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CHECKING THE CORRECTNESS OF NUMERICAL SOLUTIONS TO EQUATIONS OF MOTION

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Abstract

A method for checking the correctness of numerical solutions to equations of motion for conservative and non-conservative mechanical systems expressed in terms of generalised coordinates has been presented. The suggested method consists in checking the energy level in a system under consideration. An application of the verification method of the solving procedure in numerical computations consists in supplementing equations of motion with an additional differential equation describing changes in the total energy and in tracing time histories of the balance of the energy supplied to the system, produced and dissipated in it, as well as the energy transferred to the surroundings. The equation of the energy balance changes is written by means of a certain function, whose derivative fulfils the condition $C'(t)=0$.

The solution accuracy is shown by a time history of the function $C(t)$, *which should remain constant versus time. A way the function* $C(t)$ *is derived for holonomic systems described by Lagrange's equations of the second kind, for systems with kinematic constraints described by Lagrange's equations of the second kind with Lagrange multipliers, as well as by Maggi's equations and canonical equations, is given. The examples of applications presented concern, first of all, models of engineering machines.*

Introduction

It is difficult to state whether a solution is correct on the basis of analysis of the results of the numerical solution. The character of the time histories of solutions which are formally incorrect can be close to the time histories obtained for a correct solution.

The presented method of checking the correctness is an expansion of the idea presented by Kane and Levinson [9, 10] and discussed in many references [1, 2, 3, 4, 5, 7, 13, 14]. An application of this method in numerical computations consists in supplementing equations of motions with an additional differential equation describing changes in the total energy and in tracing the time history of the balance of the energy supplied to the system, produced and dissipated in it, as well as the energy transferred to the surroundings. The equation of the energy balance changes is written by means of a certain function (C), whose derivative satisfies the condition

$$\dot{C} = 0 . \tag{1}$$

The accuracy of the solution is manifested by the time history of the function C which should remain constant. Any changes in this function during its integration with a step-by-step method indicate errors in the solution, which can follow from errors in the algorithm (incorrect or oversimplified equations), program (divergent procedures) or data (a wrong integration time step has been selected, etc.) We shall consider that a numerical solution (which is, of course, approximate) is correct if the maximum value of the control function $C(t)$ is considerably smaller than the values of the maximum total mechanical energy E, the potential energy V or the kinetic energy T. Formally, such a condition can be presented in the form

$$\delta_C = \frac{\sup_t |C|}{\sup_t(max(|E|, |T|, |V|))} \ll 1 . \tag{2}$$

In the present paper a way in which the function C occurring in the balance equation is derived for holonomic systems described by Lagrange's equations of the second kind [8, 11, 12], for systems with kinematic constraints described by Lagrange's equations of the second kind with Lagrange multipliers [8, 12], and for systems analysed by means of Maggi's equations and canonical equations, is shown.

Lagrange's equations of the second kind

Using Lagrange's equations of the second kind in analysis of the mechanical system motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{f} - \frac{\partial V}{\partial \mathbf{q}} , \tag{3}$$

it can be shown [15] that in this case the balance equation assumes the following form

$$\dot{C}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) \equiv \dot{E} - (2\dot{T}_0 + \dot{T}_1) + \dot{Z} = 0 . \tag{4}$$

This equation has been obtained on the assumption that the potential energy (V) is a function of generalised displacements and time, and it does not depend on the velocity ($V = V(\mathbf{q}, t)$). The velocities of the points of the system under consideration are linear forms with respect to generalised velocities. The velocity vectors ($\mathbf{v}$) of the points can be presented as

$$\mathbf{v} = \mathbf{U}\dot{\mathbf{q}} + \mathbf{v}_t \,, \tag{5}$$

where: $\mathbf{U} = \mathbf{U}(\mathbf{q}, t)$, $\mathbf{v}_t = \mathbf{v}_t(\mathbf{q}, t)$.

The angular velocities of rigid bodies are expressed as:

$$\omega = \mathbf{W}\dot{\mathbf{q}} + \omega_t \,, \tag{6}$$

where $\mathbf{W}(\mathbf{q}, t)$ and $\omega_t(\mathbf{q}, t)$ are (like $\mathbf{U}$ and $\mathbf{v}_t$) independent of the generalised velocities $\dot{\mathbf{q}}$. On such assumptions, the system kinetic energy $T = T(\mathbf{q}, \dot{\mathbf{q}}, t)$ can be represented as a sum of three components, namely:

T_0 - term independent of the generalised velocities $\dot{\mathbf{q}}$,
T_1 - term linearly dependent on the velocities,
T_2 - term dependent on the square of the generalised velocities, i.e.:

$$T = T_0 + T_1 + T_2 \,, \tag{7}$$

where:

$$T_0 = \frac{1}{2}(\mathbf{v}_t^T \mathbf{M}_0 \mathbf{v}_t + \boldsymbol{\omega}_t^T \mathbf{J}_0 \boldsymbol{\omega}_t) \,, \tag{8}$$

$$T_1 = \frac{1}{2}(\mathbf{v}_t^T \mathbf{M}_1 \mathbf{U}\dot{\mathbf{q}} + \omega_t^T \mathbf{J}_1 \mathbf{W}\dot{\mathbf{q}}) \,, \tag{9}$$

$$T_2 = \frac{1}{2}\dot{\mathbf{q}}^T \left(\mathbf{U}^T \mathbf{M}_2 \mathbf{U} + \mathbf{W}^T \mathbf{J}_2 \mathbf{W}\right) \dot{\mathbf{q}} \,. \tag{10}$$

The kinetic energy derivative $\dot{T} = \dot{T}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$ and the potential energy derivative $\dot{V} = \dot{V}(\mathbf{q}, \dot{\mathbf{q}}, t)$ of the system can be presented as a sum of three terms

$$\dot{T} = \frac{\partial T}{\partial t} + \dot{\mathbf{q}}^{\mathrm{T}} \frac{\partial T}{\partial \mathbf{q}} + \ddot{\mathbf{q}}^{\mathrm{T}} \frac{\partial T}{\partial \dot{\mathbf{q}}} \,, \tag{11}$$

and two terms

$$\dot{V} = \frac{\partial V}{\partial t} + \dot{\mathbf{q}}^{\mathrm{T}} \frac{\partial \dot{T}}{\partial \mathbf{q}} \,. \tag{12}$$

Premultiplying Lagrange's equations (3) by $\dot{\mathbf{q}}^{\mathrm{T}}$

$$\dot{\mathbf{q}}^{\mathrm{T}} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \dot{\mathbf{q}}^{\mathrm{T}} \frac{\partial T}{\partial \mathbf{q}} = \dot{\mathbf{q}}^{\mathrm{T}} \mathbf{f} - \dot{\mathbf{q}}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{q}} \,, \tag{13}$$

and using the identity

$$\frac{d}{dt}\left(\dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) = \ddot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}} + \dot{\mathbf{q}}^{\mathrm{T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) \tag{14}$$

or

$$\dot{\mathbf{q}}^{\mathrm{T}}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) = \frac{d}{dt}\left(\dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \ddot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}} \tag{15}$$

and substituting relations (13), (11) and (12) into (15) we obtain

$$\frac{d}{dt}\left(\dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \dot{T} + \frac{\partial T}{\partial t} = \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} - \dot{V} + \frac{\partial V}{\partial t}\,, \tag{16}$$

where $\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f}$ is a power of the nonpotential forces $\mathbf{f}$, which can depend on the coordinates, generalised velocities and time, i.e. $\mathbf{f} = \mathbf{f}(\mathbf{q},\ \dot{\mathbf{q}},\ t)$. Employing the relationship resulting from the Euler's theorem concerning the uniform form in equation (16)

$$\dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}} = 0\cdot T_0 + 1\cdot T_1 + 2\cdot T_2\,, \tag{17}$$

we get the power balance equation of the system during its motion in the form

$$\dot{T}_1 + 2\dot{T}_2 - \dot{T} + \frac{\partial T}{\partial t} = \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} - \dot{V} + \frac{\partial V}{\partial t} \tag{18}$$

or, substituting (7)

$$\dot{T} - (2\dot{T}_0 + \dot{T}_1) + \frac{\partial T}{\partial t} = \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} - \dot{V} + \frac{\partial V}{\partial t}T\,. \tag{19}$$

If we introduce the total energy of the system $E = T + V$, then we will obtain

$$\dot{E} - (2\dot{T}_0 + \dot{T}_1) + \frac{\partial T}{\partial t} - \frac{\partial V}{\partial t} - \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} = 0\,. \tag{20}$$

Finally, having introduced the notation

$$\dot{Z} = -\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} - \frac{\partial V}{\partial t} + \frac{\partial T}{\partial t}\,, \tag{21}$$

the balance equation assumes the form:

$$\dot{C} = \dot{E} - (2\dot{T}_0 + \dot{T}_1) + \dot{Z} = 0. \tag{22}$$

We state that the function $C(t)$ defined in such a way

$$C(t) = E - (2T_0 + T_1) - Z \tag{23}$$

has to remain constant during integration of the equations of motion.

If the Lagrange's function $L = T - V$ is used in the equations, then we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{f}\,, \tag{24}$$

$$\dot{Z} = -\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} + \frac{\partial L}{\partial t}\,, \tag{25}$$

$$\dot{C} = \dot{L} - (2\dot{L}_0 + \dot{L}_1) + \dot{Z} = 0. \tag{26}$$

For the potential system ($\mathbf{f} = \mathbf{0}$) and the scleronomic system ($T_0 = T_1 = L_0 = L_1 = 0$), we get

$$\dot{Z} = 0, \tag{27}$$

it means

$$\dot{C} = \dot{E} = \dot{V} + \dot{T} = 0. \tag{28}$$

Such a system is conservative (the principle of conservation of the total mechanical energy of the system is satisfied). Thus, the relation $C(t) = const$ is a generalisation of the energy conservation principle over nonconservative systems with time-dependent constraints and loaded by nonpotentional forces, and can be used to check the correctness of numerical solutions of equations of motion.

Lagrange's equations of the second kindwith multipliers

In many mechanical systems the constraints imposed on a system depend on velocity or geometrical constraints are presented in the form of kinematic constraints of the first order. They appear, first of all, in the issues connected with body motion control. The examples of machines in which kinematic constraints occur are devices operating in automated production and transportation processes, such as robots, overhead cranes, etc. For the system with kinematic constraints

$$\boldsymbol{\psi}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{G}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{G}_1\dot{\mathbf{q}}_1 + \mathbf{G}_2\dot{\mathbf{q}}_2 + \mathbf{g} = \mathbf{0}\,, \tag{29}$$

Lagrange's equations of the second kind have the form [6]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}_1} - \frac{\partial T}{\partial \mathbf{q}_1} - \mathbf{f}_1 - \mathbf{G}_1^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}\,, \tag{30}$$

where the indeterminate multipliers $\boldsymbol{\lambda}$ are determined as follows

$$\boldsymbol{\lambda} = \mathbf{G}_2^{-\mathrm{T}}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}_2} - \frac{\partial T}{\partial \mathbf{q}_2} - \mathbf{f}_2\right). \tag{31}$$

The quantities with the subscript $(\cdot)_1$ and $(\cdot)_2$ are related to independent and dependent velocities, respectively. Below, a way in which the function Z occurring in the balance equation is derived for nonholonomic systems described by Lagrange's equations of the second kind with Lagrange multipliers is presented. It is assumed that the potential energy is a function of displacements and time, and it does not depend on the generalised velocites ($V = V(\mathbf{q}, t)$), and that the kinetic energy of the system can be written as the following sum

$$T(\mathbf{q}, \dot{\mathbf{q}}, t) = T_0(\mathbf{q}, t) + T_1(\mathbf{q}, \dot{\mathbf{q}}, t) + T_2(\mathbf{q}, \dot{\mathbf{q}}^2, t) , \tag{32}$$

where T_0 does not depend on the velocity, T_1 is a linear form of the generalised velocities, and T_2 is a square form of the generalised velocities. Determining the derivative of the total mechanical energy and employing Lagrange's equations with multipliers (30), one obtains

$$\dot{E} = \frac{dE}{dt} = \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{f} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\boldsymbol{\lambda} . \tag{33}$$

This relation is used in mechanics in derivation of the principles of conservation of energy [8, 11]. Introducing the notation

$$\dot{Z} = \frac{\partial T}{\partial t} - \frac{\partial V}{\partial t} - \dot{\mathbf{q}}^{\mathrm{T}} \left(\mathbf{f} + \mathbf{G}^{\mathrm{T}}\boldsymbol{\lambda}\right) , \tag{34}$$

the power balance of the system can be expressed:
– by means of the total energy as

$$\dot{E} - \dot{T}_1 - 2\dot{T}_0 + \dot{Z} = 0 , \tag{35}$$

– by means of the potential energy with the relation

$$\dot{V} - \dot{T}_0 + \dot{T}_2 + \dot{Z} = 0 , \tag{36}$$

– by means of the Hamiltonian function ($H \equiv E - 2T_0 - T_1$) with the formula

$$\dot{H} + \dot{Z} = 0 . \tag{37}$$

Defining the function C as

$$C \equiv E - 2T_0 - T_1 + Z \equiv V - T_0 + T_2 + Z \equiv H + Z , \tag{38}$$

we find that it remains constant in time [1].

[1] For the potential system, ($\mathbf{f} = \mathbf{0}$), with the time-independent constraints ($T_0 = T_1 = 0$; $T = T_2$) and the energy independent of time in an explicit way ($\frac{\partial T}{\partial t} = \frac{\partial V}{\partial t} = 0$), we obtain $\dot{Z} = 0$ and $\dot{E} = 0$, i.e. the principle of total mechanical energy conservation (conservative system) is fulfilled.

Maggi's equations

Maggi's equations are obtained after elimination of indeterminate multipliers from Lagrange's equations [6]

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}_1} - \frac{\partial T}{\partial \mathbf{q}_1} - \mathbf{f}_1\right) - \mathbf{G}_1^{\mathrm{T}}\mathbf{G}_2^{-\mathrm{T}}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}_2} - \frac{\partial T}{\partial \mathbf{q}_2} - \mathbf{f}_2\right) = \mathbf{0}\,. \tag{39}$$

Having eliminated the vector of the Lagrange multipliers ($\boldsymbol{\lambda}$) (i.e. substituting (31) into (34)), the function $\dot{Z}$ can be presented in the form

$$\dot{Z} = \frac{\partial T}{\partial t} - \frac{\partial V}{\partial t} - \dot{\mathbf{q}}_1^{\mathrm{T}}\left(\mathbf{f}_1 + \mathbf{G}_1^{\mathrm{T}}\mathbf{G}_2^{-\mathrm{T}}\mathbf{f}_2\right) +$$

$$-\left(\dot{\mathbf{q}}_1^{\mathrm{T}}\mathbf{G}_1^{\mathrm{T}}\mathbf{G}_2^{-\mathrm{T}} + \dot{\mathbf{q}}_2^{\mathrm{T}}\right)\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}_2} - \frac{\partial T}{\partial \mathbf{q}_2}\right). \tag{40}$$

Finally, the power balance of the system can be expressed:
– by means of the total energy as

$$\dot{E} - \dot{T}_1 - 2\dot{T}_0 + \dot{Z} = 0\,, \tag{41}$$

– by means of the potential energy with the relation

$$\dot{V} - \dot{T}_0 + \dot{T}_2 + \dot{Z} = 0\,, \tag{42}$$

– by means of the Hamiltonian function ($H \equiv E - 2T_0 - T_1 \equiv V - T_0 + T_2$) as

$$\dot{H} + \dot{Z} = 0\,. \tag{43}$$

Defining the function $C(t)$ as

$$C(t) \equiv E - 2T_0 - T_1 + Z \equiv V - T_0 + T_2 + Z \equiv H + Z\,, \tag{44}$$

and on the basis of (43), we find that it remains constant in time.

Canonical equations

Introducing into analysis the function $H(t, \mathbf{q}, \dot{\mathbf{q}})$ determined by the relationship

$$H(t, \mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T\mathbf{p} - L\,, \tag{45}$$

(where $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ denotes generalised momenta), calculating a variation from the Hamiltonian function ($\delta H(t, \mathbf{q}, \dot{\mathbf{q}}) = \delta(\dot{\mathbf{q}}^T\mathbf{p}) - \delta L$) and employing the relations resulting from Lagrange's equations (3), we obtain canonical equations of motion:

$$\frac{\partial H}{\partial \mathbf{q}} = -(\dot{\mathbf{p}} - \mathbf{f})\,, \tag{46}$$

$$\frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}} \, . \tag{47}$$

Below, a way the function C is derived for the systems described by Hamilton's canonical equations is presented. The method discussed consists in checking the energy (power) balance changes in the system. As has been done previously, it is assumed that the potential energy (V) is a function of displacements and time, and it does not depend on the velocity ($V = V(\mathbf{q}, t)$), and that the Hamiltonian function for the system can be written as the following sum

$$H(\mathbf{q}, \mathbf{p}, t) = H_0(\mathbf{q}, t) + H_1(\mathbf{q}, \mathbf{p}, t) + H_2(\mathbf{q}, \mathbf{p}^2, t) \, , \tag{48}$$

where H_0 does not depend on momenta, H_1 and H_2 is a linear and square uniform form of momenta, respectively. Presenting the derivative of the Hamiltonian function H of the system with respect to time as

$$\dot{H} = \frac{\partial H}{\partial t} + \dot{\mathbf{q}}^T \frac{\partial H}{\partial \mathbf{q}} + \dot{\mathbf{p}}^T \frac{\partial H}{\partial \mathbf{p}} \tag{49}$$

and using the relations resulting from canonical equations (46) and (47)

$$\dot{\mathbf{p}}^T \frac{\partial H}{\partial \mathbf{p}} = -\dot{\mathbf{q}}^T \left(\frac{\partial H}{\partial \mathbf{q}} - \mathbf{f} \right) , \tag{50}$$

we arrive at the relation:

$$\dot{H} = \frac{\partial H}{\partial t} + \dot{\mathbf{q}}^T \frac{\partial H}{\partial \mathbf{q}} - \dot{\mathbf{q}}^T \left(\frac{\partial H}{\partial \mathbf{q}} - \mathbf{f} \right) . \tag{51}$$

Finally, we obtain

$$\dot{H} = \frac{\partial H}{\partial t} + \dot{\mathbf{q}}^T \mathbf{f} . \tag{52}$$

Defining $\dot{C}(t)$ as

$$\dot{C} \equiv \dot{H} - \left(\frac{\partial H}{\partial t} + \dot{\mathbf{q}}^T \mathbf{f} \right) = 0 \tag{53}$$

and using the notations employed in this work so far, we have

$$\dot{Z} = -\frac{\partial H}{\partial t} - \dot{\mathbf{q}}^T \mathbf{f} \, , \tag{54}$$

$$\dot{C}(t) \equiv \dot{H} + \dot{Z} = 0 \, . \tag{55}$$

We find that the function $C(t) = H + Z$ has preserved a constant value during the system motion. The values of the function $C(t)$ at each integration step of the motion equations can be checked in order to test the accuracy of the solution.

Sample calculations

1. Model of the mechanism

For the mechanism shown in Fig. 1, an application of the method for testing the correctness of numerical solutions of equations of motion has been shown in detail. The model under consideration consists of three rigid bodies, namely: body (1) with the mass m_1 moving along the axis x, body (2) connected by a cylindrical joint with body (1) and having the mass m_2 and the moment of inertia J_2, and the material point m_3. This point is connected with (2) by means of inextensible string. The kinematic excitation of the element with the mass m_2 is defined by the equations of constraints $\dot{\beta} - f(t) = 0$.

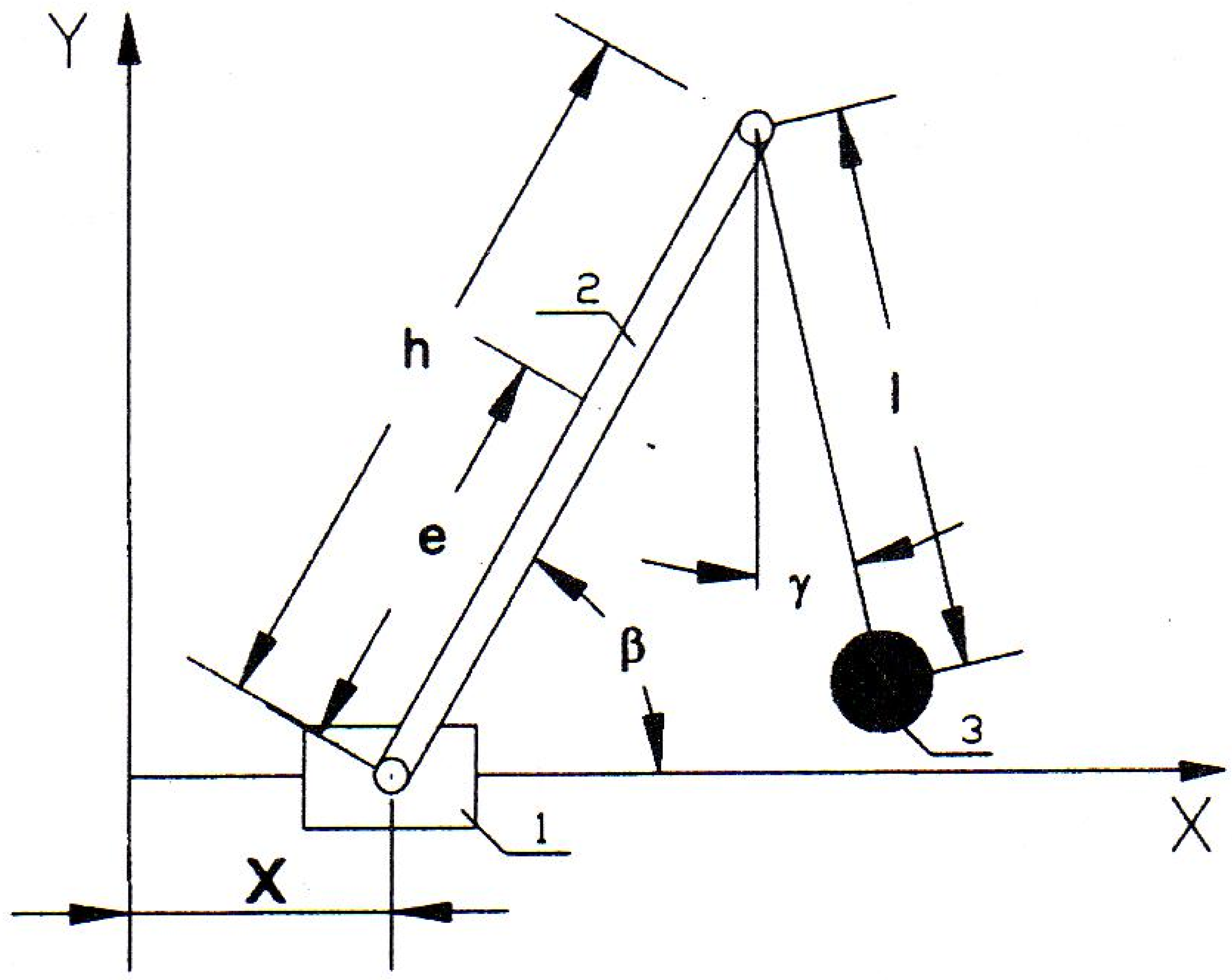

Figure 1: Computational model of the mechanism

The balance equation allowing for checking the correctness of numerical solutions requires that the previously defined function $C(t) = E - (2T_0 + T_1) + Z$, which has to remain constant during integration of the equations of motion, should be calculated. For the data assumed for the mechanism under consideration, we have

$$T_2 = \frac{1}{2}\begin{bmatrix} \dot{x} \\ \dot{\gamma} \end{bmatrix}^T \begin{bmatrix} m_1 + m_2 + m_3 & m_3 l \cos\gamma \\ m_3 l \cos\gamma & m_3 l^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\gamma} \end{bmatrix},$$

$$T_1 = \dot{\beta}\begin{bmatrix} (-m_2 e - m_3 h)\sin\beta & m_3 h l \sin(\gamma - \beta) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\gamma} \end{bmatrix},$$

$$T_0 = \frac{1}{2}\dot{\beta}\left[\ (-m_2 e^2 + J_2 + m_3 h^2)\ \right]\dot{\beta}\ ,$$

$$V = m_2 g e \sin\beta + m_3 g (h \sin\beta - l \cos\gamma) \ .$$

Having determined the partial derivatives from the kinetic and potential energy with respect to time, one obtains

$$\dot{Z} = (m_2 e^2 + J_2 + m_3 h^2)\dot{\beta}\ddot{\beta} + (-m_2 e - m_3 h)\cos\beta\dot{x}\dot{\beta}^2 +$$

$$(-m_2 e - m_3 h)\sin\beta\ddot{x}\dot{\beta}^2 + m_3 h l \ddot{\beta}\dot{\gamma}\sin(\gamma - \beta) +$$

$$-m_3 h l \dot{\beta}^2\dot{\gamma}\cos(\gamma - \beta) + -(m_2 e + m_3 h) g \dot{\beta}\cos\beta \ .$$

This equation is integrated simultaneously with the equations of motion of the system under analysis. Knowing the value $Z(t)$, the value of the function $C(t)$ (23) is determined in each integration time step. The time history of $C(t)$ testifies the correctness of the results of the numerical solution (if its value is constant), or an occurrence of error (variable values of $C(t)$).

Numerical calculations have been carried out for the following data $f(t) = 5\cos 5t$ [rd], $m_1 = m_2 = m_3 = 2.5$ [kg], $J_2 = 2.5$ [kgm^2], $h = 2$ [m], $e = 0.5$ [m] oraz $l = 0.1$ [m].

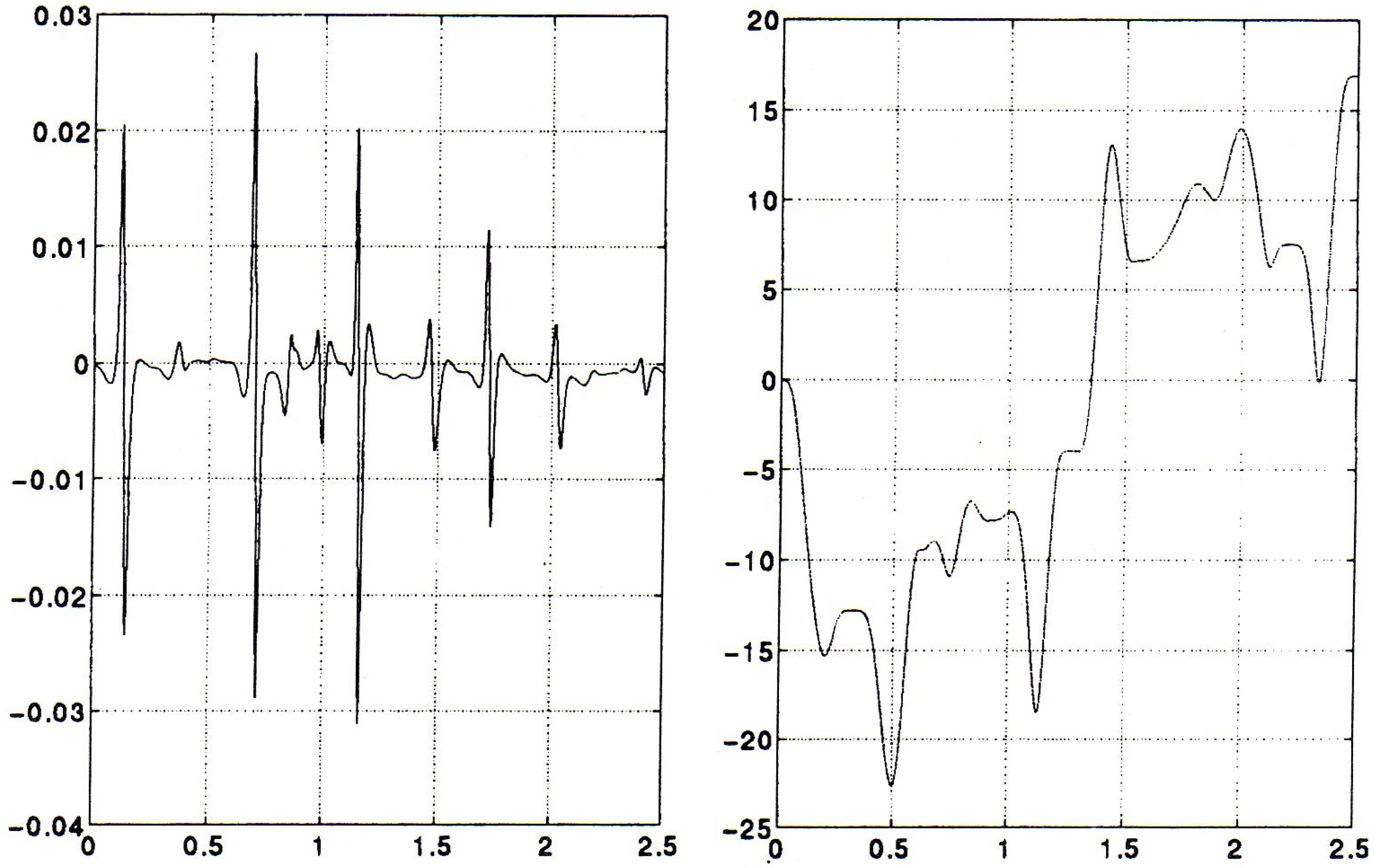

Figure 2: Function C for a) correct, and b) wrong equations of motion

The obtained results are illustrated in figures (Figs. 2-3). The co-ordinates $x(t)$ and $\gamma(t)$ for a correct and wrong solution have been compared. Analysing the character of the time histories only, one cannot distinguish between a correct and incorrect solution. Such a diagnosis is possible when the time histories of the function $C(t)$ for both the cases are compared [2]. As the procedures used in numerical solutions impose a given accuracy, in practice we obtain variable values of $C(t)$. In Fig. 2 the diagrams of the function $C(t)$ in the scale which makes it possible to evaluate a range of its changes during calculations have been presented. The ratio of maximum changes for both the cases presented is $\frac{\Delta C_b}{\Delta C_a} \sim 670$. In Fig. 3 the functions $C(t), E(t)$ and $Z(t)$ for a correct and wrong solution have been compared. In the case a correct system of equations of motion, program and integration procedure are used, these changes are insignificant. It is assumed that a solution is correct if the time history of the function $C(t)$ is constant on the diagram of the total energy of the system (Fig. 3b), i.e. changes in this function are negligibly small in comparison with changes in the total energy of the system.

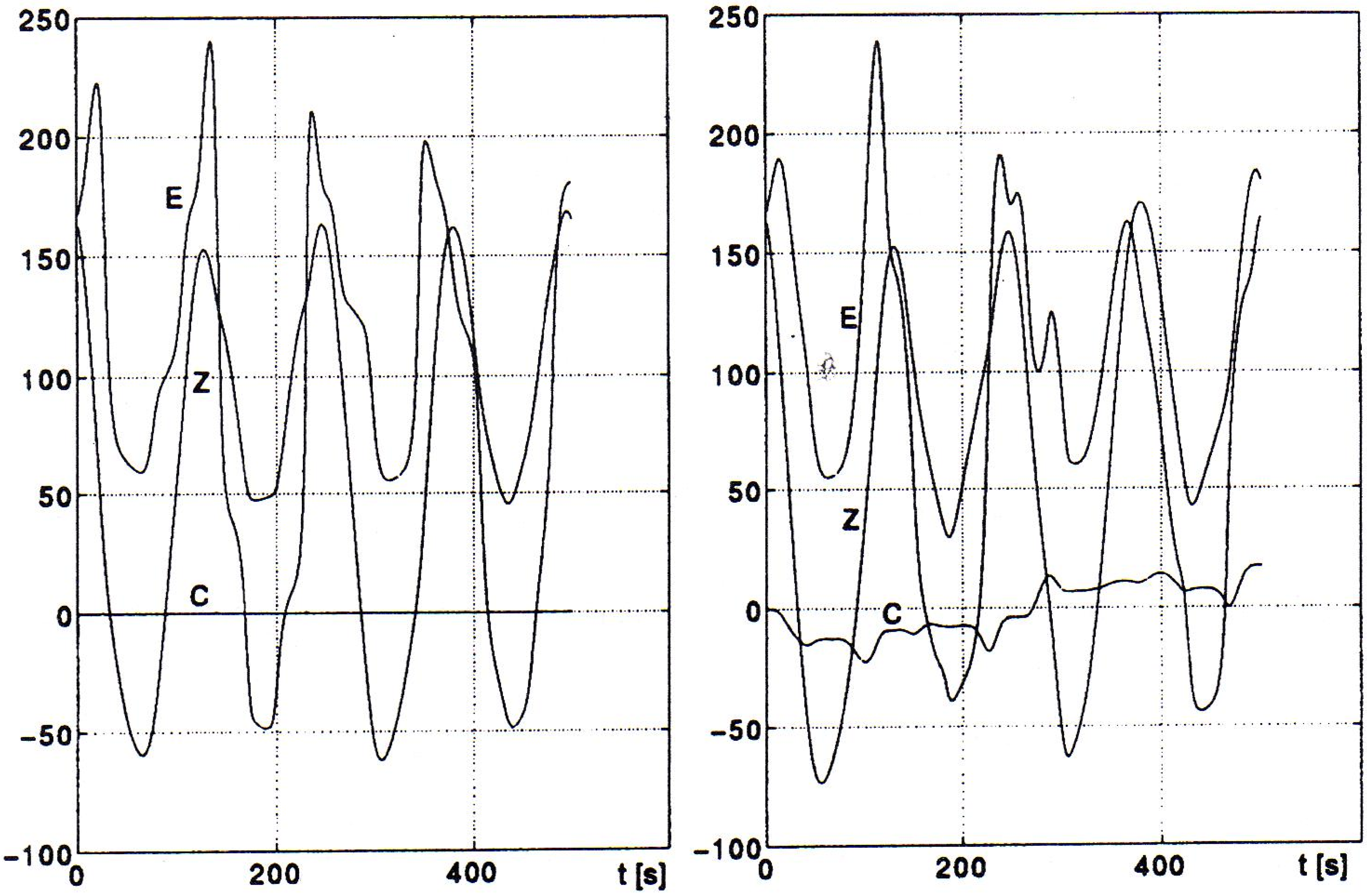

Figure 3: Function C, Z and energy E for a) correct, and b) wrong equations of motion

2. Three-dimensional model of a crane

The described method has been used to solve the equations of motion of the three-dimensional model of a crane with six degrees of freedom. The model assumed in analysis consists of two rigid bodies connected with each other and

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[2] The error has been introduced on purpose into the equations of motion - the sign of one of the terms has been changed.

elastically connected to the base. It is a simplified model of the system described in [4]. The analysis comprises significant displacements of the system elements. In order to test the correctness of numerical computations, one should determine the quantities occurring in the formula of the function $C(t)$

$$C(t) = T_2 - T_0 + V + Z \,, \tag{56}$$

where the derivative of the function Z is reduced to the form (only potential forces act)

$$\dot{Z} = \frac{dZ}{dt} = \frac{\partial T}{\partial t} - \frac{\partial V}{\partial t} \,. \tag{57}$$

Finally, the derivative of the function Z assumes the form

$$\begin{aligned}\dot{Z} = &\frac{\partial \mathbf{\Omega}_{21}^T}{\partial t}\mathbf{J}_2\mathbf{\Omega}_{21}+\\ &+\frac{1}{2}\frac{\partial \mathbf{\Omega}_{21}^T}{\partial t}\mathbf{J}_2\mathbf{\Omega}_{20} + \frac{1}{2}\mathbf{\Omega}_{20}^T\mathbf{J}_2\frac{\partial \mathbf{\Omega}_{21}^T}{\partial t}+\\ &+\mathbf{V}_1^T\frac{\partial \mathbf{Q}_2}{\partial t}\mathbf{\Omega}_{21} + \mathbf{V}_1^T\mathbf{Q}_2\frac{\partial \mathbf{\Omega}_{21}}{\partial t}+\\ &-m_2 g\,[0,\ 0,\ 1]\,(\mathbf{A}_1\frac{\partial \mathbf{A}_{\alpha\beta}}{\partial t}\mathbf{W}_{s2}) \,. \end{aligned} \tag{58}$$

In the case when the nonstationary constraints are linear functions of time, relation (58) is reduced significantly

$$\dot{Z} = \mathbf{V}_1^T\frac{\partial \mathbf{Q}_2}{\partial t}\mathbf{\Omega}_{21} - m_2 g\,[0,\ 0,\ 1]\,(\mathbf{A}_1\frac{\partial \mathbf{A}_{\alpha\beta}}{\partial t}\mathbf{W}_{s2}) \,. \tag{59}$$

The calculations have been carried out for a crane characterised by the following data: $m_1 = 16000$ $[kg]$, $m_2 = 1900$ $[kg]$, $m_3 = 3900$ $[kg]$, $m_4 = 3500$ $[kg]$, $J_1 = 62000$ $[kgm^2]$, $J_2 = 1200$ $[kgm^2]$, $J_3 = 100000$ $[kgm^2]$.

Four computation variants have been performed for different integration time steps, whereas other parameters have remained constant. The results are illustrated in Fig. 4.

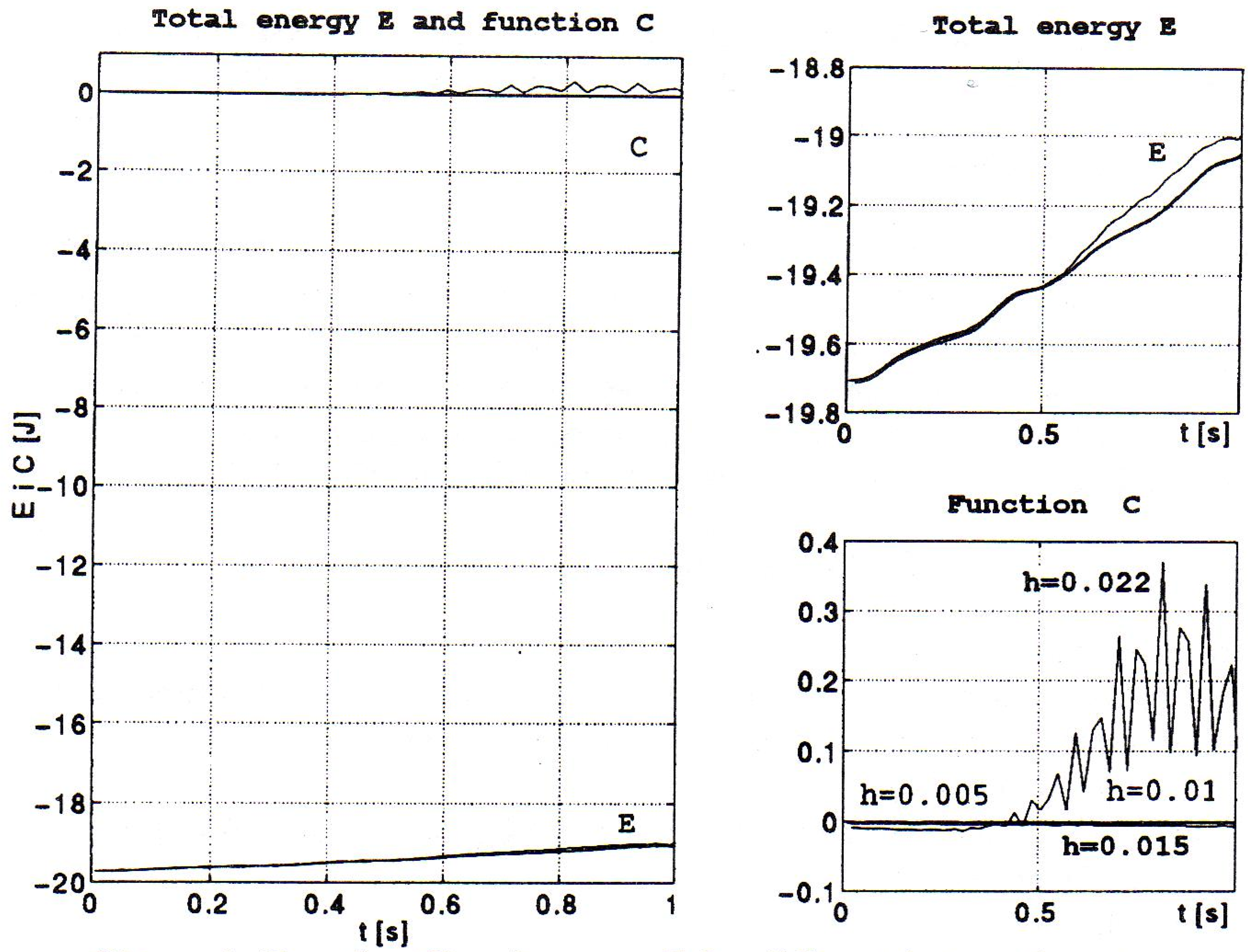

Figure 4: Function C and energy E for different integration steps

3. Model of the load – container

In the next example, the correctness of equations of motion and their numerical solutions was tested for a load - container motion. In this example, a model made of a container with the dimensions $a \times b \times c$, hung on the inextensible rope of the length L, was developed. In order to generate equations of motion, Lagrange's equations of the second kind with multipliers were used. To describe the position of the system, three vectors of the co-ordinates were introduced, namely:

$\mathbf{q}_1^T = [x_B \ \ y_B \ \ z_B]$ – vector of the position of the point in which the load is conected with the rope,

$\mathbf{q}_2^T = [\psi \ \ \vartheta \ \ \varphi]$ – vector of the rotation of the load around the axes $X_1Y_1Z_1$,

$\mathbf{q}_3^T = [x_A \ \ y_A \ \ z_A]$ – vector of the position of the movable end of the rope.

The rope of a constant length constitutes the constraints imposed on the motion of the rigid body. This condition can be presented analytically by the relation

$$(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2 - l^2 = 0 \ . \tag{60}$$

The equation of constraints have been reduced to the kinematic form (much more convenient for the further calculation procedure)

$$\psi(\mathbf{q}, \dot{\mathbf{q}}, t) = (\dot{x}_B - \dot{x}_A)(x_B - x_A) + (\dot{y}_B - \dot{y}_A)(y_B - y_A) +$$

$$+ (\dot{z}_B - \dot{z}_A)(z_B - z_A) = 0 \ . \tag{61}$$

The vector of the equations of constraints $\psi = \mathbf{G}\dot{\mathbf{q}} + \mathbf{g}$ has got one component ($\psi = [\psi_1]$). The matrix occurring in further calculations has the form

$$\mathbf{G} = [(x_B - x_A) \quad (y_B - y_A) \quad (z_B - z_A) \quad 0 \quad 0 \quad 0]\,. \tag{62}$$

The number of degrees of freedom of the system under consideration is equal to $k = 6 - 1 = 5$, whereas the number of the co-ordinates necessary to determine the position is equal to 6 (is higher than the number of degrees of freedom by the number of equations of kinematic constraints). In order to test the correctness of a numerical solution to the equations of motion obtained on the basis of Lagrange's equations with multipliers, the function $C(t)$ described by the relation

$$C(t) = T_2 - T_0 + V + Z$$

should be calculated.

Numerical computations have been performed for the following numerical data:

- integration step in the Runge-Kutta-Gaer method $h = 0.04\ [s]$,
- mass $m = 5000\ [kg]$,
- moments of inertia of the load: $J_x = 5000\ [kgm^2]$, $J_y = 6250\ [kgm^2]$, $J_z = 10000\ [kgm^2]$,
- rope lenth $l = 8\ [m]$,
- position of the mass centre $\xi_c = 0.05\ [m]$, $\eta_c = 0.05\ [m]$, $\zeta_c = 0.5\ [m]$.

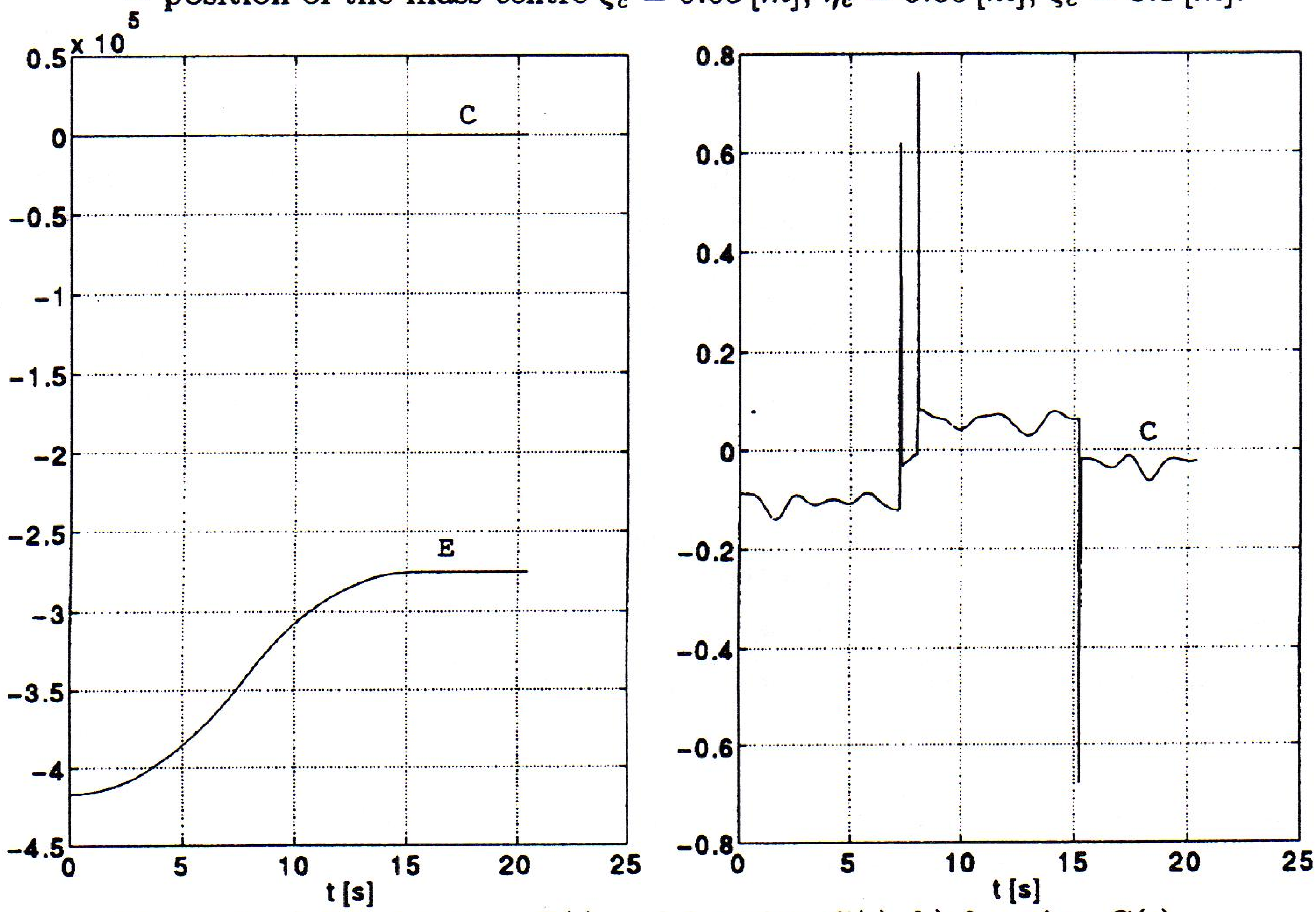

Figure 5: a) Total energy $E(t)$ and function $C(t)$, b) function $C(t)$

The computation results obtained after the system of equations have been solved are presented in Fig.5. If we compare the total energy $E(t)$ and the function $C(t)$ (Fig.5a), we can state that the numerical computations do not include any formal errors. It follows from the time history of the function $C(t)$ (Fig.5b) that in the moments in which a change of the acceleration of the rope end occurs, the accuracy of calculations changes as well.

Further computations have been performed for the case when the rope end (point A) moves along the circle (jib of the length $L = 12$ $[m]$ rotates with a constant angular velocity $\omega = 0.15$ $[\frac{rd}{s}]$). The computation results are presented in Fig. 6. The compared amounts of the total energy $E(t)$ and the function $C(t)$ (Fig.6a) show that the numerical computations are correct (although one can see clearly that a continuos increase in the absolute value of C occurs on the time history of the function $C(t)$ throughout the time interval under analysis (Fig. 6b).

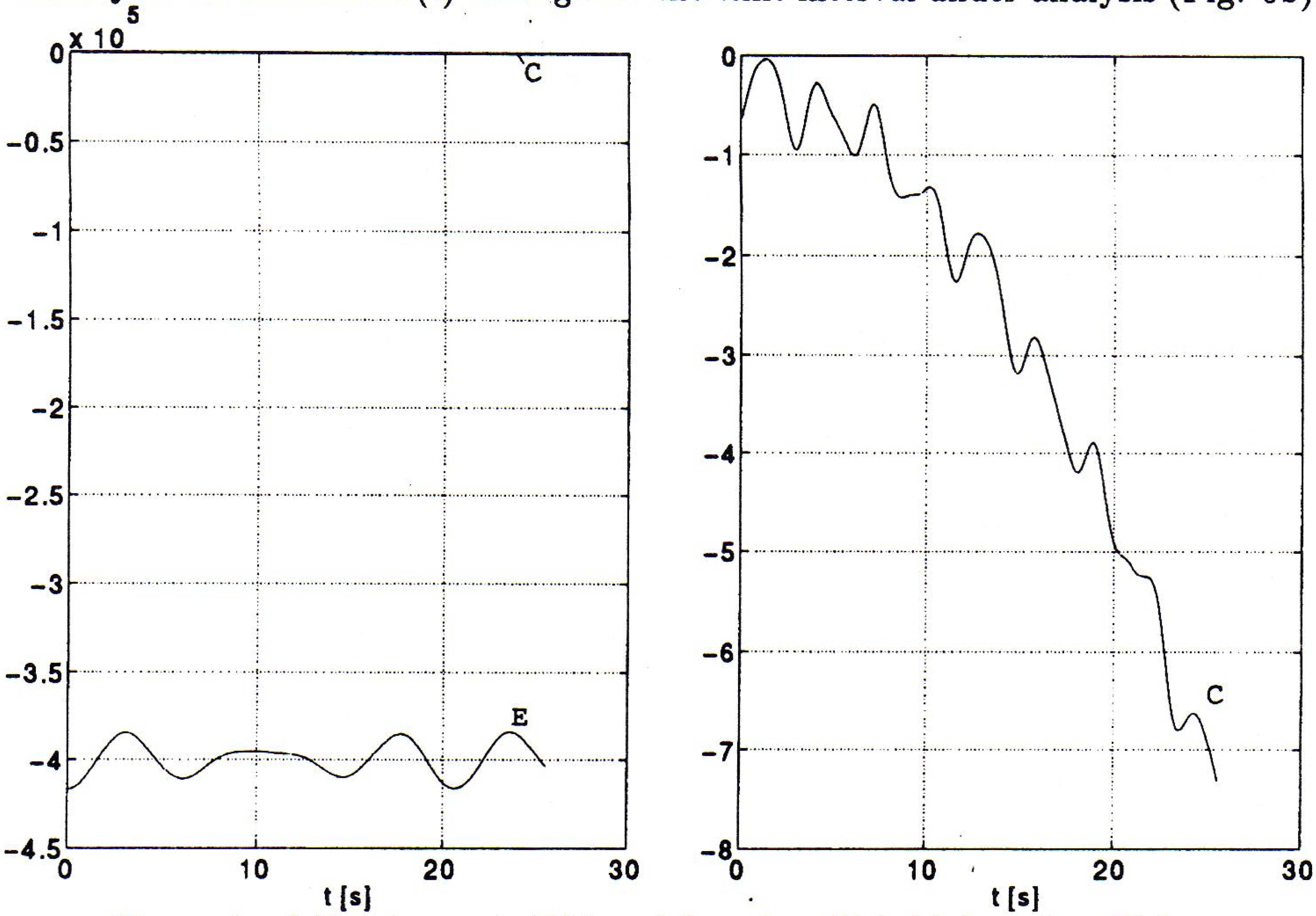

Figure 6: a) Total energy $E(t)$ and function $C(t)$, b) function $C(t)$

A complete series of computations with different integration time steps ($h = 0.01 \div 0.1$ $[s]$) have been performed, whereas other calculation parameters have remained constant. Additionally, calculations for wrong equations of motion (a sign of one of the terms describing the derivative of the potential energy has been changed) have been performed as well. It has turned out that an influence (on φ and $\dot{\varphi}$) of the term with a changed sign is insignificant in the time interval under analysis and for the numerical data assumed (Fig. 7).

In Figure 8a, the diagrams of the energy $E(t)$ and the function $C(t)$ (multiplied

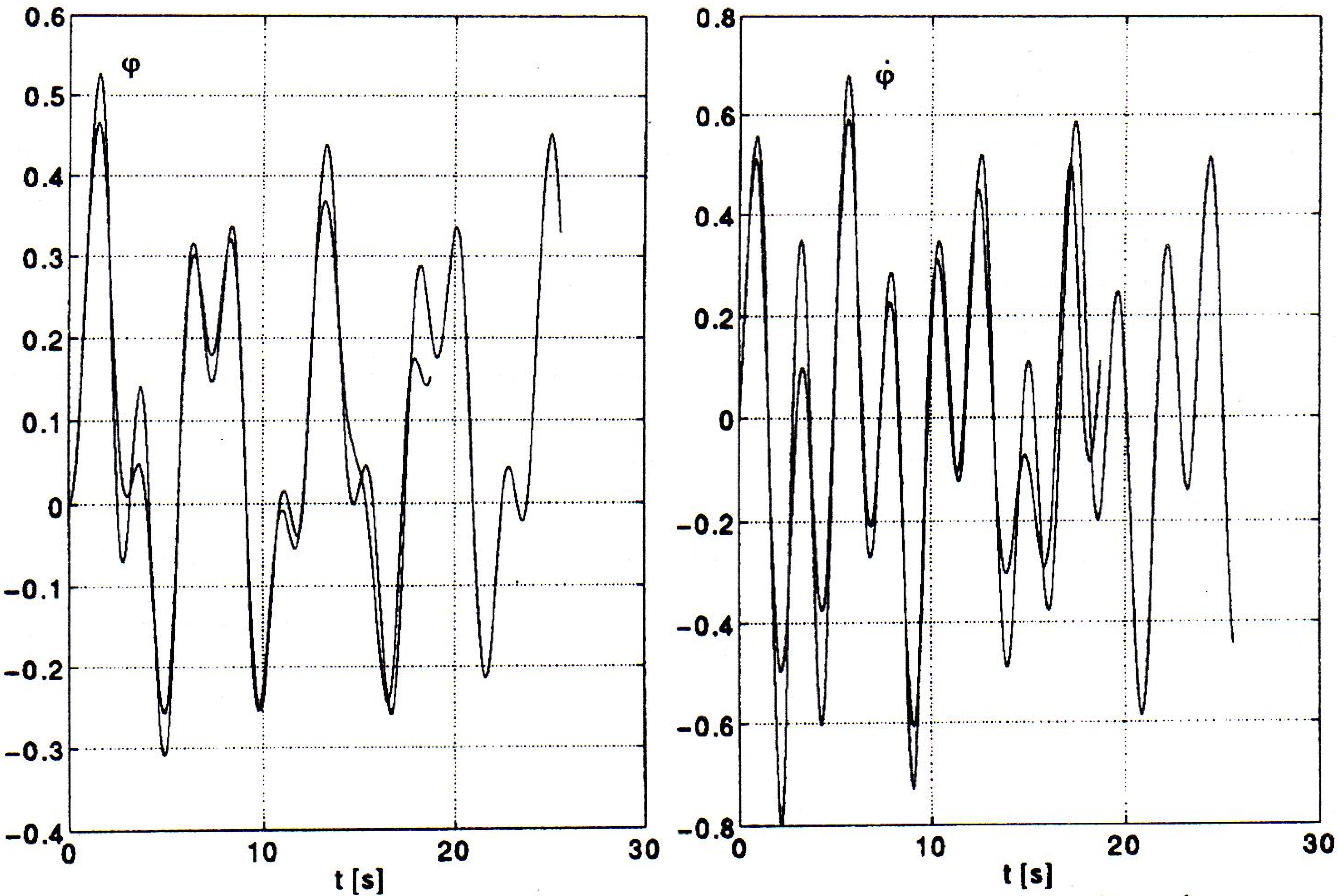

Figure 7: Functions φ and $\dot{\varphi}$ for correct and wrong equations of motion

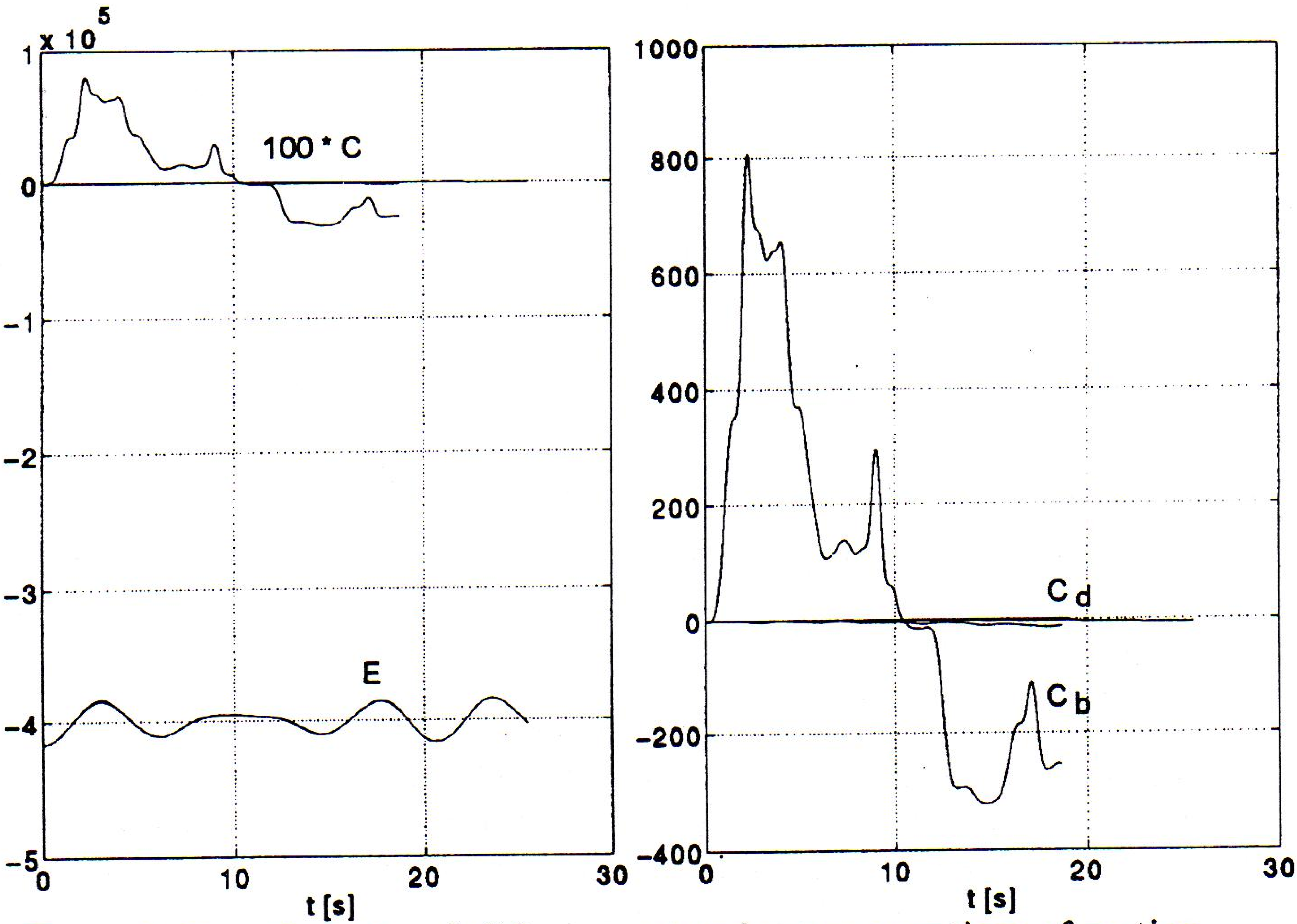

Figure 8: Functions E and C for correct and wrong equations of motion

by 100 in order to improve the legibility of the diagram) have been compared. The diagram of the function $C(t)$ obtained for wrong equations of motion differs distinctly from the remaining ones, the calculated value $\delta_C = 1.94\ 10^{-3}$ shows that there was an error in the calculations. In the case of correct equations, δC has a significantly smaller value (~ 3000 times) and is equal, depending on the integration time step, to $\delta_C \in (10^{-6},\ 3.5\ 10^{-5})$. The differences between time histories of the function $C(t)$ for correct and wrong calculations are shown in Fig. 8b.

Conclusions

The problem discussed in the present work is of great practical importance. A proper development of a mathematical model is a basis for designing, for establishing optimal operating conditions and for designing a control system. Owing to complexity of real objects, it is very difficult to develop an optimal mathematical model. Such a model must be verified.

The first step is to eliminate formal and random errors, in which the method presented here can be very helpful. The examples shown provide evidence that "insignificant" errors and simplifications do not exert any visible and noticeable influence on the results in certain calculation ranges during the analysis of results (even experimental verification, limited to a certain range of parameters, can be insufficient). However, if the calculation parameters are changed, these errors can exert not only a quantitative but also qualitative influence on the behaviour of a model.

Engineering machines, being typical examples of mechanical systems, most often consist of four basic kinds of elements: mass elements, elastic elements, damping elements and sources of energy. An exchange of energy, which is inseparably accompanied by energy dissipation in damping elements, occurs during operation of mechanical systems. It consists in conversion of mechanical energy into the thermal one and in its transfer to the surroundings. An inflow of the energy from the sources of energy and through the nonstationary constraints imposed on the system is also possible. It means that the consistence of the energy balance - taking into account the above mentioned phenomena - can testify the correctness of the calculations which have been carried out.

The shown results of theoretical considerations - concerning the investigations of the correctness of solutions - and of numerical simulation of machine models allow one to state that the proposed methodology of calculations is very efficient. Conservative and nonconservative systems have been investigated. Lagrange's equations of the second kind, and, in the case of kinematic constraints, also Lagrange's equations of the second kind with multipliers or Maggi's equations have been used to develop a mathematical model.

The presented results illustrate possibilities of application of the approach discussed in this study. It is possible to trace a motion of an arbitrary model of the

Table 1: Comparison of the δ_C values

Item	Figure	Value of δ_C
1	2a, 3a	$9.4\ 10^{-2}$
2	4	$1.9\ 10^{-2}$
3	8	$1.94\ 10^{-3}$
4	4	$5.1\ 10^{-4}$
5	4	$1.3\ 10^{-4}$
6	2b, 3b	$1.1\ 10^{-4}$
7	6	$1.7\ 10^{-5}$
8	5	$1.9\ 10^{-6}$

engineering machine in emergency situations when permissible loads are exceeded and cause large displacements of the base. The method of testing the correctness has a global character - the whole calculation process, starting from the verification of equations of the model motion, through checking the algorithm and computer code, and finally a selection of the integration time step, is controlled. An application of the control function in calculations yields a number of practical benefits. While a mathematical model is being developed, it allows for verification of equations and solving procedures. Moreover, it enables diagnosis and identification of errors. While numerical simulation is being carried out, it allows one to choose a proper integration time step. It also makes it possible to perform calculations with presumed accuracy. The introduced measure of error δ_C defines a level of calculation accuracy.

The values of the coefficient δ_C obtained during the calculations have been collected in Table 6.1. On the basis of the results included in this table, it can be assumed that in order to arrive at a correct mathematical model, this coefficient value has to fulfil the following condition: $\delta_C \leq 1.0\ 10^{-3}$.

The suggested accuracy criterion does not have a firm character and does not depend on the method the equations of motion are obtained. The final evaluation of the correctness of computations based on it depends on the person who performs the calculations or on the person who commissions them. Depending on an application, i.e. on the degree of accuracy of operation of the device being designed, its reliability or dangers caused by not enough precise analysis of dynamic conditions, the admissible measure of error of calculations can be decreased (which yields higher accuracy of calculations). The final decision whether to accept the dynamic computations carried out during the designing process or not is to be made by a man; it is crucial to provide him/her with a proper tool which allows him/her to evaluate the results rationally (a virtual prototype of the machine should be reliable).

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