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STOCHASTIC PARAMETRIC INSTABILITY OF THIN CYLINDRICAL SHELL USING VARIOUS SHELL THEORIES

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Abstract

The dynamic stability problem is solved for thin cylindrical shells compressed by time-dependent deterministic or stochastic membrane forces using three common thin shell theories, namely Donnell's, Love's and Flügge's shell theories. The asymptotic stability and almost sure asymptotic stability criteria involving a damping coefficient and loading parameters are derived using Liapunov's direct method. The present paper compares effects of the use of the various shell theories on the dynamic stability regions.

Introduction

Dynamics of laminated composites have been an object of considerable attention over the past quarter of century. Numerous papers are available on laminated plates and shells under constant and periodic forces. The first analysis of the parametric instability of cylindrical shells subjected to periodic deterministic membrane forces with a constant frequency is due to to Bolotin [3]. Birman [2] published a study on the dynamic stability of unsymmetrically laminated rectangular plates subjected to in-plane harmonic forces using a single model approach of transverse displacement. Instability regions as functions of the load amplitude and frequency were obtained analyzing the Mathieu differential equation. All papers

have applied finite dimensional or modal approximations in analysis of vibration and stability. The Liapunov direct method is a quite different approach and can be successfully used to analyze continuous systems described by partial differential equations. A significant advantage is offered by the method in that the equations of motion do not have to be solved in order to examine the stability. The Liapunov functional method together with Donnell's shell theory were utilized by the present author to determine the stability regions of antisymmetrically laminated cross-ply cylindrical shell under time-dependent membrane forces (Tylikowski [9], [10]). The stability analysis of structures under time-dependent forces strongly depends on a dissipation energy. The simplest model of viscous damping with constant coefficient was commonly assumed in previous papers despite the fact that there are another more sophisticated theories of energy dissipation (see eg. Siu and Bert [8]) according to which different engineering constant have different dissipative properties.
The literature search showed that a study comparing the instability regions generated using Donnell's, Love's and Flügge's shell theories have been done by Lam and Loy [7] for free vibration of a rotating multi-layered cylindrical shell and by Ng and Lam [8] for the dynamic stability of thin, laminated cross-ply shells under the axial force with a constant and periodic components.

In the present paper three shell theories for the dynamic stability analysis of a thin-walled laminated cylindrical shell are compared. They are the Donnell's, Love's and Flügge's theories for thin cylindrical shells. The plates are destabilized by compressive deterministic or stochastic in-plane forces. The viscous model of external damping with constant coefficient β is assumed. Using the appropriate energy-like Liapunov functional sufficient conditions for the asymptotic stability and the almost sure asymptotic stability of undeflected form of the shell are derived.

Problem Formulation

Let us consider a closed, elastic, cross-ply laminated cylindrical shell of radius a, length ℓ and total thickness h, $a >> h$, $\ell >> h$. The shell consists of an even number of equal thickness orthotropic layers antisymmetrically laminated with respect to its midsurface from both the geometric and the material property standpoint. The Kirchhoff-Love hypothesis on nondeformable normal element is taken into account. The x-axis is taken along a generator, the circumferential are angle is denoted by Θ. The z - axis is directed radially inwards. The components of the displacement of the shell with reference to this coordinate system are denoted by u, v and w in the x, Θ and z directions respectively. The shell consisting of an even number n of elastic orthotropic layers antisymmetrically laminated about its middle surface from both a geometric and a material property standpoint is compressed by the time-dependent force f. The governing partial differential equations are given as [1]

$$N_{x,x} + \frac{1}{a}N_{\Theta x,\Theta} = \rho h u_{,tt} + 2\rho h\beta u_{,t} + \xi_1 f\frac{1}{a}(v_{,x\Theta} + w_{,x}) \tag{1}$$

$$\frac{1}{a}N_{\Theta,\Theta} + N_{x\Theta,x} + \zeta_1 M_{x\Theta,x} + \zeta_2\frac{1}{a}M_{\Theta,\Theta} = \rho h v_{,tt} + 2\rho h\beta v_{,t} + \xi_2 f\frac{1}{a}u_{,x\Theta} \tag{2}$$

$$\frac{1}{a^2}M_{\Theta,\Theta\Theta} + \frac{1}{a}M_{x\Theta,x\Theta} + \frac{1}{a}M_{\Theta x,x\Theta} + M_{x,xx} - \frac{1}{a}N_\Theta = \rho h w_{,tt} + 2\rho h\beta w_{,t} + f w_{,xx}$$
$$(x,\Theta) \in \Omega \equiv (0,\ell) \times (0,2\pi) \tag{3}$$

where ξ_1, ξ_2 are tracers employed to unify the equation of motion from the three shell theories and ρ is the mean density of plate material. Membrane forces and moments are expressed by the displacements as follows [1]

$$N_x = \big(A_{11} + \delta_5\frac{B_{11}}{a}\big)u_{,x} + \frac{A_{12}}{a}\big(v_{,\Theta} + w\big) - \big(B_{11} + \delta_5\frac{D_{11}}{a}\big)w_{,xx} +$$
$$+\delta_5\frac{D_{12}}{a^3}\big(w_{,\Theta\Theta} - \delta_1 v_{,\Theta} + \delta_2 w\big) \tag{4}$$

$$N_\Theta = A_{12}u_{,x} + \frac{A_{22}}{a}\big(v_{,\Theta} + w\big) + \frac{B_{11}}{a^2}\big(w_{,\Theta\Theta} - \delta_1 v_{,\Theta} + \delta_2 w\big) \tag{5}$$

$$N_{x\Theta} = A_{66}\big(v_{,x} + \frac{1}{a}u_{,\Theta}\big) + \delta_5\frac{D_{66}}{a^2}\big(- 2w_{,x\Theta} + \delta_3 v_{,x} - \frac{1}{a}\delta_4 u_{,\Theta}\big) \tag{6}$$

$$N_{\Theta x} = A_{66}\big(v_{,x} + \frac{1}{a}u_{,\Theta}\big) \tag{7}$$

$$M_x = \big(B_{11} - \delta_5\frac{D_{11}}{a}\big)u_{,x} - \delta_5\frac{D_{12}}{a^2}\big(v_{,x} + \frac{1}{a}u_{,\Theta}\big) - D_{11}w_{,xx} + -\frac{D_{12}}{a^2}\big(w_{,\Theta\Theta} - \delta_1 v_{,\Theta} + \delta_2 w\big) \tag{8}$$

$$M_\Theta = -\frac{B_{11}}{a}\big(v_{,x} + \frac{1}{a}u_{,\Theta}\big) - D_{12}w_{,xx} - \frac{D_{22}}{a^2}\big(w_{,\Theta\Theta} - \delta_1 v_{,\Theta} + \delta_2 w\big) \tag{9}$$

$$M_{x\Theta} = \frac{D_{66}}{a}\big(- 2w_{,x\Theta} + \delta_3 v_{,x} - \frac{1}{a}\delta_4 u_{,\Theta}\big) + \delta_5\frac{D_{66}}{a}\big(v_{,\Theta} + w\big) \tag{10}$$

$$M_{x\Theta} = \frac{D_{66}}{a}\big(- 2w_{,x\Theta} + \delta_3 v_{,x} - \frac{1}{a}\delta_4 u_{,\Theta}\big) \tag{11}$$

in which the extensional, coupling and bending stiffnesses are denoted respectively by A_{ij}, B_{ij}, D_{ij}. Tracers δ_i, $i = 1, ..., 5$ and ζ_1, ζ_2 used to unify the three shell theories are given in Table 1. It is necessary to notice that due to the antisymmetrical cross-ply configuration $A_{16} = A_{26} = 0$, $B_{16} = B_{26} = B_{66} = B_{12} = 0$, $D_{16} = D_{26} = 0$.

The reduced in-plane stiffnesses $\bar{Q}_{ij}$ of an individual lamina can be calculated using the lamina principal material properties E_1, E_2, G_{12}, ν_{12} and the lamination angle 0, $\pi/2$ [1].
The shell is assumed to be simply supported along edges $x = 0, \ell$. The conditions imposed on displacements and forces for simply supported movable in the tangential direction to the edge of the shell have the form

$$
\begin{aligned}
w &= 0 \quad m_x = 0 \quad u = 0 \quad n_{xy} = 0 \quad at \quad x = 0, a \\
w &= 0 \quad m_y = 0 \quad v = 0 \quad n_{xy} = 0 \quad at \quad y = 0, b.
\end{aligned}
\tag{12}
$$

We assume that the solution of equations (1) - (3) exists and belongs to an appropriate Hilbert space. The purpose of the present paper is to derive criteria for solving the following problem: will the deviations of shell surface from the unperturbed state (equilibrium state) be sufficiently small in some mathematical sense in the case when membrane forces are time-dependent. The shell dynamically buckles when the membrane forces get so large that the plate does not oscillate about the unperturbed shell state and a new increasing mode of oscillations occurs. To estimate a perturbed solution of equations (1) - (3) we introduce a measure of distance $\| \, . \, \|$ of the solution of equations (1) - (3) with nontrivial initial conditions from the trivial one. We shall say that the trivial solution of equations (1) - (3) is almost sure asymptotically stable if a measure of distance between the perturbed solution and the trivial one tends to zero with probability one as time tends to infinity

$$
\mathbf{P}(\lim_{t\to\infty} \| w(.,t) \| = 0) = 1. \tag{13}
$$

In the deterministic case the trivial solution is called asymptotically stable if for all solution of equation (1) - (3) with arbitrary initial conditions a measure of distance between the perturbed solution and the trivial one tends to zero as time tends to infinity

Shell theory	Donnel	Love	Flügge
ξ_1	0	0	1
ξ_2	0	1	1
ζ_1	0	1	1
ζ_2	0	1	1
δ_1	0	1	0
δ_2	0	0	1
δ_3	0	2	1
δ_4	0	0	1
δ_5	0	0	1

Table 1. Values of tracers for Donnell's, Love's and Flügge's shell theories

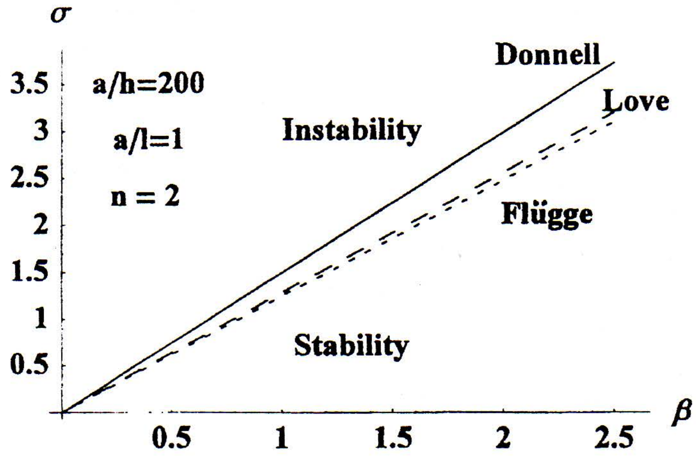

Fig. 1. Stability regions for different shell theories

$$\lim_{t\to\infty} \| w(.,t) \| = 0. \tag{14}$$

In the present analysis the direct Liapunov method is proposed to establish criteria for the asymptotic and the almost sure asymptotic stability of the unperturbed (trivial) solution of antisymmetrically laminated cross-ply shells treated as the infinite - dimensional system subjected to the membrane deterministic and stochastic forces with the known probability distribution.

Stability Analysis

In order to examine the stability of trivial solution $u = v = w = 0$ we construct the Liapunov functional as a sum of modified kinetic energy and the elastic energy of the shell [9]

$$\mathcal{V} = \frac{1}{2}\int_0^{\ell}\int_0^{2\pi} \Big(u_{,t}^2 + v_{,t}^2 + w_{,t}^2 + 2\beta(uu_{,t} + vv_{,t} + ww_{,t}) + 2\beta^2(u^2 + v^2 + w^2) +$$
$$-M_x w_{,xx} - M_\Theta \frac{1}{a^2}(w_{,\Theta\Theta} - \delta_1 v_{,\Theta} + \delta_2 w) - (M_{x\Theta} + M_{\Theta x})\frac{1}{a}(-2w_{,x\Theta} + \delta_3 v_{,x} - \frac{1}{a}\delta_4 u_{,\Theta}) +$$
$$+ N_x u_{,x} + N_\Theta \frac{1}{a}(v_{,\Theta} + w) + \frac{1}{2}(N_{x\Theta} + N_{\Theta x})(v_{,x} + \frac{1}{a}u_{,\Theta})\Big) a dx d\Theta \tag{15}$$

The functional (15) is called *the best* as it gives the greatest almost sure stability domain in the class of twoparameter functional (γ, δ)

$$\mathcal{V} = \frac{1}{2}\int_0^{\ell}\int_0^{2\pi} (u_{,t}^2 + v_{,t}^2 + w_{,t}^2 + \gamma(uu_{,t} + ...) + \delta(u^2 + v^2 + w^2) - M_x w_{,xx} + ...) d\Omega$$

for the ergodic Gaussian forces.
The functional (15) is positive-definite as the terms of the integrand can be rearranged as a sum of squares, and the measure of distance can be chosen as the square root of functional

$$\| w(.,t) \| = V^{1/2}. \tag{16}$$

If the forces acting in the shell middle surface are stationary and continuous with probability one the classic differentiation rule can be applied to calculate the time-derivative of functional (15)

$$\frac{d\mathcal{V}}{dt} = -2\beta\mathcal{V} + 2\mathcal{U} \tag{17}$$

where an auxiliary functional U is defined as follows

$$U = \frac{1}{2}\int_0^{\ell}\int_0^{2\pi}\Big(2\beta^2(uu_{,t} + vv_{,t} + ww_{,t}) + 2\beta^3(u^2 + v^2 + w^2) +$$

$$+\xi_1(u_{,t} + \beta u)\frac{f}{a}(v_{,x\Theta} + w_{,x}) + \xi_2(v_{,t} + \beta v)\frac{f}{a}u_{,x\Theta} + (w_{,t} + \beta w)fw_{,xx}\Big)d\Omega. \tag{18}$$

In order to find a function λ satisfying inequality

$$U \leq \lambda V \tag{19}$$

we look for a stationary point of functional $U - \lambda V$, which is equivalent to inequality (19) for the second order functionals. Solving the Euler auxiliary variational problem $\delta(U - \lambda V) = 0$ we can find the appropriate function λ. In the case of simply supported edges described by the boundary conditions (12) there exists a solution of equations (1) - (3) in the form of infinite series

$$\begin{aligned} u(x,\Theta,t) &= \sum_{m,n=1}^{\infty} F_{mn}(t)\cos\frac{m\pi}{\ell}x\cos n\Theta \\ v(x,\Theta,t) &= \sum_{m,n=1}^{\infty} G_{mn}(t)\sin\frac{m\pi}{\ell}x\sin n\Theta \\ w(x,\Theta,t) &= \sum_{m,n=1}^{\infty} H_{mn}(t)\sin\frac{m\pi}{\ell}x\cos n\Theta \end{aligned} \tag{20}$$

Therefore, their time-derivatives have the form

$$u_{,t}(x,\Theta,t) = \sum_{m,n=1}^{\infty} R_{mn}(t)\cos\frac{m\pi}{\ell}x\cos n\Theta$$

$$v_{,t}(x,\Theta,t) = \sum_{m,n=1}^{\infty} S_{mn}(t)\sin\frac{m\pi}{\ell}x\sin n\Theta \tag{21}$$

$$w_{,t}(x,\Theta,t) = \sum_{m,n=1}^{\infty} T_{mn}(t)\sin\frac{m\pi}{\ell}x\cos n\Theta$$

where the infinite sequence of functions F_{mn}, G_{mn}, H_{mn}, R_{mn}, S_{mn}, T_{mn}, $m,n = 1,2,...$ is not known.

Due to the orthogonality of series present in equations (20), (21) the value of functionals can be calculated as a sum of the suitable quadratic terms

$$\mathcal{V} = \sum_{m,n=1}^{\infty} \mathcal{V}_{mn} \tag{22}$$

$$\mathcal{U} = \sum_{m,n=1}^{\infty} \mathcal{U}_{mn} \tag{23}$$

where $\mathcal{V}_{mn}$ and $\mathcal{U}_{mn}$ are calculated from formula (15) and (18) for a single term of the expansion, respectively. If λ_{mn}, which satisfies a single term inequality, is known

$$\frac{d\mathcal{V}_{mn}}{dt} + 2\beta\mathcal{V}_{mn} = \mathcal{U}_{mn} \leq \lambda_{mn}\mathcal{V}_{mn} \tag{24}$$

then the following chain of inequalities is true

$$\mathcal{U} = \sum_{m,n=1}^{\infty} \mathcal{U}_{mn} \leq \sum_{m,n=1}^{\infty} \lambda_{mn}\mathcal{V}_{mn} \leq \Big(\max_{m,n=1,2,...} \lambda_{mn}\Big)\mathcal{V} = \lambda\mathcal{V} \tag{25}$$

Therefore, the function λ can be effectively calculated. Substituting the mn-th terms of expansions (20)-(21) into inequality (19) we obtain the second order quadratic inequality with respect to the six variables F_{mn}, G_{mn}, H_{mn}, R_{mn}, S_{mn}, T_{mn}. The inequality solution is equivalent to finding the smallest root of the six-th order algebraic equation.

Using a property of function λ in equality (19) leads to the first order differential inequality the solution of which has the form

$$V(t) \leq V(0)\exp\left[-\left(\beta - \frac{1}{t}\int_0^t \lambda(s)ds\right)t\right]. \tag{26}$$

Therefore, the sufficient criterion of the asymptotic stability has the form

$$\beta \geq \lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda(s) ds. \tag{27}$$

If the process f satisfies an ergodic property guaranteeing the equality of time and assemble averages with probability one the sufficient condition of the almost sure asymptotic stability can be written as follows

$$\beta \geq \mathbf{E}\lambda \tag{28}$$

where $\mathbf{E}$ denotes the mathematical expectation.

Numerical Results

Inequality (28) gives us a possibility to obtain minimal damping coefficients β guaranteeing the asymptotic and the almost sure asymptotic stability called critical damping coefficients. A domain where damping coefficients are greater than the critical damping coefficient is called the stability region or the almost sure asymptotic stability region. The stability regions as functions of loading variance, damping coefficient, plate aspect ratio, lamination angle, constant component of in-plane forces and properties of plate material are calculated numerically using an approximate method. First, discrete values of force f are chosen, the sixth order algebraic equation is solved, the largest value λ is determined and the expectation is calculated numerically integrating the product of λ by the probability density function of loading. This is accomplished for various values of parameters by choosing the variance and varying the damping coefficient until inequality (28) will be satisfied. Numerical calculations are performed for the Gaussian process with the mean value of force f and variance σ and for the harmonic process with an amplitude A. In order to compare both processes the variance of harmonic process $\sigma^2 = A^2/2$ is used.
The almost sure asymptotic stability regions of cylindrical two- layered cross-ply shell , unidirectionally loaded by the Gaussian process for the graphite-epoxy material are shown in Fig.1. It is seen that stability regions situated under curves and calculated using Donnell's, Love's and Flügge's theory differ substantially.

Conclusions

The dynamic stability of thin, laminated cylindrical shells under time-dependent deterministic or stochastic axial force uniformly distributed along the shell edge has been examined using three different shell theories - Donnell's, Love's and Flügge's. Using the partial differential equations of motion, the direct Liapunov method without finite dimensional or modal approximations the stability regions have been obtained. It has been observed that the stabilty regions obtained by Donnel's theory are significantly different from those obtained from Love's and Flügge's theories which show a rough egreement. The stability regions do not change qualitatively in going from the Gaussian process to the harmonic one, but the Gaussian loading needs the greater damping coefficient that the harmonic loading.

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References

1. Ambarcumian, S. A., Theory of Anisotropic Shells, Fizmatgiz, Moscow, 1961.
2. Birman, V., Dynamic Stability of Unsymmetrically Laminated Rectangular Plates, Mech. Res. Commun., **12**, 1985, 81-86.
3. Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden Day, San Francisco, 1964.
4. Donell, L. H., Beams, Plates and Shells, Mc Graw Hill, N.Y., 1976.
5. Flügge, W., Statik und Dynamik der Schallen, Julius Springer, Berlin, 1934.
6. Lam, K. Y., Loy, C. T., Analysis of Rotating Laminated Cylindrical Shells by Different Thin Shell Theories, Journal of Sound and Vibration, **186**, 1995, 23-35.
7. Ng, T. Y., Lam, K. Y., Dynamic Stability Analysis of Cross-Ply Laminated Cylindrical Shells Using Different Thin Shell Theories, Acta Mechanica, **132**, 1999, 147-167.
8. Siu, C. C., Bert, C. W., Sinusoidal Response of Composite-Material Plates with Material Damping, ASME J. Engng. Ind., **96**, 1974, 603-610.
9. Tylikowski, A., Dynamic Stability of Antisymmetricall Laminated Cross-Ply Cylindrical Shells, Mechanika Teoretyczna i Stosowana, **26**, 1988, 655-661.
10. Tylikowski, A., Dynamic Stability of Viscoelastic Shells Under Time-Dependent Membrane Loads, **31**, 1989, 591-597.