MECHANICS AND MECHANICAL ENGINEERING
International Journal
Vol. 4 No. 2 2000

FREE VIBRATIONS OF A STEPPED BEAM WITH TWO UNIFORM AND/OR TAPERED PARTS

S.S. El-Din, A.A. Mahmoud, M.A. Nassar
Eng. Math. & Physics Dept.
Fac. Of Eng.
Cairo Univ., Giza, Egypt

ABSTRACT

Closed form solutions for free vibrations of a stepped beam of two parts are given. Each part is of rectangular cross-sectional area. The parts may be of uniform cross-section and/or tapered with both equal and different tapered ratio in the horizontal and vertical planes. General constraints at the ends are possible at the three ends of the beam. The equations of motion of the beam are given in terms of trigonometric functions, hyperbolic functions, and the well known Bessel functions. Various special cases are deduced from the present solution and showed complete agreement with previous closed form solutions for these special cases.

KEYWORDS

Free-vibrations-beams; Vibrations-tapered-beams; Vibrations-stepped-beams.

1. INTRODUCTION

Several works dealing with the transverse vibrations of beams with uniform and tapered cross-sections have been published proposing both approximate and closed types of approaches.

With the solution based on Bessels functions, Lee [1] dealt with a cantilever with a mass at one end. The solution of the equations of motion are given in terms of Bessel functions and he presented tables of frequencies for combinations of clamped, pinned and free boundary conditions. Sato [2] studied the transverse vibration of tapered beams with linearly varying cross-sectional area in the presence of an axial force acting on the beam. Craver and Jampala [3] studied the vibration of a linearly tapered cantilever beam elastically constrained at an arbitrary position along the length of the beam. The beam has a rectangular cross-section with equal taper in the horizontal and vertical planes and the constraint is a translational spring. The natural frequencies were given in tabular form. Auciello [4] studied the free vibrations of tapered beams of rectangular cross sections. The study is extended to beams made up of two sections with different cross-sectional variations. El-Din *et al.*[5] presented a closed form solutions for the transverse vibrations of a uniform beam with a tapered cantilever. Sanger [6] considered a class of non-uniform beams, the geometry of which made it possible to express the solution in terms of Bessel functions of order n. For the types of non-uniform beams which do not admit solution in terms of Bessel functions, Wange [7] proposed a solution based on hypergeometric functions for the transverse vibrations of a class of non-uniform beams. A direct solution based on the method of Frobenius was suggested by Naguleswaran [8].

Several numerical solutions have been published, mainly for cantilevers with a linear taper. Among these are those of Rao [9] (Galerkin), Carnegie and Thomas [10] (finite difference), Krynicki and Mazurliewicz [11] (Rayleigh-Ritz), Klein [12] (Rayleigh-Ritz/finite element), Kim and Dickinson [13] (finite element), and Lee *et al.* [14] (approximating by finite number of step functions).

In this paper, we study the transverse vibrations of a stepped beam of two parts. We consider three main case studies. In the first case, the two parts are uniform. In the second, one part is uniform and the other is tapered. In the third, the parts are tapered. In all cases, a stepped ratio is allowed and the beam is elastically restrained with translational and rotational springs at its three ends. The solutions of the equations of motion are given using trigonometric functions, hyperbolic functions and Bessel functions. The nondimensional frequencies of the beam are given in terms of the cross-sectional area and the flexural rigidity at the first end of the beam. Various special cases are given and compared with the solutions of these special cases given in previous works.

2. PROBLEM FORMULATION

Figure1 illustrates the problem being considered. βL is the length of part (1), $(1-\beta)L$ is the length of part (2), whereas L is the length of the combined beam.

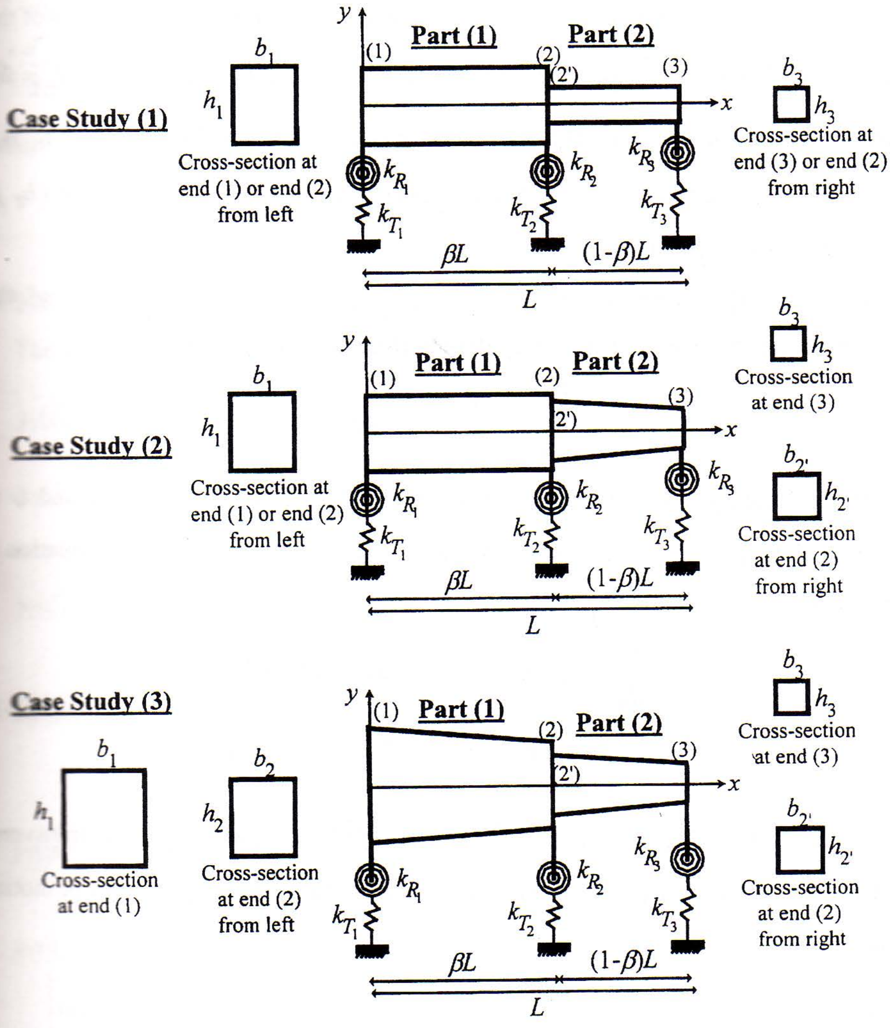

Fig. 1: The Problem being considered

k_{T_i} $(i = 1,2,3)$ and k_{R_i} are the constants of translational and rotational springs, respectively, at the end (i). The beam is of rectangular cross-section of width h_i $(i = 1,3)$ and breadth b_i $(i = 1,3)$ at end (i). h_2 and b_2 are the width and the breadth of the beam at end (2) from left whereas $h_{2'}$ and $b_{2'}$ are the width and breadth of the beam at end (2) from right (2'). $A_i = b_i h_i$ $(i = 1,2,2',3)$ is the cross-sectional area of the beam at end (i) and $I_i = b_i h_i^3 / 12$ is the moment of inertia at this end. $D_i = EI_i$ is the flexural rigidity of the beam at end (i) where E is Young's modulus. $\gamma = \sqrt{h_1 / h_{2'}}$ is a ***width*** ratio for the beam, $\alpha_1 = h_2 / h_1$ is the ***tapered*** ratio of part (1) and $\alpha_2 = h_3 / h_{2'}$ is the ***tapered*** ratio of part (2).

The differential equation for small amplitude, free bending vibrations of the constrained beam shown in Fig.1, according to Bernoulli-Euler theory, is ;

$$\frac{d^2}{dx^2}\left(D\frac{d^2 y}{dx^2}\right) - \rho\omega^2 A y = 0$$

where y is the transverse displacement, x is abscissa, ρ is the mass density, and ω is the natural frequency of vibration. For case study (1), this differential equation is equivalent to :

$$\left.\begin{aligned} &\frac{d^4 y_1}{d\zeta_1^4} - p^4 y_1 = 0 && \text{for } \zeta_1 \in [0, \beta] && \text{(a)} \\ &\frac{d^4 y_2}{d\zeta_2^4} - \gamma^4 p^4 y_2 = 0 && \text{for } \zeta_2 \in [0, 1-\beta] && \text{(b)} \end{aligned}\right\} \qquad (1\text{-}1)$$

where $p^4 = \rho L^4 \omega^2 (A_1 / D_1)$, $\zeta_1 = x / L$, $\zeta_2 = (x - \beta L)/L$, y_1 and y_2 are the transverse deflections of the two parts of the beam. For case study (2), the equations of motion take the forms :

$$\left.\begin{aligned} &\frac{d^4 y_1}{d\zeta_1^4} - p^4 y_1 = 0 && \text{for } \zeta_1 \in [0, \beta] && \text{(a)} \\ &\eta_2^4 \frac{d^4 y_2}{d\eta_2^4} + 2(n_2 + 2)\eta_2^3 \frac{d^3 y_2}{d\eta_2^3} + 6 n_2 \eta_2^2 \frac{d^2 y_2}{d\eta_2^2} - q_2^4 y_2 = 0 && \text{for } \eta_2 \in [1, \alpha_2] && \text{(b)} \end{aligned}\right\} \qquad (2\text{-}1)$$

where $\eta_2 = 1 + Q_2(x - \beta L)/L$, $Q_2 = (\alpha_2 - 1)/(1-\beta)$, $q_2 = \gamma p/|Q_2|$ and $n_2 = 1$ if $b_3 = b_{2'}$ whereas $n_2 = 2$ if $b_3/b_{2'} = \alpha_2$. For case study (3), The equations of motion are :

$$\left.\begin{aligned}
&\eta_1^4 \frac{d^4 y_1}{d\eta_1^4} + 2(n_1+2)\eta_1^3 \frac{d^3 y_1}{d\eta_1^3} + 6n_1\eta_1^2 \frac{d^2 y_1}{d\eta_1^2} - q_1^4 y_1 = 0 \qquad \text{for } \eta_1 \in [1,\alpha_1] \qquad \text{(a)}\\
&\eta_2^4 \frac{d^4 y_2}{d\eta_2^4} + 2(n_2+2)\eta_2^3 \frac{d^3 y_2}{d\eta_2^3} + 6n_2\eta_2^2 \frac{d^2 y_2}{d\eta_2^2} - q_2^4 y_2 = 0 \qquad \text{for } \eta_2 \in [1,\alpha_2] \qquad \text{(b)}
\end{aligned}\right\} \text{(3-1)}$$

where $\eta_1 = 1 + Q_1(x/L)$, $Q_1 = (\alpha_1 - 1)/\beta$, $q_1 = p/|Q_1|$ and $n_1 = 1$ if $b_2 = b_1$ whereas $n_1 = 2$ if $b_2/b_1 = \alpha_1$.

3. SOLUTION OF THE GOVERNING DIFFERENTIAL EQUATIONS

The solutions to Eqs. (1-1), (2-1) and (3-1), respectively, can be obtained as :

$$\left.\begin{aligned}
y_1(\zeta_1) &= a_1\cosh(p\zeta_1) + a_2\sinh(p\zeta_1) + a_3\cos(p\zeta_1) + a_4\sin(p\zeta_1), \qquad 0 \le \zeta_1 \le \beta \qquad \text{(a)}\\
y_2(\zeta_2) &= a_5\cosh(\gamma p\zeta_2) + a_6\sinh(\gamma p\zeta_2) + a_7\cos(\gamma p\zeta_2) + a_8\sin(\gamma p\zeta_2) \qquad 0 \le \zeta_2 \le 1-\beta \qquad \text{(b)}
\end{aligned}\right\} \text{(1-2)}$$

$$\left.\begin{aligned}
y_1(\zeta_1) &= a_1\cosh(p\zeta_1) + a_2\sinh(p\zeta_1) + a_3\cos(p\zeta_1) + a_4\sin(p\zeta_1) \qquad 0 \le \zeta_1 \le \beta \qquad \text{(a)}\\
y_2(\eta_2) &= \eta_2^{-n_2/2}\begin{bmatrix} a_5 \mathrm{J}_{n_2}\left(2q_2\sqrt{\eta_2}\right) + a_6 \mathrm{Y}_{n_2}\left(2q_2\sqrt{\eta_2}\right) \\ + a_7 \mathrm{I}_{n_2}\left(2q_2\sqrt{\eta_2}\right) + a_8 \mathrm{K}_{n_2}\left(2q_2\sqrt{\eta_2}\right) \end{bmatrix} \qquad \eta_2 \in [1,\alpha_2] \qquad \text{(b)}
\end{aligned}\right\} \text{(2-2)}$$

$$\left.\begin{aligned}
y_1(\eta_1) &= \eta_1^{-n_1/2}\begin{bmatrix} a_1 \mathrm{J}_{n_1}\left(2q_1\sqrt{\eta_1}\right) + a_2 \mathrm{Y}_{n_1}\left(2q_1\sqrt{\eta_1}\right) \\ + a_3 \mathrm{I}_{n_1}\left(2q_1\sqrt{\eta_1}\right) + a_4 \mathrm{K}_{n_1}\left(2q_1\sqrt{\eta_1}\right) \end{bmatrix} \qquad \eta_1 \in [1,\alpha_1] \qquad \text{(a)}\\
y_2(\eta_2) &= \eta_2^{-n_2/2}\begin{bmatrix} a_5 \mathrm{J}_{n_2}\left(2q_2\sqrt{\eta_2}\right) + a_6 \mathrm{Y}_{n_2}\left(2q_2\sqrt{\eta_2}\right) \\ + a_7 \mathrm{I}_{n_2}\left(2q_2\sqrt{\eta_2}\right) + a_8 \mathrm{K}_{n_2}\left(2q_2\sqrt{\eta_2}\right) \end{bmatrix} \qquad \eta_2 \in [1,\alpha_2] \qquad \text{(b)}
\end{aligned}\right\} \text{(3-2)}$$

where $a_1, \cdots, a_8$ are constants and $\mathrm{J}, \mathrm{Y}, \mathrm{I}, \mathrm{K}$ are Bessel functions. From equations (1-2), (2-2) and (3-2), the rotation dy/dx, the bending moment $M\left(= -Dd^2y/dx^2\right)$ and the shearing force S $(= -dM/dx)$ for the three cases illustrated in Fig.1, can be obtained.

The boundary conditions at the three ends of the beam together with the continuity conditions at the intermediate end lead to a set of eight equations in the constants $a_1,\cdots,a_8$ into the form $\left(c_{ij}a_j = 0 \text{ where } i,j = 1,2,\cdots,8\right)$ and the characteristic equation of the problem will be $\left(\det\left|c_{ij}\right| = 0\right)$. This characteristic equation can be solved for the dimensionless frequencies $\{p_m\}$ for different values of $\beta,\gamma,\alpha_1,\alpha_2,n_1,n_2$ and for different boundary conditions. Also, the effects of the translational and rotational spring constants k_{T_i} and k_{R_i} at the three ends of the beam on the frequencies $\{p_m\}$ can also be studied.

Let us consider each case study independently;

Case Study (1) :

Referring to Fig.1 and Eqs. (1-2), the boundary conditions at end (1) ($\zeta_1 = 0$) imply that :

$$k_{R_1}\frac{dy_1(0)}{dx} = D_1\frac{d^2y_1(0)}{dx^2} \quad , \quad k_{T_1}y_1(0) = -D_1\frac{d^3y_1(0)}{dx^3}$$

The boundary and continuity conditions at the intermediate end (2) ($\zeta_1 = \beta$, $\zeta_2 = 0$) imply that :

$$y_1(\beta) = y_2(0) \quad , \quad \frac{dy_1(\beta)}{dx} = \frac{dy_2(0)}{dx}$$

$$k_{R_2}\frac{dy_1(\beta)}{dx} = D_1\left[-\frac{d^2y_1(\beta)}{dx^2} + \delta_{2'}\frac{d^2y_2(0)}{dx^2}\right] \quad , \quad k_{T_2}y_1(\beta) = D_1\left[\frac{d^3y_1(\beta)}{dx^3} - \delta_{2'}\frac{d^3y_2(0)}{dx^3}\right]$$

where $\delta_{2'} = D_{2'}/D_1$. Finally, the boundary conditions at end (3) ($\zeta_2 = 1-\beta$) imply that :

$$k_{R_3}\frac{dy_2(1-\beta)}{dx} = -\delta_3 D_1\frac{d^2y_2(1-\beta)}{dx^2} \quad , \quad k_{T_3}y_2(1-\beta) = \delta_3 D_1\frac{d^3y_2(1-\beta)}{dx^3}$$

where $\delta_3 = D_3/D_1 = \delta_{2'}$. These boundary and continuity conditions lead to :

$$\begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & U_{23} & U_{24} & 0 & 0 & 0 & 0 \\ U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\ U_{41} & U_{42} & U_{43} & U_{44} & U_{45} & U_{45} & U_{45} & U_{46} \\ U_{51} & U_{52} & U_{53} & U_{54} & U_{55} & U_{56} & U_{57} & U_{58} \\ U_{61} & U_{62} & U_{63} & U_{64} & U_{65} & U_{66} & U_{67} & U_{68} \\ 0 & 0 & 0 & 0 & U_{75} & U_{76} & U_{77} & U_{78} \\ 0 & 0 & 0 & 0 & U_{85} & U_{86} & U_{87} & U_{88} \end{vmatrix} = 0 \qquad (1\text{-}3a)$$

or equivalently;

$$\begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & U_{23} & U_{24} & 0 & 0 & 0 & 0 \\ U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} & U_{37} & U_{38} \\ U_{41} & U_{42} & U_{43} & U_{44} & U_{45} & U_{45} & U_{45} & U_{46} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & u_{56} & u_{57} & u_{58} \\ u_{61} & u_{62} & u_{63} & u_{64} & u_{65} & u_{66} & u_{67} & u_{68} \\ 0 & 0 & 0 & 0 & U_{75} & U_{76} & U_{77} & U_{78} \\ 0 & 0 & 0 & 0 & U_{85} & U_{86} & U_{87} & U_{88} \end{vmatrix} = 0 \qquad (1\text{-}3b)$$

where

$$\left.\begin{aligned} &U_{11} = -pC_{R_1}, && U_{12} = 1, && U_{13} = pC_{R_1}, && U_{14} = 1 \\ &U_{21} = 1, && U_{22} = p^3C_{T_1}, && U_{23} = 1, && U_{24} = -p^3C_{T_1} \end{aligned}\right\} \qquad (1\text{-}4a)$$

$$\left.\begin{aligned} &U_{31} = \cosh p\beta, && U_{32} = \sinh p\beta \\ &U_{33} = \cos p\beta, && U_{34} = \sin p\beta \\ &U_{41} = \sinh p\beta, && U_{42} = \cosh p\beta \\ &U_{43} = -\sin p\beta, && U_{44} = \cos p\beta \\ &U_{51} = \sinh p\beta + pC_{R_2}\cosh p\beta, && U_{52} = \cosh p\beta + pC_{R_2}\sinh p\beta \\ &U_{53} = -\sin p\beta - pC_{R_2}\cos p\beta, && U_{54} = \cos p\beta - pC_{R_2}\sin p\beta \\ &U_{61} = \cosh p\beta - p^3C_{T_2}\sinh p\beta, && U_{62} = \sinh p\beta - p^3C_{T_2}\cosh p\beta \\ &U_{63} = \cos p\beta - p^3C_{T_2}\sin p\beta, && U_{64} = \sin p\beta + p^3C_{T_2}\cos p\beta \\ &u_{51} = U_{51} - U_{41} = pC_{R_2}\cosh p\beta, && u_{52} = U_{52} - U_{42} = pC_{R_2}\sinh p\beta \\ &u_{53} = U_{53} - U_{43} = -pC_{R_2}\cos p\beta, && u_{54} = U_{54} - U_{44} = -pC_{R_2}\sin p\beta \\ &u_{61} = U_{61} - U_{31} = -p^3C_{T_2}\sinh p\beta, && u_{62} = U_{62} - U_{32} = -p^3C_{T_2}\cosh p\beta \\ &u_{63} = U_{63} - U_{33} = -p^3C_{T_2}\sin p\beta, && u_{64} = U_{64} - U_{34} = p^3C_{T_2}\cos p\beta \end{aligned}\right\} \qquad (1\text{-}4b)$$

$$\left.\begin{array}{llll}
U_{35}=-1 & , U_{36}=0 & , U_{37}=-1 & , U_{38}=0 \\
U_{45}=0 & , U_{46}=-\gamma & , U_{47}=0 & , U_{48}=-\gamma \\
U_{55}=-\gamma^2 p\delta_{2'}C_{R_2} & , U_{56}=0 & & \\
U_{57}=\gamma^2 p\delta_{2'}C_{R_2} & , U_{58}=0 & & \\
U_{65}=0 & , U_{66}=\gamma^3 p^3\delta_{2'}C_{T_2} & & \\
U_{67}=0 & , U_{68}=-\gamma^3 p^3\delta_{2'}C_{T_2} & & \\
u_{55}=U_{55}-U_{45}=-\gamma^2 p\delta_{2'}C_{R_2} & , u_{56}=U_{56}-U_{46}=\gamma & & \\
u_{57}=U_{57}-U_{47}=\gamma^2 p\delta_{2'}C_{R_2} & , u_{58}=U_{58}-U_{48}=\gamma & & \\
u_{65}=U_{65}-U_{35}=1 & , u_{66}=U_{66}-U_{36}=\gamma^3 p^3\delta_{2'}C_{T_2} & & \\
u_{67}=U_{67}-U_{37}=1 & , u_{68}=U_{68}-U_{38}=-\gamma^3 p^3\delta_{2'}C_{T_2} & &
\end{array}\right\} \quad (1\text{-}4c)$$

$$\left.\begin{array}{l}
U_{75}=\sinh\gamma p(1-\beta)+\gamma p\delta_3 C_{R_3}\cosh\gamma p(1-\beta) \\
U_{76}=\cosh\gamma p(1-\beta)+\gamma p\delta_3 C_{R_3}\sinh\gamma p(1-\beta) \\
U_{77}=-\sin\gamma p(1-\beta)-\gamma p\delta_3 C_{R_3}\cos\gamma p(1-\beta) \\
U_{78}=\cos\gamma p(1-\beta)-\gamma p\delta_3 C_{R_3}\sin\gamma p(1-\beta) \\
U_{85}=\cosh\gamma p(1-\beta)-\gamma^3 p^3\delta_3 C_{T_3}\sinh\gamma p(1-\beta) \\
U_{86}=\sinh\gamma p(1-\beta)-\gamma^3 p^3\delta_3 C_{T_3}\cosh\gamma p(1-\beta) \\
U_{87}=\cos\gamma p(1-\beta)-\gamma^3 p^3\delta_3 C_{T_3}\sin\gamma p(1-\beta) \\
U_{88}=\sin\gamma p(1-\beta)+\gamma^3 p^3\delta_3 C_{T_3}\cos\gamma p(1-\beta)
\end{array}\right\} \quad (1\text{-}4d)$$

and $$C_{R_i}=\frac{D_1}{k_{R_i}L} \quad , \quad C_{T_i}=\frac{D_1}{k_{T_i}L^3} \qquad (i=1,2,3)$$

The first two rows of (1-3a and 1-3b) represent the boundary conditions at end (1). Since the part (1) of the beam is Uniform, U_{ij} are used to denote the elements of these rows. The next four rows (the third to the sixth rows) represent the boundary and continuity conditions at the intermediate end (2). Since the parts on both sides of this end are Uniform, U_{ij} and u_{ij} are also used to denote the elements of these rows. Similarly, the last two rows represent the boundary conditions at end (3) and U_{ij} are used to denote the elements of these rows since part (2) is Uniform. The case of a uniform continuous beam (Fig.2a) is a special case of this case study. In addition, if end (2) is free $\left(C_{R_2} \to \infty \text{ and } C_{T_2} \to \infty\right)$, the beam will be a simple uniform beam (Fig.2b) and the characteristic equation can be reduced to :

$$\begin{vmatrix} -pC_{R_1} & 1 & pC_{R_1} & 1 \\ 1 & p^3C_{T_1} & 1 & -p^3C_{T_1} \\ \sinh p + pC_{R_3}\cosh p & \cosh p + pC_{R_3}\sinh p & -\sin p - pC_{R_3}\cos p & \cos p - pC_{R_3}\sin p \\ \cosh p - p^3C_{T_3}\sinh p & \sinh p - p^3C_{T_3}\cosh p & \cos p - p^3C_{T_3}\sin p & \sin p + p^3C_{T_3}\cos p \end{vmatrix} = 0 \tag{1-5}$$

which is the characteristic equation of a simple uniform beam. Also, the cases : $\beta = 1$ (Fig.2c) and $\beta = 0$ (Fig.2d) can be deduced from Eqs. (1-3), respectively as :

$$\left.\begin{aligned} &\begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ \hline U_{51} & U_{52} & U_{53} & U_{54} \\ U_{61} & U_{62} & U_{63} & U_{64} \end{vmatrix}_{\beta=1} = 0 \quad (c) \quad \text{and} \quad \begin{vmatrix} u_{55} & u_{56} & u_{57} & u_{58} \\ u_{65} & u_{66} & u_{67} & u_{68} \\ \hline U_{75} & U_{76} & U_{77} & U_{78} \\ U_{85} & U_{86} & U_{86} & U_{88} \end{vmatrix}_{\substack{\beta=0,\gamma=1 \\ \delta_{2'},\delta_3=1}} = 0 \quad (d) \end{aligned}\right\} \tag{1-6}$$

which agree with (1-5).

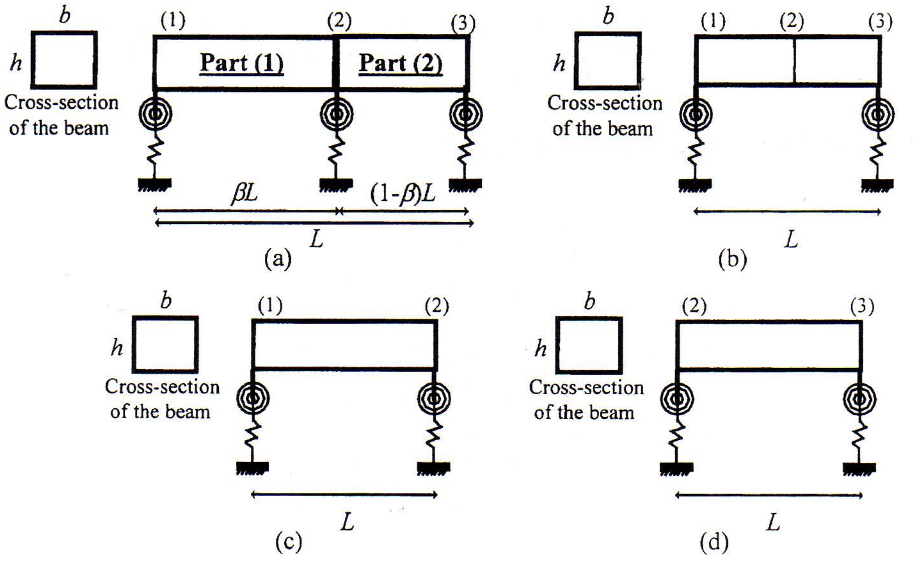

Fig. 2 : Special cases of Case Study (1)

Case Study (2) :

Referring to Fig.1 and Eqs. (2-2), the boundary conditions at end (1) ($\zeta_1 = 0$) imply that :

$$k_{R_1} \frac{dy_1(0)}{dx} = D_1 \frac{d^2 y_1(0)}{dx^2} \quad , \quad k_{T_1} y_1(0) = -D_1 \frac{d^3 y_1(0)}{dx^3}$$

The boundary and continuity conditions at the intermediate end (2) ($\zeta_1 = \beta$, $\eta_2 = 1$) imply that :

$$y_1(\beta) = y_2(1) \quad , \quad \frac{dy_1(\beta)}{dx} = \frac{dy_2(1)}{dx}$$

$$k_{R_2} \frac{dy_1(\beta)}{dx} = D_1 \left[-\frac{d^2 y_1(\beta)}{dx^2} + \delta_{2'} \frac{d^2 y_2(1)}{dx^2} \right]$$

$$k_{T_2} y_1(\beta) = D_1 \left[\frac{d^3 y_1(\beta)}{dx^3} - \left\{ \frac{d^3 y_2(1)}{dx^3} + \left(\frac{1}{L} \right) \delta_{2'} (n_2 + 2) Q_2 \frac{d^2 y_2(1)}{dx^2} \right\} \right]$$

The boundary conditions at end (3) ($\eta_2 = \alpha_2$) imply that :

$$k_{R_3} \frac{dy_2(\alpha_2)}{dx} = -\delta_3 D_1 \frac{d^2 y_2(\alpha_2)}{dx^2}$$

$$k_{T_3} y_2(\alpha_2) = \delta_3 D_1 \left[\frac{d^3 y_2(\alpha_2)}{dx^3} + \left(\frac{1}{L} \right) (n_2 + 2) Q_2 \frac{d^2 y_2(\alpha_2)}{dx^2} \right]$$

These boundary and continuity conditions lead to :

$$\begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & U_{23} & U_{24} & 0 & 0 & 0 & 0 \\ U_{31} & U_{32} & U_{33} & U_{34} & T_{35} & T_{36} & T_{37} & T_{38} \\ U_{41} & U_{42} & U_{43} & U_{44} & T_{45} & T_{45} & T_{45} & T_{46} \\ U_{51} & U_{52} & U_{53} & U_{54} & T_{55} & T_{56} & T_{57} & T_{58} \\ U_{61} & U_{62} & U_{63} & U_{64} & T_{65} & T_{66} & T_{67} & T_{68} \\ 0 & 0 & 0 & 0 & T_{75} & T_{76} & T_{77} & T_{78} \\ 0 & 0 & 0 & 0 & T_{85} & T_{86} & T_{87} & T_{88} \end{vmatrix} = 0 \qquad (2\text{-}3a)$$

or equivalently;

$$\begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & U_{23} & U_{24} & 0 & 0 & 0 & 0 \\ U_{31} & U_{32} & U_{33} & U_{34} & T_{35} & T_{36} & T_{37} & T_{38} \\ U_{41} & U_{42} & U_{43} & U_{44} & T_{45} & T_{45} & T_{45} & T_{46} \\ u_{51} & u_{52} & u_{53} & u_{54} & t_{55} & t_{56} & t_{57} & t_{58} \\ u_{61} & u_{62} & u_{63} & u_{64} & t_{65} & t_{66} & t_{67} & t_{68} \\ 0 & 0 & 0 & 0 & T_{75} & T_{76} & T_{77} & T_{78} \\ 0 & 0 & 0 & 0 & T_{85} & T_{86} & T_{87} & T_{88} \end{vmatrix} = 0 \qquad (2\text{-}3b)$$

where the elements denoted by U_{ij} and u_{ij} are given by (1-4a,b) whereas the elements denoted by T_{ij} and t_{ij} are given as :

$$\left.\begin{aligned}
&T_{35} = -\mathrm{J}_{n_2}(2q_2) \quad , \quad T_{36} = -\mathrm{Y}_{n_2}(2q_2) \\
&T_{37} = -\mathrm{I}_{n_2}(2q_2) \quad , \quad T_{38} = -\mathrm{K}_{n_2}(2q_2) \\
&T_{45} = \gamma \mathrm{J}_{n_2+1}(2q_2) \quad , \quad T_{46} = \gamma \mathrm{Y}_{n_2+1}(2q_2) \\
&T_{47} = -\gamma \mathrm{I}_{n_2+1}(2q_2) \quad , \quad T_{48} = \gamma \mathrm{K}_{n_2+1}(2q_2) \\
&T_{55} = -\gamma^2 p\delta_{2'} C_{R_2} \mathrm{J}_{n_2+2}(2q_2) \quad , \quad T_{56} = -\gamma^2 p\delta_{2'} C_{R_2} \mathrm{Y}_{n_2+2}(2q_2) \\
&T_{57} = -\gamma^2 p\delta_{2'} C_{R_2} \mathrm{I}_{n_2+2}(2q_2) \quad , \quad T_{58} = -\gamma^2 p\delta_{2'} C_{R_2} \mathrm{K}_{n_2+2}(2q_2) \\
&T_{65} = -\gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{J}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{J}_{n_2+2}(2q_2)\right] \\
&T_{66} = -\gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{Y}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{Y}_{n_2+2}(2q_2)\right] \\
&T_{67} = -\gamma^2 p^2 \delta_{2'} C_{T_2} \left[-\gamma p \mathrm{I}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{I}_{n_2+2}(2q_2)\right] \\
&T_{68} = -\gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{K}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{K}_{n_2+2}(2q_2)\right] \\
&t_{55} = T_{55} - T_{45} = -\gamma \mathrm{J}_{n_2+1}(2q_2) - \gamma^2 p\delta_{2'} C_{R_2} \mathrm{J}_{n_2+2}(2q_2) \\
&t_{56} = T_{56} - T_{46} = -\gamma \mathrm{Y}_{n_2+1}(2q_2) - \gamma^2 p\delta_{2'} C_{R_2} \mathrm{Y}_{n_2+2}(2q_2) \\
&t_{57} = T_{57} - T_{47} = \gamma \mathrm{I}_{n_2+1}(2q_2) - \gamma^2 p\delta_{2'} C_{R_2} \mathrm{I}_{n_2+2}(2q_2) \\
&t_{58} = T_{58} - T_{48} = -\gamma \mathrm{K}_{n_2+1}(2q_2) - \gamma^2 p\delta_{2'} C_{R_2} \mathrm{K}_{n_2+2}(2q_2) \\
&t_{65} = T_{65} - T_{35} = \mathrm{J}_{n_2}(2q_2) - \gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{J}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{J}_{n_2+2}(2q_2)\right] \\
&t_{66} = T_{66} - T_{36} = \mathrm{Y}_{n_2}(2q_2) - \gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{Y}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{Y}_{n_2+2}(2q_2)\right] \\
&t_{67} = T_{67} - T_{37} = \mathrm{I}_{n_2}(2q_2) - \gamma^2 p^2 \delta_{2'} C_{T_2} \left[-\gamma p \mathrm{I}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{I}_{n_2+2}(2q_2)\right] \\
&t_{68} = T_{68} - T_{38} = \mathrm{K}_{n_2}(2q_2) - \gamma^2 p^2 \delta_{2'} C_{T_2} \left[\gamma p \mathrm{K}_{n_2+3}(2q_2) - (n_2+2) Q_2 \mathrm{K}_{n_2+2}(2q_2)\right]
\end{aligned}\right\} \qquad (2\text{-}4a)$$

$$\left.\begin{aligned}
T_{75} &= -\sqrt{\alpha_2}\,\mathrm{J}_{n_2+1}(2q_2') + \gamma p\delta_3 C_{R_3}\mathrm{J}_{n_2+2}(2q_2')\\
T_{76} &= -\sqrt{\alpha_2}\,\mathrm{Y}_{n_2+1}(2q_2') + \gamma p\delta_3 C_{R_3}\mathrm{Y}_{n_2+2}(2q_2')\\
T_{77} &= \sqrt{\alpha_2}\,\mathrm{I}_{n_2+1}(2q_2') + \gamma p\delta_3 C_{R_3}\mathrm{I}_{n_2+2}(2q_2')\\
T_{78} &= -\sqrt{\alpha_2}\,\mathrm{K}_{n_2+1}(2q_2') + \gamma p\delta_3 C_{R_3}\mathrm{K}_{n_2+2}(2q_2')\\
T_{85} &= \alpha_2^2\mathrm{J}_{n_2}(2q_2') + \gamma^2p^2\delta_3 C_{T_3}\left[\gamma p\sqrt{\alpha_2}\,\mathrm{J}_{n_2+3}(2q_2') - (n_2+2)Q_2\mathrm{J}_{n_2+2}(2q_2')\right]\\
T_{86} &= \alpha_2^2\mathrm{Y}_{n_2}(2q_2') + \gamma^2p^2\delta_3 C_{T_3}\left[\gamma p\sqrt{\alpha_2}\,\mathrm{Y}_{n_2+3}(2q_2') - (n_2+2)Q_2\mathrm{Y}_{n_2+2}(2q_2')\right]\\
T_{87} &= \alpha_2^2\mathrm{I}_{n_2}(2q_2') + \gamma^2p^2\delta_3 C_{T_3}\left[-\gamma p\sqrt{\alpha_2}\,\mathrm{I}_{n_2+3}(2q_2') - (n_2+2)Q_2\mathrm{I}_{n_2+2}(2q_2')\right]\\
T_{88} &= \alpha_2^2\mathrm{K}_{n_2}(2q_2') + \gamma^2p^2\delta_3 C_{T_3}\left[\gamma p\sqrt{\alpha_2}\,\mathrm{K}_{n_2+3}(2q_2') - (n_2+2)Q_2\mathrm{K}_{n_2+2}(2q_2')\right]
\end{aligned}\right\}\quad (2\text{-}4b)$$

and $q_2' = q_2\sqrt{\alpha_2}$. Again, the last two rows of (2-3a) and (2-3b) represent the boundary conditions at end (3) and their elements are denoted by T_{ij} since part (2) of the beam is Tapered. The intermediate four rows (the third to the sixth rows) represent the boundary and continuity conditions at the intermediate end (2). The elements of the first four columns of these rows are denoted by U_{ij} and/or u_{ij} whereas the elements of the last four columns are denoted by T_{ij} and/or t_{ij} (since the part to the left of this end is Uniform and the part to the right is Tapered). Special cases for this case study are illustrated in Fig.3 .

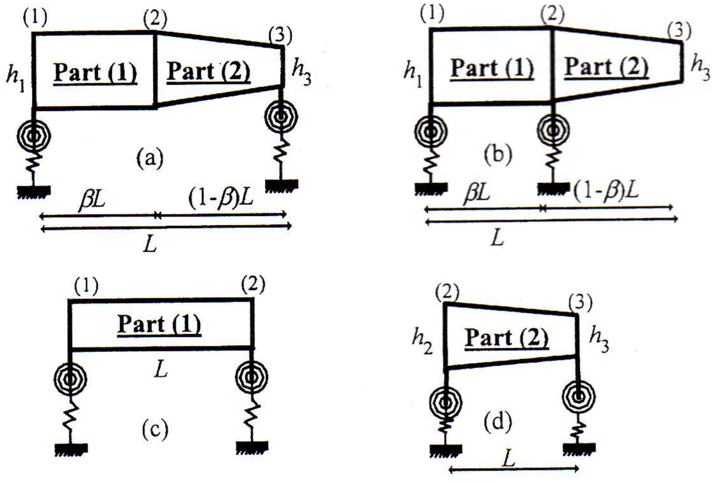

Fig.3 : Special cases of Case Study (2)

Figures (3a,3b) represent the special case in which $\delta_2 = \delta_{2'} = 1$ and $\gamma = 1$. In Fig.3a [4] end (2) is free ($C_{R_2} \to \infty$ and $C_{T_2} \to \infty$). In Fig.3b [5] end (3) is free ($C_{R_3} \to \infty$ and $C_{T_3} \to \infty$). Also, the cases of simple uniform beam (Fig.3c) and simple tapered beam (Fig.3d) can be deduced from the characteristic Eqs. (2-3), respectively, as :

$$\left.\begin{array}{l} \begin{vmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{51} & U_{52} & U_{53} & U_{54} \\ U_{61} & U_{62} & U_{63} & U_{64} \end{vmatrix}_{\substack{\beta=1 \\ \delta_{2'}=1}} = 0 \text{ (c)} \quad \text{and} \quad \begin{vmatrix} t_{55} & t_{56} & t_{57} & t_{58} \\ t_{65} & t_{66} & t_{67} & t_{68} \\ T_{75} & T_{76} & T_{77} & T_{78} \\ T_{85} & T_{86} & T_{86} & T_{88} \end{vmatrix}_{\substack{\beta=0,\,\gamma=1 \\ \delta_{2'},\,\delta_3=1}} = 0 \text{ (d)} \end{array}\right\} \quad (2\text{-}5)$$

It is clear that (2-5c) is the same as (1-6c).

Case Study (3) :

Referring to Fig.1 and Eqs. (3-2), the boundary conditions at end (1) ($\eta_1 = 1$) imply that :

$$k_{R_1}\frac{dy_1(1)}{dx} = D_1\frac{d^2y_1(1)}{dx^2}$$

$$k_{T_1}y_1(1) = -D_1\left[\frac{d^3y_1(1)}{dx^3} + \left(\frac{1}{L}\right)(n_1+2)Q_1\frac{d^2y_1(1)}{dx^2}\right]$$

The boundary and continuity conditions at the intermediate end (2) ($\eta_1 = \alpha_1$, $\eta_2 = 1$) imply that :

$$y_1(\alpha_1) = y_2(1) \quad , \quad \frac{dy_1(\alpha_1)}{dx} = \frac{dy_2(1)}{dx}$$

$$k_{R_2}\frac{dy_1(\alpha_1)}{dx} = D_1\left[-\frac{d^2y_1(\alpha_1)}{dx^2} + \delta_{2'}\frac{d^2y_2(1)}{dx^2}\right]$$

$$k_{T_2}y_1(\alpha_1) = D_1\left[\begin{array}{l} \left\{\dfrac{d^3y_1(\alpha_1)}{dx^3} + \left(\dfrac{1}{L}\right)\delta_2(n_1+2)Q_1\dfrac{d^2y_1(\alpha_1)}{dx^2}\right\} \\ -\left\{\dfrac{d^3y_2(1)}{dx^3} + \left(\dfrac{1}{L}\right)\delta_{2'}(n_2+2)Q_2\dfrac{d^2y_2(1)}{dx^2}\right\} \end{array}\right] \backslash$$

The boundary conditions at end (3) ($\eta_2 = \alpha_2$) imply that :

$$k_{R_3}\frac{dy_2(\alpha_2)}{dx} = -\delta_3 D_1 \frac{d^2 y_2(\alpha_2)}{dx^2}$$

$$k_{T_3} y_2(\alpha_2) = \delta_3 D_1 \left[\frac{d^3 y_2(\alpha_2)}{dx^3} + \left(\frac{1}{L}\right)(n_2+2)Q_2 \frac{d^2 y_2(\alpha_2)}{dx^2}\right]$$

These boundary and continuity conditions lead to :

$$\begin{vmatrix} T_{11} & T_{12} & T_{13} & T_{14} & 0 & 0 & 0 & 0 \\ T_{21} & T_{22} & T_{23} & T_{24} & 0 & 0 & 0 & 0 \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} & T_{37} & T_{38} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{45} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} & T_{57} & T_{58} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} & T_{67} & T_{68} \\ 0 & 0 & 0 & 0 & T_{75} & T_{76} & T_{77} & T_{78} \\ 0 & 0 & 0 & 0 & T_{85} & T_{86} & T_{87} & T_{88} \end{vmatrix} = 0 \qquad (3\text{-}3a)$$

or equivalently;

$$\begin{vmatrix} T_{11} & T_{12} & T_{13} & T_{14} & 0 & 0 & 0 & 0 \\ T_{21} & T_{22} & T_{23} & T_{24} & 0 & 0 & 0 & 0 \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} & T_{37} & T_{38} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{45} & T_{45} & T_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} & t_{57} & t_{58} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} & t_{67} & t_{68} \\ 0 & 0 & 0 & 0 & T_{75} & T_{76} & T_{77} & T_{78} \\ 0 & 0 & 0 & 0 & T_{85} & T_{86} & T_{87} & T_{88} \end{vmatrix} = 0 \qquad (3\text{-}3b)$$

where :

$$\left.\begin{array}{l} T_{11} = -\mathrm{J}_{n_1+1}(2q_1) - pC_{R_1}\mathrm{J}_{n_1+2}(2q_1) \quad , \quad T_{12} = -\mathrm{Y}_{n_1+1}(2q_1) - pC_{R_1}\mathrm{Y}_{n_1+2}(2q_1) \\ T_{13} = \mathrm{I}_{n_1+1}(2q_1) - pC_{R_1}\mathrm{I}_{n_1+2}(2q_1) \quad , \quad T_{14} = -\mathrm{K}_{n_1+1}(2q_1) - pC_{R_1}\mathrm{K}_{n_1+2}(2q_1) \\ T_{21} = \mathrm{J}_{n_1}(2q_1) - p^2 C_{T_1}\left[p\mathrm{J}_{n_1+3}(2q_1) - (n_1+2)Q_1\mathrm{J}_{n_1+2}(2q_1)\right] \\ T_{22} = \mathrm{Y}_{n_1}(2q_1) - p^2 C_{T_1}\left[p\mathrm{Y}_{n_1+3}(2q_1) - (n_1+2)Q_1\mathrm{Y}_{n_1+2}(2q_1)\right] \\ T_{23} = \mathrm{I}_{n_1}(2q_1) - p^2 C_{T_1}\left[-p\mathrm{I}_{n_1+3}(2q_1) - (n_1+2)Q_1\mathrm{I}_{n_1+2}(2q_1)\right] \\ T_{24} = \mathrm{K}_{n_1}(2q_1) - p^2 C_{T_1}\left[p\mathrm{K}_{n_1+3}(2q_1) - (n_1+2)Q_1\mathrm{K}_{n_1+2}(2q_1)\right] \end{array}\right\}$$

(3-4a)

$$\left.\begin{aligned}
T_{31} &= \alpha_1^{-n_1/2}\mathsf{J}_{n_1}(2q_1') \quad , \quad T_{32} = \alpha_1^{-n_1/2}\mathsf{Y}_{n_1}(2q_1') \\
T_{33} &= \alpha_1^{-n_1/2}\mathsf{I}_{n_1}(2q_1') \quad , \quad T_{34} = \alpha_1^{-n_1/2}\mathsf{K}_{n_1}(2q_1') \\
T_{41} &= -\alpha_1^{-(n_1+1)/2}\mathsf{J}_{n_1+1}(2q_1') \quad , \quad T_{42} = -\alpha_1^{-(n_1+1)/2}\mathsf{Y}_{n_1+1}(2q_1') \\
T_{43} &= \alpha_1^{-(n_1+1)/2}\mathsf{I}_{n_1+1}(2q_1') \quad , \quad T_{44} = -\alpha_1^{-(n_1+1)/2}\mathsf{K}_{n_1+1}(2q_1') \\
T_{51} &= \alpha_1^{-(n_1+2)/2}\left[-\sqrt{\alpha_1}\mathsf{J}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{J}_{n_1+2}(2q_1')\right] \\
T_{52} &= \alpha_1^{-(n_1+2)/2}\left[-\sqrt{\alpha_1}\mathsf{Y}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{Y}_{n_1+2}(2q_1')\right] \\
T_{53} &= \alpha_1^{-(n_1+2)/2}\left[\sqrt{\alpha_1}\mathsf{I}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{I}_{n_1+2}(2q_1')\right] \\
T_{54} &= \alpha_1^{-(n_1+2)/2}\left[-\sqrt{\alpha_1}\mathsf{K}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{K}_{n_1+2}(2q_1')\right] \\
T_{61} &= \alpha_1^{-(n_1+4)/2}\left[\alpha_1^2\mathsf{J}_{n_1}(2q_1') + p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{J}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{J}_{n_1+2}(2q_1')\right\}\right] \\
T_{62} &= \alpha_1^{-(n_1+4)/2}\left[\alpha_1^2\mathsf{Y}_{n_1}(2q_1') + p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{Y}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{Y}_{n_1+2}(2q_1')\right\}\right] \\
T_{63} &= \alpha_1^{-(n_1+4)/2}\left[\alpha_1^2\mathsf{I}_{n_1}(2q_1') + p^2\delta_2 C_{T_2}\left\{-p\sqrt{\alpha_1}\mathsf{I}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{I}_{n_1+2}(2q_1')\right\}\right] \\
T_{64} &= \alpha_1^{-(n_1+4)/2}\left[\alpha_1^2\mathsf{K}_{n_1}(2q_1') + p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{K}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{K}_{n_1+2}(2q_1')\right\}\right] \\
t_{51} &= T_{51} - T_{41} = \alpha_1^{-(n_1+2)/2}\left[\left(1-\sqrt{\alpha_1}\right)\mathsf{J}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{J}_{n_1+2}(2q_1')\right] \\
t_{52} &= T_{52} - T_{42} = \alpha_1^{-(n_1+2)/2}\left[\left(1-\sqrt{\alpha_1}\right)\mathsf{Y}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{Y}_{n_1+2}(2q_1')\right] \\
t_{53} &= T_{53} - T_{43} = \alpha_1^{-(n_1+2)/2}\left[-\left(1-\sqrt{\alpha_1}\right)\mathsf{I}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{I}_{n_1+2}(2q_1')\right] \\
t_{54} &= T_{54} - T_{44} = \alpha_1^{-(n_1+2)/2}\left[\left(1-\sqrt{\alpha_1}\right)\mathsf{K}_{n_1+1}(2q_1') + p\delta_2 C_{R_2}\mathsf{K}_{n_1+2}(2q_1')\right] \\
t_{61} &= T_{61} - T_{31} = \alpha_1^{-(n_1+4)/2}\left[p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{J}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{J}_{n_1+2}(2q_1')\right\}\right] \\
t_{62} &= T_{62} - T_{32} = \alpha_1^{-(n_1+4)/2}\left[p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{Y}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{Y}_{n_1+2}(2q_1')\right\}\right] \\
t_{63} &= T_{63} - T_{33} = \alpha_1^{-(n_1+4)/2}\left[p^2\delta_2 C_{T_2}\left\{-p\sqrt{\alpha_1}\mathsf{I}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{I}_{n_1+2}(2q_1')\right\}\right] \\
t_{64} &= T_{64} - T_{34} = \alpha_1^{-(n_1+4)/2}\left[p^2\delta_2 C_{T_2}\left\{p\sqrt{\alpha_1}\mathsf{K}_{n_1+3}(2q_1') - (n_1+2)Q_1\mathsf{K}_{n_1+2}(2q_1')\right\}\right]
\end{aligned}\right\} \tag{3-4b}$$

with $q_1' = q_1\sqrt{\alpha_1}$. The remaining elements denoted by T_{ij} and t_{ij} are given by (2-4a,b).

Special cases for this case study are illustrated in Fig.4. Fig.4b illustrates studied by Craver [3] ($C_{R_1} = 0$, $C_{T_1} = 0$, $C_{R_3} \to \infty$, $C_{T_3} \to \infty$, $C_{R_2} \to \infty$, $C_{T_2} = \delta_2 D_1 / kL^3$). If end (2) is free (Fig.4c : $C_{R_2} \to \infty$, $C_{T_2} \to \infty$), or $\beta = 1$ (Fig.4d), or $\beta = 1$ (Fig.4e), it can be shown, from Eqs. (3-3), that the characteristic equation reduces to that of a simple tapered beam (2-5d).

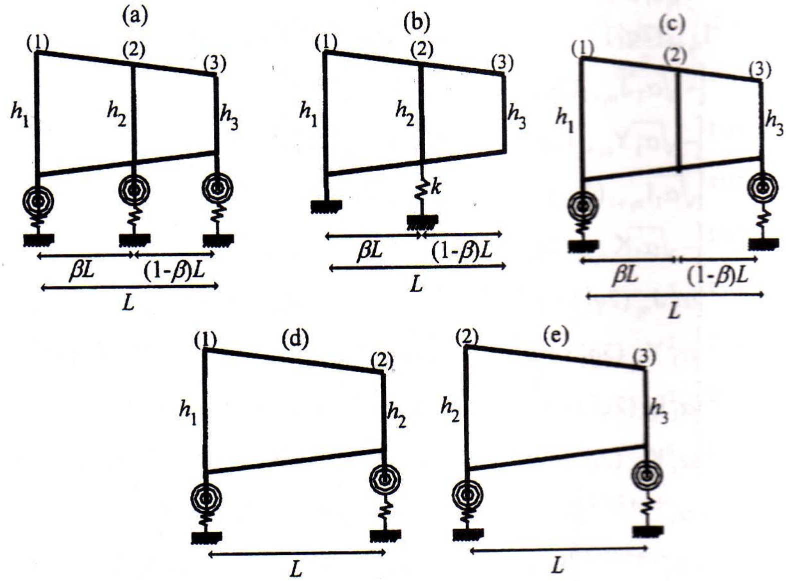

Fig. 4 : Special cases of Case Study (3)

4. CONCLUSION

The characteristic equation for the transverse vibrations of a stepped beam of two parts is given. The solution of this characteristic equation gives the nondimensionless frequencies $\{p_m\}$ of the beam and allows for the study of several parameters on these frequencies (length ratio β, width ratio γ and tapered ratios α_1 and α_2). The present work allows to study the effect of the constraints at the three ends of the beam on the natural frequencies. The nondimensional frequencies of the beam are given in terms of the cross-sectional area and the flexural rigidity at the first end of the beam. The problem of a simple uniform beam and simple tapered beam with/without an intermediate translational/rotational spring can be deduced from the present work. For tapered part, the breadth of beam may be constant or varying linearly with the length of the beam.

REFERENCES

1. Lee, T.W. (1976), "Transverse vibrations of a tapered beam carrying a concentrated mass". Trans. ASME, J. of Applied Mech. (Series E) 43,366-367.
2. Sato, K. (1980), "Transverse vibration of linearly tapered beams with ends restrained elastically against rotation subject to axial force". International Journal of Mechanical Engineering Science 22,109-115.
3. Craver, W.L., Jr. and Jampala, P. (1993), "Transverse vibrations of linearly tapered cantilever beam with constraining spring". Journal of Sound and Vibration 166,521-529.
4. Auciello, N.M. (1996), "Exact solution for the transverse vibration of a beam a part of which is a taper beam and other part and other part is a uniform beam". Journal of Sound and Vibration 187,724-726.
5. El-Din, S.S., El-Gammal, E., Mahmoud, A.A. (2000), "Free vibration of a uniform beam, with tapered cantilever". Submitted to Journal of Engineering and Applied Science, Fac. of Eng., Cairo Univ.
6. Sanger, D.J. (1968), "Transverse vibration of a class of non--uniform beams". Journal of Mechanical Engineering Science 10,111-120.
7. Wange, H.C. (1967), "Generalized hypergeometric function solutions on the transverse vibrations of a class of non-uniform beams". Trans. ASME, J. of Applied Mech. (Series E) 34,702-708.
8. Naguleswaran, S. (1994), "A direct solution for the transverse vibration of Euler-Bernoulli wedge and beams". Journal of Sound and Vibration 172,289-394.
9. Rao, J.S. (1965), "The fundamental flexural vibration of a cantilever beam of rectangular cross-section with uniform taper ". The Aeronautical Quarterly 18,139-144.
10. Carnegie, W. and Thomas, J. (1967), "Natural frequencies of a long tapered cantilever", The Aeronautical Quarterly 18,309-320.
11. Krynicki, E. and Mazurkiewicz, Z. (1962), "Free vibration of a simply supported bar with a linearly varying height of cross-section". Trans. ASME, J. of Applied Mech. (Series E) 29,497-501.
12. Klein, L. (1974), "Transverse vibrations of non--uniform beams". Journal of Sound and Vibration 37,491-505.
13. Kim, C.S. and Dickinson, S.M. (1988), "On the analysis of laterally vibrating slender beams subject to various complicating effects". Journal of Sound and Vibration 122,441-455.
14. Lee, S.Y., Ke, H.Y. and Kuo, Y.H. (1990), "Analysis of non-uniform beams". Journal of Sound and vibration 142,15-29.