbation magnitude of individual degrees of freedom. We can distinguish here the cases mentioned in the section entitled Theoretical Introduction.

In the case of a high configuration of the manipulator and at the generalised co-ordinate $q_2 = 0$, as well as in the case of a low configuration of the manipulator and at the generalised co-ordinate $q_2 = \pi$, Eq.(12) assumes the form:

$$E^*_{p2} = \frac{1}{24}\cdot B\cdot\varphi_2^4 - \frac{1}{2}\cdot B\cdot\varphi_2^2 + \frac{1}{120}\cdot D\cdot\varphi_2^5\cdot\varphi_3 + \\ - \frac{1}{6}\cdot D\cdot\varphi_2^3\cdot\varphi_3 + D\cdot\varphi_2\cdot\varphi_3 \tag{13}$$

In the case of a high configuration of the manipulator and at the generalised co-ordinate $q_2 = \pi$, as well as in the case of a low configuration of the manipulator and at the generalised co-ordinate $q_2 = 0$, Eq. (12) assumes, in turn, the form:

$$E^*_{p2} = -\frac{1}{24}\cdot B\cdot\varphi_2^4 + \frac{1}{2}\cdot B\cdot\varphi_2^2 - \frac{1}{120}\cdot D\cdot\varphi_2^5\cdot\varphi_3 + \\ + \frac{1}{6}\cdot D\cdot\varphi_2^3\cdot\varphi_3 - D\cdot\varphi_2\cdot\varphi_3 \tag{14}$$

Examples of numerical analysis results

Analysing Figs. 2, 3, 4, 5, one can observe the changes in the potential energy of the manipulator location perturbation as a function of the perturbation magnitude and the case when:

- a perturbation of one of the generalised co-ordinates of the manipulator under consideration occurs; in this case it is the generalised co-ordinate q_2, Figs. 2, 3;

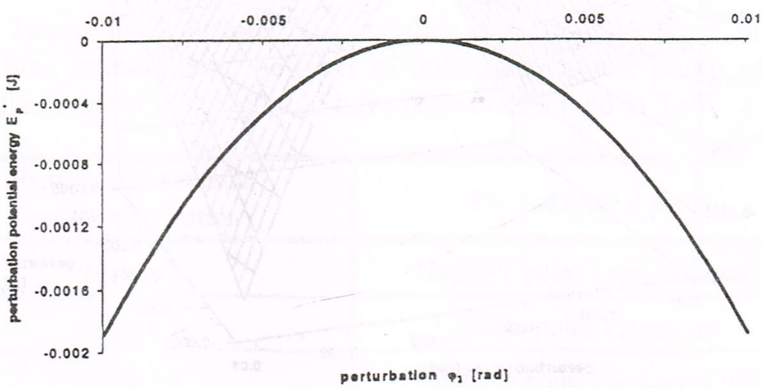

Fig. 2. Perturbation potential energy distribution in case of unstable point.

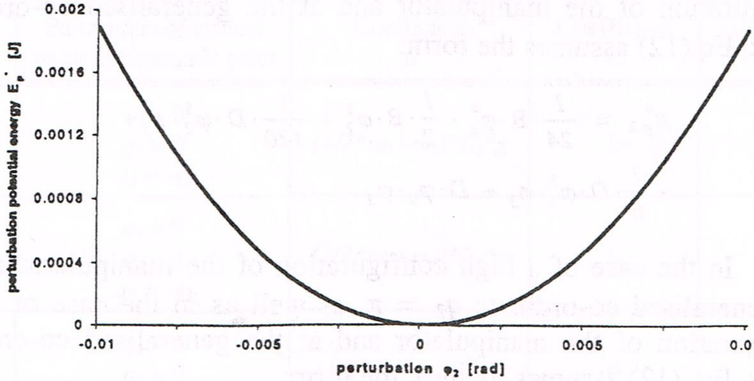

Fig. 3. Perturbation potential energy distribution in case of stable point.

- perturbations of a higher number of generalised co-ordinates occur; in the case considered these are the generalised co-ordinates q_2 and q_3, Figs. 4, 5.

In Figs. (4) and (5) a distribution of the perturbation potential energy as a function of the two-dimensional perturbation φ_2 and φ_3 for selected cases of equilibrium points has been presented.

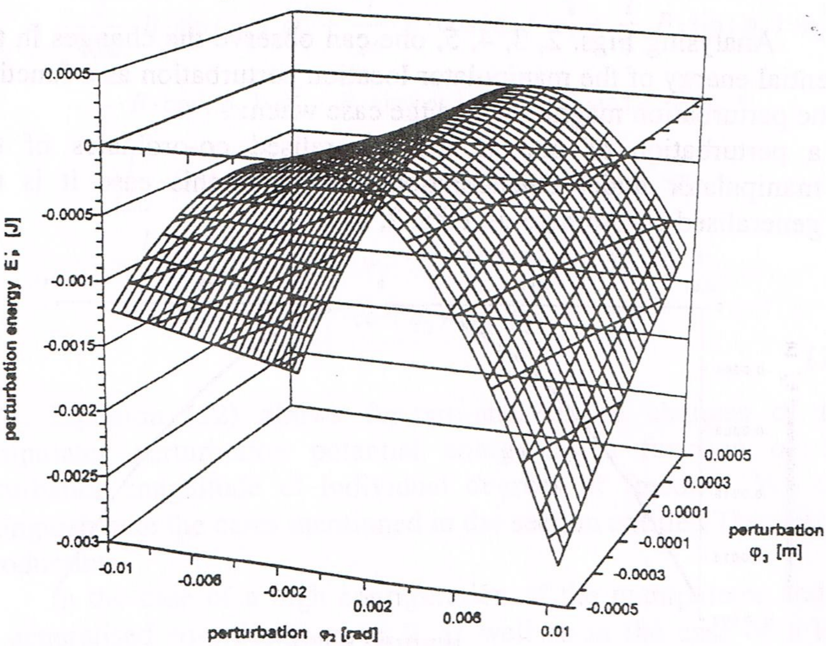

Fig. 4. Perturbation potential energy space distribution in case of unstable point.

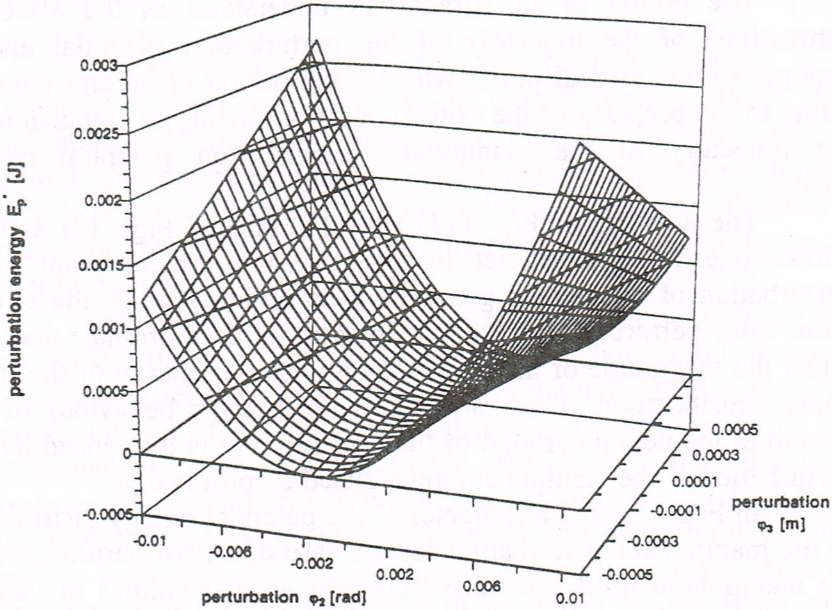

Fig. 5. Perturbation potential energy space distribution in case of stable point.

The perturbations of the location of the manipulator links in the following ranges:

$$\varphi_2 = \pm\ 0.01\ \ rad \quad , \quad \varphi_3 = \pm\ 0.0005\ \ rad \qquad (15)$$

have been introduced into the system under analysis.

Taking into account the stability criteria mentioned in the Theoretical Introduction, the types of individual critical points of the manipulator under consideration have been determined in Table 2.

Table 2

Perturbation potential energy distribution	Kind of critical point
Figure 2	Unstable point (anti-attractor)
Figure 3	Stable point (attractor)
Figure 4	Unstable point (anti-attractor)
Figure 5	Stable point (attractor)

The notion of an attractor is understood as the effect of "attraction" of the trajectory of the perturbation potential energy changes by the critical point, whereas the notion of an anti-attractor refers to the property of the critical point consisting in "repulsion" of the trajectory of the manipulator perturbation potential energy changes.

The analysis of Eqs. (13) and (14) and of Figs. (4) and (5) allows one to observe that in the case the critical point value perturbation of the second generalised co-ordinate lacks, the critical point value perturbation of the manipulator third co-ordinate does not affect the magnitude of the perturbation potential energy of the robot under consideration. Thus, one can state that the behaviour of the system in the neighbourhood of the critical point is influenced by the perturbation of the manipulator generalised co-ordinate q_2.

In Figs. 2 and 3 a character of the potential energy distribution of the manipulator perturbation for the case of a zero perturbation of the manipulator third generalised co-ordinate in vicinities of various types of critical points, in the second stage of the robot motion has been presented. The figures confirm the obtained results referring to types of the manipulator individual critical points determined by the method of eigenvalues and by the motion perturbation method [3, 7].

Physical and geometrical data of the analysed manipulator are included in Table 3.

Table 3

		Manipulator link number i:		
		1	2	3
Kind of kinematic pair:		R – rotational	R – rotational	P – prismatic
Link mass m_i [kg]		12.7	12.7	15
Link length L_i [m]		0.17	0.18	0.42
Position of the centre of gravity	x^r_{si} [m]	$-L_1/2$	$-L_2/2$	0
	y^r_{si} [m]	0	0	0
	z^r_{si} [m]	0	0	$-L_3/2$
$S_3 = 0.18$ m				

Conclusions

The presented method for the stability analysis of the dynamical system in vicinities of critical points allows for a qualitative and quantitative evaluation of the system behaviour. The method allows one to answer the question how the system will behave when it is out of the equilibrium position and how the vibration will change in the vicinity of the equilibrium point, for instance as a function of time.

The forms and kinds of critical points are closely related to the manipulator model which has been assumed. One should observe the qualitative differences in the behaviour of the dynamical system in the neighbourhood of individual critical points, depending on the robot mathematical model assumed. In the case of ideal manipulators, the characteristic type of critical points is *a centre* or *an unsteady saddle-type attractor*, whereas with manipulators with damping in kinematic pairs, *a node-type* and *a focus-type attractor* or *a saddle-type anti-attractor* occurs. The consistency of the types of individual critical points obtained with the method of eigenvalues and the method of motion perturbations have been found. Similar properties of the system under analysis in the vicinity of individual critical points are shown by means of the method of potential energy variations of the manipulator configuration perturbation, proposed in the present paper.

Additionally, the proposed method of perturbation potential energy variations allows one to answer the question concerning the qualitative character of the dynamical system behaviour (in the neighbourhood of the equilibrium point), when this system is out of the equilibrium position, as a function of the perturbation of one or more degrees of freedom.

The proposed method allows for an analysis of the dynamical system behaviour in real time of its motion. The method makes it possible to simulate the system behaviour for different cases of dynamical system motion perturbations. It can be applied for determination of motion control algorithms of subsequent generalised co-ordinates of the dynamical system in order to maintain the conditions of its stability.

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