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THE SOLVABILITY OF PARAMETRICALLY FORCED OSCILLATORS USING WHEP TECHNIQUE

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In this paper, the WHEP technique ,which is the Wiener-Hermite Expansion combined with the perturbation technique, is used to obtain an approximate solution moments for the parametrically forced Duffing Oscillator. A general algorithm is described to get the Gaussian part of the solution process under some necessary assumptions for the solution possibilities. A case study for zero initial conditions is considered to illustrate the solvability of the obtained deterministic equations.

1. Introduction

In recent years ,many investigators studied the Duffing oscillator under different point of views. Duffing oscillator under random excitation was studied by Atkinson [1] using eigenfunction expansions ,by Spanos [2] using stochastic linearization technique, by Caughey in [3] ,by Adomian [4] using decomposition technique, by Ahmadi [5] using perturbation technique, by Gahedi and Ahmadi [6] using Wiener-Hermite Expansion (WHE) technique, by Ibrahim and Pandya [7], by Abdel-Gawad and El Tawil [8] introducing WHEP technique, by Mahmoud G. [9] using the stochastic averaging method, by Bezen and Klebaner [10] using the method of detailed balance and many others.

A general approach was described in [8] which used a combination of the WHE technique [11,12] and the perturbation technique to solve the one dimensional nonlinear stochastic differential equations. Although a combination of square and cubic nonlinearities only were examined in [8], the approach can be applied on different problems successfully. In this approach, which is called WHEP in this paper, any random function S(t) can be expanded as the following:

$$S(t;\omega) = S^{(0)}H^{(0)}(t) + \int_{-\infty}^{\infty} S^{(1)}(t,t_1).H^{(1)}(t_1;\omega)dt_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S^{(2)}(t,t_1,t_2).H^{(2)}(t_1,t_2;\omega)dt_1dt_2 \qquad \text{(A)}$$

where $S^{(i)}()$ are deterministic kernels, $\{H^{(i)}\}$ is a complete set of stochastic orthogonal functions in which

$H^{(0)} = 1$,

$H^{(1)} = n(t)$: white noise,

$H^{(2)} = H^{(1)}(t_1).H^{(1)}(t_2)$,

and generally $H^{(j)} = H^{(j-1)}.H^{(1)}(t_j) - \sum_{m=1}^{j-1} H^{j-m}(t_1,..,t_{j-m}).\delta(t_{j-m} - t_j),$

in which δ is the Dirac-delta function. The random outcome ω belongs to an arbitrary triple probability space (Ω,σ,P).

The ensemble average of S(t) is $S^{(0)}$. The first two terms in the infinite series (A) represents the Gaussian part of the stochastic process while the rest of the series represent the non-Gaussian part.

Applying the expansion in the stochastic differential equation transforms it into a stochastic integro-differential equation which is unfortunately more difficult than the original one.Taking the ensemble average after multiplying by the appropriate H function, a set of simultaneous deterministic integro-differential equations is obtained. They are still not simple to be solved analytically in the majority of the cases. WHEP technique uses the perturbation technique to solve this set and it proves to be successful in obtaining an approximate solution depending on the order of approximations required. The existence of the perturbation series can be proved through the use of the successive approximations. The obtained set of deterministic equations can always be solved successively. The technique was examined successively in [6] for a simple Duffing oscillator, in [8] for mixed square and cubic nonlinearities and in [13) for a Van der Pol oscillator and in [18] for Mathieu equation using the multiple scale method instead of the direct expansion method. The general algorithm of WHEP and more details on the expansion can be reviewed in [8].

2. Problem Description and Solution

The equation of the parametrically forced Duffing oscillator of the following form is considered in this paper,

$$L x + c x^3 + \varepsilon \cos t(x^3 + x') = n(t) \tag{1}$$

where (') denotes differentiation w.r.t. t,

$$L = \frac{d^2}{dt^2} + a\frac{d}{dt} + b, \tag{2}$$

and ε is a parameter with a small value; $|\varepsilon| \le 1$,

The white noise has average and Dirac-delta function as correlation.

For a small t and small oscillations, the following assumption is used through all the paper,

$$\cos t(x^3 + x') \approx 1 - t^2(x^3 + x')^2 / 2 \tag{3}$$

Applying WHE, the solution process can be expanded as the following:

$$x(t) = x^{(0)} + {}_0\int^t x^{(1)}(t,t_1)n(t_1)\, dt_1, \quad 0 \le t_1 \le t \tag{4}$$

where the Gaussian part only is considered.

Applying equation (4) in equation (1),the following equation is obtained:

$$Lx^{(0)} + LI + c\left(x^{(0)}\right)^3 + 3\left(x^{(0)}\right)^2 I + \varepsilon - \frac{\varepsilon t^2}{2}\Big[\left(x^{(0)}\right)^6 + \left(x^{(0)}\right)^3 x'^{(0)} + \left(x'^{(0)}\right)^2 + \left\{6\left(x^{(0)}\right)^5 + 3\left(x^{(0)}\right)^2 x'^{(0)}\right\} I + \left\{\left(x^{(0)}\right)^3 + 2x'^{(0)}\right\} I' \Big] = n(t) \tag{5}$$

where

$$I = \int_0^t x^{(1)}(t,t_1)\, n(t_1)\, dt_1 \,. \tag{6}$$

Taking the ensemble average of equation (5), the following equation is obtained:

$$\begin{aligned} & L\, x^{(0)} + c(x^{(0)})^3 + \\ & + \varepsilon - \frac{\varepsilon\, t^2}{2}[(x^{(0)})^6 + (x^{(0)})^3 . x'^{(0)} + (x'^{(0)})^2] = 0 \end{aligned} \tag{7}$$

Multiplying equation (7) by $n(t_2)$,taking the ensemble average ,using the statistical properties of white noise n(t) and then replacing t_2 by t_1 ,tne following equation is obtained:

$$\begin{aligned} & L\, x^{(1)}(t,t_1) + 3c(x^{(0)})^2 . x^{(1)}(t,t_1) \\ & - \frac{\varepsilon\, t^2}{2}[\{6(x^{(0)})^5 + 3(x^{(0)})^2 . x'^{(0)}\} . x^{(1)}(t,t_1) \\ & \qquad + \{(x^{(0)})^3 + 2x'^{(0)}\} . x'^{(1)}(t,t_1)] = \delta(t - t_1) \end{aligned} \tag{8}$$

Equations (7) and (8) are deterministic integro-differential equations in the unknown kernels. They are still difficult to be solved analytically and approximate methods or numerical techniques should be approached. Equation (7) is a nonlinear equation while equation (8) is a linear equation with variable coefficient. If we rewrite equation (7) in the following form:

$$\begin{aligned} & L\, x_{j+1}^{(0)} = -c(x_j^{(0)})^3 \\ & - \varepsilon + \frac{\varepsilon\, t^2}{2}[(x_j^{(0)})^6 + (x_j^{(0)})^3 . x_j'^{(0)} + (x_j'^{(0)})^2] \quad , j \geq 0 \,, \end{aligned} \tag{9}$$

we can prove the existence of the series:

$$x^{(0)} = \sum_{j=0}^{\infty} \varepsilon^j . x_j^{(0)} \tag{10}$$

Similarly, if we put equation (8) in the following iterative form: '

$$\begin{aligned} & L\, x_{j+1}^{(1)}(t,t_1) = -3c(x_{j+1}^{(0)})^2 . x_j^{(1)}(t,t_1) \\ & + \frac{\varepsilon\, t^2}{2}[\{6(x_{j+1}^{(0)})^5 + 3(x_{j+1}^{(0)})^2 . x_{j+1}'^{(0)}\} . x_j^{(1)}(t,t_1) \\ & \qquad + \{(x_{j+1}^{(0)})^3 + 2x_{j+1}'^{(0)}\} . x_j'^{(1)}(t,t_1)] + \delta(t - t_1) \quad , j \geq 0, \end{aligned} \tag{11}$$

the following series exists:

$$x^{(1)}(t,t_1) = \sum_{j=0}^{\infty} \varepsilon^j . x_j^{(1)}(t,t_1) \,. \tag{12}$$

If we consider only second order perturbations, equation (7) yields the following three equations:

$$L\, x_0^{(0)} + c(x_0^{(0)})^3 = 0 \quad , \tag{13}$$

$$L x_1^{(0)} + 3c(x_0^{(0)})^2 x_1^{(0)} + \\ +1 - \frac{t^2}{2}[(x_0^{(0)})^6 + (x_0^{(0)})^3 . x_0'^{(0)} + (x_0'^{(0)})^2] = 0 \ , \tag{14}$$

$$L x_2^{(0)} + 3c(x_0^{(0)})^2 x_2^{(0)} + 3c(x_1^{(0)})^2 x_0^{(0)} \\ - \frac{t^2}{2}[6(x_0^{(0)})^5 x_1^{(0)} + (x_0^{(0)})^3 . x_1'^{(0)} + 2x_0'^{(0)} x_1'^{(0)} \\ + 3(x_0^{(0)})^2 x_1^{(0)} x_0'^{(0)}] = 0 \tag{15}$$

Similarly, equation (8) yields the following three equations:

$$L x_0^{(1)}(t,t_1) + 3c(x_0^{(0)})^2 . x_0^{(1)}(t,t_1) = \delta(t - t_1) \ , \tag{16}$$

$$L x_1^{(1)}(t,t_1) + 3c(x_0^{(0)})^2 . x_1^{(1)}(t,t_1) + 6c x_0^{(0)} x_1^{(0)} x_0^{(1)} \\ - \frac{t^2}{2}[\{6(x_0^{(0)})^5 + 3(x_0^{(0)})^2 . x_0'^{(0)}\}. x_0^{(1)}(t,t_1) \\ + \{(x_0^{(0)})^3 + 2x_0'^{(0)}\}. x_0'^{(1)}(t,t_1)] = 0 \ , \tag{17}$$

$$L x_2^{(1)}(t,t_1) + 3c(x_0^{(0)})^2 . x_2^{(1)}(t,t_1) + 6c x_0^{(0)} x_1^{(0)} x_1^{(1)} \\ + 3c(x_1^{(0)})^2 x_0^{(1)} + 6c x_0^{(0)} x_2^{(0)} x_0^{(1)} \\ - \frac{t^2}{2}[\{6(x_0^{(0)})^5 + 3(x_0^{(0)})^2 . x_0'^{(0)}\}. x_1^{(1)}(t,t_1) \\ + \{(x_0^{(0)})^3 + 2x_0'^{(0)}\}. x_1'^{(1)}(t,t_1) \\ + \{30(x_0^{(0)})^4 x_1^{(0)} x_0^{(1)} + 3(x_0^{(0)})^2 . x_1'^{(0)} + 6x_0^{(0)} x_0'^{(0)} x_1^{(0)} x_0^{(1)}\}. x_0^{(1)}(t,t_1) \\ + \{3(x_0^{(0)})^2 x_1^{(0)} + 2x_1'^{(0)}\}. x_0'^{(1)}(t,t_1)] = 0 \ . \tag{18}$$

3. The Solution Algorithm

Equation (13) is a simple deterministic Duffing equation. The following approximate solution [14] can be used:

$$x_0^{(0)} = D\exp(-at/2).\cos(\sqrt{b}.t + \theta - (3c/(8a\sqrt{b})).\exp(-at)) \tag{19}$$

where D and 8 are constants which can be computed from zero initial conditions. The rest of the equations, equations (14),...,(18), have the following general form:

$$L y + f^2(t) y = G_j(t); \ j = 1,2,--5, \tag{20}$$

where

$$f(t) = \sqrt{3c}. x_0^{(0)} \tag{21}$$

and the excitation function G depends on the the right hand side of equations (14),...,(18). Equation (20) has an oscillatory solution [15,16] since the time varient coefficient is always positive. We do not have a general closed form solution for equation (20). Accordingly, we may be forced again to approach approximate methods or numerical techniques. However, the solution possibility depends on the initial conditions of the original problem and the complexity of the shape of the excitation functions. Seeking a general solution for equation (20) is beyond the scope of this paper.

The general computations algorithm can be summarised as follows:

Step-1: Solving the linear equation (13).
Output: $x_0^{(0)}(t)$.
Step-2: Solving the linear equation (14).
Output: $x_1^{(0)}(t)$.
Step-3: Solving the linear equation (15).
Output: $x_2^{(0)}(t)$.
Step-4: Solving the linear equation (16).
Output: $x_0^{(1)}(t)$.
Step-5: Solving the linear equation (17).
Output: $x_1^{(1)}(t)$,
Step-6: Solving the linear equation (18).
Output: $x_2^{(1)}(t)$.
Step-7: Computing the solution moments.

$$\mathbf{E\,x} = x^{(0)}(t) = \sum_{j=0}^{2} \varepsilon^j x_j^{(0)}(t), \tag{22}$$

where E denotes the ensemble average.

$$\begin{aligned}\mathbf{Var}\ x &= \int_0^t (x^{(1)}(t,t_1))^2 dt_1 \\ &= \int_0^t [(x_0^{(1)})^2 + \varepsilon^2 (x_1^{(1)})^2 + 2\varepsilon .x_0^{(1)} x_1^{(1)} + 2\varepsilon^2 x_0^{(1)} x_2^{(1)}] dt_1 .\end{aligned} \tag{23}$$

where Var denotes the variance. The previous moments are sufficient to determine the probability density function of the Gaussian solution process proposed in this paper.

4. Case Study

Considering zero initial conditions ,the following results are obtained:

$$x_0^{(0)} = 0 \tag{24}$$

$$x_1^{(0)} = -1/b \tag{25}$$

$$x_2^{(0)} = 0 \tag{26}$$

$$Ex = -\varepsilon/b \tag{27}$$

$$x_0^{(1)}(t, t_1) = h(t - t_1), \tag{28}$$

where h (-) is the impulse response of the equation $Lx_0^{(1)} = \delta(t - t_1)$,

$$x_1^{(1)}(t,t_1)=0\ , \tag{29}$$

$$x_2^{(1)}(t,t_1)=\frac{-3c}{b^2}\int_0^t h(t-s).h(s-t_1)ds\ . \tag{30}$$

Accordingly, the variance of x is

$$\begin{aligned}\text{Var } x = &\int_0^t h^2(t-t_1)dt_1 - \frac{6c\varepsilon^2}{b^2}\int_0^t h(t-t_1).\\ &\int_0^t h(t-s).h(s-t_1)\, ds\, dt_1\\ &+\frac{9c^2\varepsilon^2}{b^4}\int_0^t\left[\int_0^t h(t-s).h(s-t_1)\, ds\right]^2 dt_1\ .\end{aligned} \tag{31}$$

5.Illustrative Example

Let a=0 and b=1,i.e. the Duffng oscillator takes the following form:

$$L\,x + c\,x^3 + \varepsilon\cos t(x^3 + x') = n(t) \tag{32}$$

$$L=\frac{d^2}{dt^2}+b\ . \tag{33}$$

The variance takes the following final form:

$$\begin{aligned}\text{Var } x = &.5\,t - .25\sin(2t)\\ &+9c^2\varepsilon^2\left[\frac{t^3}{32}+\frac{1}{8}\sin^3 t.\cos t-\frac{3}{64}t^2\sin 2t\right]\ .\end{aligned} \tag{34}$$

The graph of the root mean square of x is shown in fig.1.

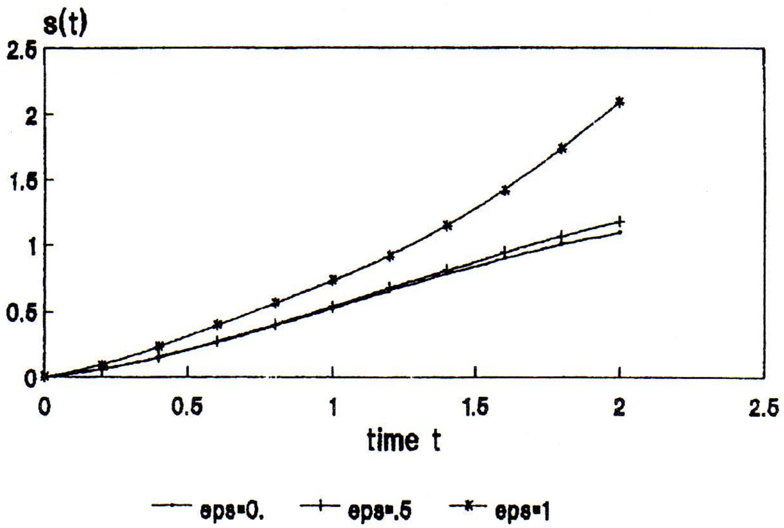

Fig.1. The Root M.S. of x; a=0, b=0, c=1; The root mean square of x, s(t), versus time t for different eps. Levels.

6. List of Assumptions

The following is the list of assumptions used in this paper:
1. Small t and small oscillation which insure the use of the first two terms in the cosine series.
2. Considering the Gaussian part only of the solution process.
3. Taking only second order pertubations in consideration.
4. Using an approximate formula as a solution of the simple Duffing equation, equation (19).

7. Suggestions for modifications

The following suggestions can be considered for developing this research:
1. Taking more terms of the cosine series. This leads to some relaxation in the first assumption.
2. Taking some terms to represent the non-Gaussian art of the solution process. This leads to more realistic computations.
3. Increasing the order of perturbations. This leads to modify the present solution moments in this paper and the other modified cases.
4. Using the solution of the simple deterministic Duffing oscillator in terms of the Jacobian elliptic functions [17].

All these modifications can be processed through the WHEP technique used in this paper.

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