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NONLINEAR INTERFACIAL INSTABILITY OF TWO ELECTRIFIED MISCIBLE FLUIDS

Y. D. Mahmoud[1], G.M. Moatimid[2], Y.O. El-Dib[2]
[1] P.O. Box 475, Ismailia 41511, Egypt
[2] Department of Mathematics, Faculty of Education, Ain Shams University, Heliopolis, Cairo, Egypt

Abstract:

In a previous paper [1], a simplified formulation was presented to deal with the interfacial linear stability problem with mass and heat transfer, considering the presence of a periodic electric field. The present paper treats the same problem with a nonlinear approach. This approach is achieved by considering the multiple time scale method. The analysis reveals the existence of both resonant and non – resonant cases. Three types of nonlinear Schödinger equations are derived. The necessary and sufficient stability of conditions is obtained and the results are confirmed numerically. Graphs are drawn to illustrate the stability regions.

1.Introduction:

The instability of the plane interface between two superposed fluids of different densities is called the Rayleigh - Taylor (R - T) instability. The R- T instability was first investigated by Rayleigh [2] and then by Taylor [3]. The Kelvin - Helmholtz (K - H) instability arises when adjacent layers of fluids are in a relative motion. Chandrasekhar [4] gave an introduction to the classical K-H instability. He discussed the effects of surface tension, variable density, streaming velocity, rotation and application of magnetic field on the stability behaviour. The study of electrodydrodynamic (EHD) K-H instability of free surface charges, separating two semi - infinite dielectric streaming fluids and influenced by an electric field, has been discussed by Melcher [5]. The main difference between R - T and K - H is the inclusion of $(\underline{V}.\nabla)\underline{V}$ which is a nonlinear term, in the perturbation equations.

The nonlinear perturbation of stability analysis of the above models was investigated by many authors. Drazin [6] has been investigated the nonlinear developments of K - H instability in electrically nonconducting incompressible fluids. The method of multiple time scales introduced by Nayfeh [7] has the advantage that it leads to two nonlinear Schrödinger equations, describing the finite amplitude wave propagation through the surface, one is valid near the cutoff wave number, while the other can be used to study the stability of the system. Oron and Rosenau [8] studied the nonlinear evolution of a perturbed interface that separates two liquids of different viscosity. They showed that the interface is governed by the negularized Kuramoto - Sivashinshy equation with the use of a new approach with accounts for large gradients. El - Dib [9] studied the parametric nonlinear Schrödinger equation and its stability criterion. He gives a useful analysis of the dispersive perturbations for a periodic temporal solution.

All the pervious studies of interfacial instabilities have been based on the assumption that the fluids are immiscible, therefore, there in no mass transfer across the interface. The immiscibility condition concerns the limit case of infinite latent heat. Ordinarily, since the latent heat is very large, it is a very good approximation to treat the fluids as immiscible when the thermal effects are very small. However, when there is a strong temperature gradient in the fluid, thermal effects on the interfacial waves can be appreciable. Therefore, there is significant mass transfer across the interface and in turn transfer of heat in the fluid has been taken into consideration. Hsieh [10] has given a general formulation of the interfacial flow problem with mass and heat transfer and applied it to both R - T and K - H instabilities. The linear study of these problems, in the presence of a periodic electric field, has been studied by Moatimid [1,11]. Mohamed et al. [12,13] studied the nonlinear (EHD) R - T instabilities with mass and heat transfer.

To the best of our knowledge, no attempt has been made to examine the effect of a tangential periodic electric field to a horizontal interface admitting mass and heat transfer. We have, therefore, extended the pervious work [1] through a nonlinear perturbation analysis. The derivation of the parametric nonlinear Schrödinger equation is based on the method of multiple scales. Both of the steepness ratio of the perturbed wave and the amplitude of the periodic electric field are taken as a small parameter. The method used carried out the necessary and sufficient conditions of the stability. The results are confirmed numerically.

2. Formulation of the problem :

Two fluid layers confined between two horizontal rigid parallel plates that are infinitely long are considered. Cartisian coordinates (x, y) are used and without any loss of generality the z - axis is omitted. The y - axis being taken vertically. It is assumed, in the equilibrium state, that a hypothetical interface at $y = 0$ separates the two fluid regions. The fluids are incompressible, inviscid and homogeneous. The fluid (1) occupies the region $-h_1 < y < 0$ having the density $\rho^{(1)}$ and the dielectric constant $\tilde{\varepsilon}^{(1)}$. The fluid (2) occupies region $0 < y < h_2$, possessing density $\rho^{(2)}$ and dielectric constant $\tilde{\varepsilon}^{(2)}$. The temperatures at $y = h_2, y = -h_1$ and $y = 0$ are $T^{(2)}$, $T^{(1)}$ and $T^{(0)}$, respectively. Acceleration due to gravity (g) acts in the negative y direction. The perturbed interface is given by

$$S(x,y,t) = y - \zeta(x,t) = 0, \tag{1}$$

where $y = \zeta(x,t)$ denotes the elevation of the surface at time t. The system is stressed upon by a periodic tangential electric field in the x - direction, which has the following form :

$$\underline{E} = (E_0 + \varepsilon\hat{E}\cos\omega_0 t)\underline{e}_x, \tag{2}$$

ω_0 is the frequency of the field.

The flow is assumed to be irrotational, thus the basic equations governing the perturbed velocity potential ϕ are

$$\nabla^2\phi^{(1)} = 0, \qquad \text{for } -h_1 < y < \zeta(x,t) \tag{3}$$

$$\nabla^2\phi^{(2)} = 0, \qquad \text{for } \zeta(x,t) < y < h_2, \tag{4}$$

The perturbation produces an additional electric field, which can be derived from a scalar potential $\psi(x,y)$ such that

$$\underline{E} = (E_0 + \varepsilon\hat{E}\cos\omega_0 t)\underline{e}_x - \nabla\psi \tag{5}$$

It follows that [5]

$$\nabla^2\psi^{(1)} = 0, \qquad \text{for } -h_1 < y < \zeta(x,t), \tag{6}$$

$$\nabla^2\psi^{(2)} = 0, \qquad \text{for } \zeta(x,t) < y < h_2. \tag{7}$$

Here ϕ and ψ are the velocity and electrostatic potential of the perturbations. At the two plates, these potentials satisfy the following boundary conditions:

$$\left(\frac{\partial\phi^{(1)}}{\partial y}\right)_{y=-h_1} = \left(\frac{\partial\phi^{(2)}}{\partial y}\right)_{y=h_2} = 0, \tag{8}$$

$$\left(\frac{\partial\psi^{(1)}}{\partial x}\right)_{y=-h_1} = \left(\frac{\partial\psi^{(2)}}{\partial x}\right)_{y=h_2} = 0 \tag{9}$$

In addition, the interfacial boundary conditions between the two fluids are:

$$[[\frac{\partial\psi}{\partial x}]] + \frac{\partial\zeta}{\partial x}[[\frac{\partial\psi}{\partial y}]] = 0, \qquad \text{at } y=\zeta \tag{10}$$

$$(E_0 + \varepsilon\hat{E}\cos\omega_0 t)[[\tilde{\varepsilon}]]\frac{\partial\zeta}{\partial x} - \frac{\partial\zeta}{\partial x}[[\tilde{\varepsilon}\frac{\partial\psi}{\partial x}]] + [[\tilde{\varepsilon}\frac{\partial\psi}{\partial y}]] = 0. \qquad \text{at } y=\zeta \tag{11}$$

where [[ ]] represents the jump across the interface .

The stress tensor may be expressed as [5] :

$$\Pi_{ij} = -\Pi\delta_{ij} + \tilde{\varepsilon}E_i E_j - (1/2)\tilde{\varepsilon}E^2\delta_{ij}. \tag{12}$$

where $\Pi = P - (1/2)\tilde{\varepsilon}E_0^2$, with P is the hydrostatic pressure which can be obtained from Bernoulli's equation and δ_{ij} is the delta Kronecher.

The interfacial conditions that express the conservation of mass and momentum are given by [10]

$$\rho^{(1)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(1)}.\nabla S\right) = \rho^{(2)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(2)}.\nabla S\right) \qquad \text{at } y=\zeta\ , \tag{13}$$

and

$$\rho^{(1)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(1)}.\nabla S\right)\left(\nabla\phi^{(1)}.\nabla S\right) = \rho^{(2)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(2)}.\nabla S\right)\left(\nabla\phi^{(2)}.\nabla S\right)x$$

$$x\left[(\Pi_{ij}^{(1)} - \Pi_{ij}^{(2)})n_i - n_j\sigma(\frac{1}{R_1} + \frac{1}{R_2})\right], \qquad j=1,2, \qquad \text{at } y=\zeta, \tag{14}$$

where $\underline{n}$ is the unit normal vector to the interface , σ is the surface tension coefficient , R_1 and R_2 are the two principal radii of curvature of the interface . The radius of curvature is taken to be positive if the center of curvature lies on the side of the fluid (2) , otherwise it is negative.

Finally, the interfacial condition for energy transfer is given by

$$L\rho^{(1)}\left(\frac{\partial S}{\partial t}+\nabla\phi^{(1)}.\nabla S\right)=f(\zeta), \qquad \text{at } y=\zeta, \tag{15}$$

where L is the latent heat released when the fluid is transformed from phase (1) to phase (2). The left - hand side of (15) represents the net heat flux from the interface into the fluid regions when such a phase transformation is taking place. This quantity is taken to be approximately expressible in terms of the balance of heat fluxes in the fluid regions as if the system is instantaneously in dynamic equilibrium.

As in Ref. [14], we denote

$$f(\zeta)=\alpha(\zeta+\alpha_2\zeta^2+\alpha_3\zeta^3), \tag{16}$$

where

$$\alpha=\frac{G}{L}\left(\frac{1}{h_1}+\frac{1}{h_2}\right),$$

$$\alpha_2=\left(\frac{1}{h_2}-\frac{1}{h_1}\right),$$

$$\alpha_3=\frac{h_1^3+h_2^3}{h_1^2h_2^2(h_1+h_2)}$$

and

$$G=\frac{K^{(2)}(T^{(0)}-T^{(2)})}{h_2}=\frac{K^{(1)}(T^{(1)}-T^{(0)})}{h_1},$$

is the equilibrium heat flux.Here $K^{(1)}$ and $K^{(2)}$ represents the lower and upper thermal conductivity , respectively.

If fluid (1) is hotter than fluid (2) , then L is positive and G is positive since $T^{(1)}>T^{(0)}>T^{(2)}$. If fluid (2) is hotter than fluid (1), then L and G are both negative . Therefore, α is always positive.

3. Method of Solution :

Now, we have a well-defined boundary value problem. To investigate the nonlinear interaction of small but finite amplitude waves, let us apply the multiple scales method. To that end, we expand the various variables in ascending powers in terms of a small dimensionless parameter ε characterizing the amplitude of the periodic force. The independent variables x ,t are scaled in a like manner,

$$X_n = \varepsilon^n x \quad , \quad T_n = \varepsilon^n t \quad , n = 0,1,2,$$

(17)

$$\zeta(x,t) = \sum_{n=1}^{3} \varepsilon^n \zeta_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\varepsilon^4),$$

(18)

and the variables may expanded as

$$\Phi(x,t) = \sum_{n=1}^{3} \varepsilon^n \Phi_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\varepsilon^4),$$

(19)

where Φ stands for the physical quantities ϕ or ψ.

Since the boundary conditions (10), (11), (13)-(15) are prescribed at the interface $y = \zeta(x,t)$, therefore, we express all the involved physical quantities in terms of Maclaurin's series about $y = 0$. On using the above expansions (17) and (18) into the set of equations (3) - (15), and equating the coefficients of equal power in ε, we obtain the linear as well as the successive higher - order equations. The hierarchy of the equations for each order can be obtained with the knowledge of the previous orders.

4. The first order problem :

The solutions of the first - order problem lead to the dispersion relation

$$F(\omega, k) = 0$$

where

$$F(\omega,k) = (\omega^2 / k)(\rho^{(2)} \coth kh_2 + \rho^{(1)} \coth kh_1) + (i\alpha\omega / k)(\coth kh_2 + \coth kh_1)$$

$$-[E_0^2/\varepsilon^*(k)]k(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2 \sinh kh_2 \sinh kh_1 - g(\rho^{(1)}-\rho^{(2)}) - k^2\sigma, \tag{20}$$

where

$$\varepsilon^*(k) = \tilde{\varepsilon}^{(1)} \sinh kh_2 \cosh kh_1 + \tilde{\varepsilon}^{(2)} \sinh kh_1 \cosh kh_2$$

The dispersion relation (20) is reduced to

$$a_2(i\omega)^2 + a_1(-i\omega) + a_0 = 0, \tag{21}$$

where

$$a_2 = \rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2,$$
$$a_1 = \alpha(\coth kh_1 + \coth kh_2)$$
$$a_0 = \left(\frac{E_0^2 k^2(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{(\tilde{\varepsilon}^{(1)} \coth kh_1 + \tilde{\varepsilon}^{(2)} \coth kh_2)}\right) + k^3\sigma + gk(\rho^{(1)}-\rho^{(2)})$$

It is well known from the Routh - Hurwitz criterion [15], that necessary and sufficient conditions for stability, for the quadratic equation (21), are

$$a_1 > 0 \quad \text{and} \quad a_0 > 0, \tag{22}$$

since a_2 is always positive.

The condition $a_1 > 0$ is trivially satisfied since $\alpha > 0$, while the condition $a_0 > 0$ gives

$$E_0^2 > \frac{[\tilde{\varepsilon}^{(1)} \coth kh_1 + \tilde{\varepsilon}^{(2)} \coth kh_2][g(\rho^{(2)}-\rho^{(1)}) - k^2\sigma]}{k(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}. \tag{23}$$

It is clear, from (23), that the tangential electric field has a stabilizing influence for all values of $E_0^2 > E_*^2$, where

$$E_*^2 = \frac{[\tilde{\varepsilon}^{(1)} \coth kh_1 + \tilde{\varepsilon}^{(2)} \coth kh_2][g(\rho^{(2)}-\rho^{(1)}) - k^2\sigma]}{k(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}. \tag{24}$$

Thus, the system is linearly stable. On the other side, the periodic electric field has no implication at this order.

(4) The second order problem :

The inclusion of the periodic electric field would yield results radically different from the classical case. In classical Rayleigh - Taylor problem, admiting mass and heat transfer, the second - order surface deflection ζ_2 is modified to be

$$\zeta_2 = \zeta_{20} + \zeta_{11}, \tag{25}$$

where ζ_{20} represents the elevation in the absence of the periodic field as obtained by Mohamed et al [12].

$$\zeta_{20} = -2\alpha_2 A\bar{A} + \Omega A^2(X_1, X_2, T_1, T_2)e^{2i(kX_0-\omega T_0)} + C.C., \tag{26}$$

the addition term ζ_{11} is due to periodic force which is given by

$$\zeta_{11} = \Omega_1(X_1, X_2, T_1, T_2)e^{i(kX_0-\omega T_0)} + C.C., \tag{27}$$

while ζ_{11} denotes the forcing due to oscillation of gravity. It is found that Ω_1 should satisfy

$$\Big[(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)\frac{\partial^2}{\partial T_0^2} + \alpha(\coth kh_1 + \coth kh_2)\frac{\partial}{\partial T_0} + (\rho^{(1)}\coth kh_1 +$$

$$+\rho^{(2)}\coth kh_2)(\omega_r^2 - \omega_i^2) - 2\alpha(\coth kh_1 + \coth kh_2)(\frac{\omega_i}{2})\Big]\Omega_1 = \Big[-\frac{2k^2E_0\hat{E}}{\varepsilon^*(k)}(\tilde{\varepsilon}^{(1)}$$

$$-\tilde{\varepsilon}^{(2)})\sinh kh_1 \sinh kh_2 A\cos\omega_0 T_0 + 2i\omega_r(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)(\frac{\partial A}{\partial T_1} +$$

$$+\frac{d\omega}{dk}\frac{\partial A}{\partial X_1})\Big]e^{i(kX_0-\omega T_0)} + C.C., \tag{28}$$

Equation (28) contains terms which correspond to the factor $\exp(-i\omega_r T_0)$. The elimination of these terms leads to secular terms that lead to the solvability conditions. In omitting these terms, we need to distinguish between two cases; the first case when the external frequency ω_0 is away from the real part of the wave frequency ω (the non - resonant case), and the second one arises when the frequency ω_0 approaches $2\omega_r$ and $\omega_r (\omega = \omega_r + i\omega_i)$. Thus in the non - resonance case, the following solvability condition is obtained:

$$\frac{\partial A}{\partial T_1} + \frac{d\omega}{dk}\frac{\partial A}{\partial X_1} = 0, \tag{29}$$

where $\frac{d\omega}{dk}$ is given in the appendix.

With the solvability condition (29), in the non - resonant case, the particular solution of (28) is

$$\Omega_1 = \left[\frac{2k^2 E_0 \hat{E}(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 \sinh kh_1 \sinh kh_2}{\varepsilon^*(k)\omega_0 L_1(\omega_0^2 - 4\omega_r^2)}\right][\omega_0 \cos\omega_0 T_0$$
$$+2i\omega_r \sin\omega_0 T_0]Ae^{-i\omega T_0} + C.C \tag{30}$$

where

$$L_1 = \rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2 \tag{31}$$

In the resonance case the frequency ω_0 is assumed to approache $2\omega_r$ by introducing the detuning parameter σ_1 as defined by

$$\omega_0 \approx 2\omega_r + 2\varepsilon\sigma_1, \tag{32}$$

$$-i(\omega_0 - \omega_r)T_0 = -i(\omega_r + 2\varepsilon\sigma_1)T_0. \tag{33}$$

Thus the solvability condition, in this case, yields

$$\frac{\partial A}{\partial T_1}+\frac{d\omega}{dk}\frac{\partial A}{\partial X_1}=i\gamma_0\bar{A}e^{-2i\sigma_1 T_1},$$

(34)

where

$$\gamma_0=-\left[\frac{k^2E_0\hat{E}(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})\sinh kh_1\sinh kh_2}{2\varepsilon^*(k)\omega_r L_1}\right]$$

(35)

Equation (34) is the solvability condition in the resonant case . Therefore the particular solution of equation (28) is

$$\Omega_1=\left[\frac{k^2E_0\hat{E}(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})\sinh kh_1\sinh kh_2}{\varepsilon^*(k)\omega_0 L_1(\omega_0+2\omega_r)}\right]e^{-i(\omega_0+\omega)T_0}+C.C$$

(36)

5. The third order problem :

In this order of the investigation the surface deflection ζ_3 satisfies the following equation

$$\left[(\rho^{(1)}-\rho^{(2)})g+(\frac{L_1}{k})\frac{\partial^2}{\partial T_0^2}+\frac{L_2}{k}\frac{\partial}{\partial T_0}+\sigma k^2-\frac{kE_0^2(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2}{\varepsilon^*(k)\sinh kh_1\sinh kh_2}\right]\zeta_3=$$

$$-\left[\frac{i}{k^2}[L_1+k(\rho^{(2)}h_2\coth^2 kh_2+\rho^{(1)}h_1\coth^2 kh_1)-k(\rho^{(2)}h_2+\rho^{(1)}h_1)]\frac{\partial^3}{\partial T_0^2\partial X_1}\right.$$

$$+[(i/k^2)L_2+(i\alpha/k)(h_2\coth^2 kh_2+h_1\coth^2 kh_1)+(2L_1/k)-(i\alpha/k)(h_2+h_1)]$$

$$\times\frac{\partial^2}{\partial T_0\partial X_1}+(2L_1/k)\frac{\partial^2}{\partial T_0\partial T_1}+(L_2/k)\frac{\partial}{\partial T_1}+i\{kE_0\hat{E}/\varepsilon^*(k)\}(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2\cos\omega_0 T_0$$

$$-2k\sigma-\{k(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2/\varepsilon^{*2}(k)\}E_0^2\{\varepsilon^*(k)\sinh kh_1\sinh kh_2+(\tilde{\varepsilon}^{(1)}h_1\sinh^2 kh_2$$

$$\left.+\tilde{\varepsilon}^{(2)}h_2\sinh^2 kh_1)\}]\frac{\partial}{\partial X_1}\right]\Omega_1 e^{ikX_0}-\left[[\frac{ikE_0\hat{E}}{\varepsilon^*(k)}\cos\omega_0 T_0(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2(\tilde{\varepsilon}^{(1)}h_1\sinh^2 kh_2\right.$$

$$+\tilde{\varepsilon}^{(2)}h_2 \sinh^2 kh_1) + \frac{ikE_0\hat{E}}{\varepsilon^*(k)}\cos\omega_0 T_0(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 \sinh kh_1 \sinh kh_2]\frac{\partial}{\partial X_1}$$

$$-\frac{k\hat{E}}{\varepsilon^*(k)}(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 \cos^2\omega_0 T_0 \sinh kh_1 \sinh kh_2 \Big] A e^{i(kX_0-\omega T_0)} - \Big[-\frac{i}{k}(iL_2 + 2\omega L_1)$$

$$x\frac{\partial A}{\partial T_2} + i[(-iL_2\omega / k^2) - (\omega^2 L_1 / k^2) - 2\sigma k - \{k(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 / \varepsilon^{*3}(k)\}E_0^2\{(\tilde{\varepsilon}^{(1)}$$

$$x h_1 \sinh^2 kh_2 + \tilde{\varepsilon}^{(2)}h_2 \sinh^2 kh_1) - \sinh kh_1 \sinh kh_2\} - (h_2 / k \sinh^2 kh_2)(i\alpha\omega +$$

$$+\omega^2\rho^{(2)}) - (h_1 / k \sinh^2 kh_1)(i\alpha\omega + \omega^2\rho^{(1)})]\frac{\partial A}{\partial X_2} + (L_1 / k)\frac{\partial^2 A}{\partial T_1^2} + [(iL_2 / k^2) +$$

$$+(2\omega L_1 / k^2) + (h_1 / k \sinh^2 kh_1)(i\alpha + 2\omega\rho^{(1)}) + (h_2 / k \sinh^2 kh_2)(i\alpha + 2\omega\rho^{(2)})]x$$

$$\frac{\partial^2 A}{\partial X_1 \partial T_1} + [(i\omega L_2 / k^3) + (\omega^2 L_1 / k^3) - \sigma + \{(h_1 / k^2 \sinh^2 kh_1) + (h_1^2 \cosh kh_1 /$$

$$k \sinh^3 kh_1\}(i\alpha\omega + \omega^2\rho^{(1)}) + \{(h_2 / k^2 \sinh^2 kh_2) + (h_2^2 \cosh kh_2 / k \sinh^3 kh_1)\}x$$

$$(i\alpha\omega + \omega^2\rho^{(2)}) + \{E_0^2(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 / \varepsilon^{*3}(k)\}\{-2kh_1h_2\tilde{\varepsilon}^{(2)}\tilde{\varepsilon}^{(1)} \sinh kh_1 \sinh kh_2$$

$$-kh_1^2 \sinh^2 kh_2(\tilde{\varepsilon}^{(1)} \sinh kh_1 \sinh kh_2 + \tilde{\varepsilon}^{(2)} \cosh kh_1 \cosh kh_2) - kh_2^2 \sinh^2 kh_1 x$$

$$(\tilde{\varepsilon}^{(1)} \cosh kh_1 \cosh kh_2 + \tilde{\varepsilon}^{(2)} \sinh kh_1 \sinh kh_2) + \varepsilon^*(k)(\tilde{\varepsilon}^{(1)}h_1 \sinh^2 kh_2$$

$$+\tilde{\varepsilon}^{(2)}h_2 \sinh^2 kh_1)]\frac{\partial^2 A}{\partial X^2} - A^2\bar{A}\Theta \Big] e^{i(kX_0-\omega T_0)} + C.C. + NST, \tag{37}$$

where

$$L_2 = \coth kh_1 + \coth kh_2 ,$$

the term NST stands for terms that do not produce secular terms and Θ is given in [12]. To analyze the particular solution for equation (37), we need to avoid the non - uniformity of equation (37). Thus, we need that the secular terms to vanish . Three possible cases that produce secular terms are found , where ω_0 is away from ω_r and ω_0 approaches $2\omega_r$ and ω_r . The elimination of the secular terms produces the following solvability conditions in these cases.

(5.1) In non resonance case :

In this case the frequency ω_0 is considered away from the wave frequency ω_r. Thus the solvability condition of the third order in the non resonance case is given by

$$i\frac{\partial A}{\partial T_2}+i\frac{d\omega}{dk}\frac{\partial A}{\partial X_2}+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial X_1^2}+M_1A+(Q_1+iQ_2)A^2\bar{A}=0, \tag{38}$$

where

$$M_1=\{-k/2\omega_r L_1\}[\{2k^3E_0^2\hat{E}^2(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^4\sinh^2 kh_1\sinh^2 kh_2/\varepsilon^{*2}(k)L_1(\omega_0^2-4\omega_r^2)\}+\{k\hat{E}^2/2\varepsilon^*(k)\}(\tilde{\varepsilon}^{(2)}-\tilde{\varepsilon}^{(1)})^2\sinh kh_1\sinh kh_2], \tag{39}$$

$$Q_1=-[k\Theta_r/2(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2)\omega_r], \tag{40}$$

and

$$Q_2=-[k\Theta_i/2(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2)\omega_r]. \tag{41}$$

By using the Gardner - Morikawa transformation, equation (38) becomes :

$$i\frac{\partial A}{\partial \tau}+(P_1+iP_2)\frac{\partial^2 A}{\partial \xi^2}+M_1A+(Q_1+iQ_2)A^2\bar{A}=0, \tag{42}$$

where

$$P_1+iP_2=\frac{1}{2}\frac{d^2\omega}{dk^2}. \tag{43}$$

Equation (42) is a nonlinear Schödinger equation with complex coefficients.The stability condition for the nonlinear equation (42) is discussed in Elhefnawy et al.[16].

<u>(5.2) The resonance cases :</u>

(5.2.1) The case of ω_0 near $2\omega_r$:

Inspection of the right - hand side of equation (37) reveals that , in addition to terms proportional to the factor $\exp(\pm i\omega_r T_0)$, the secular terms are produced by the terms proportional to the factor $\exp[(\pm i(\omega_0 - \omega_r)T_0]$. In this case, we express the nearness of ω_0 to $2\omega_r$ by introducing a detuning parameter σ_1 defined according to equation (32). Therefore, $\exp[-i(\omega_0 - \omega_r)T_0] = \exp[-i\omega_r - 2i\sigma_1 T_1]$. The secular terms are eliminated from equation (37) by the help of equation (34). Thus the solvability condition in this case is given by :

$$i\frac{\partial A}{\partial T_2} + i\frac{d\omega}{dk}\frac{\partial A}{\partial X_2} + \frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial X_1^2} + R_1 A + (Q_1 + iQ_2)A^2\bar{A} + [(S_1 + iS_2)\frac{\partial \bar{A}}{\partial X_1} + \\ + F_1\bar{A}]e^{-2i\sigma_1 T_1} = 0, \tag{44}$$

where R_1, S_1, S_2 and F_1 are given in the appendix.

By using the Gardner - Morikawa transformation, equation (44) becomes :

$$i\frac{\partial A}{\partial \tau} + (P_1 + iP_2)\frac{\partial^2 A}{\partial \xi^2} + R_1 A + (Q_1 + iQ_2)A^2\bar{A} + [(S_1 + iS_2)\frac{\partial \bar{A}}{\partial \xi} \\ + F_1\bar{A}]e^{-2i\sigma_1\varepsilon^{-1}\tau} = 0, \tag{45}$$

This equation is a parametric nonlinear Schrödinger equation with complex coefficients. The stability criterion is obtained by El-Dib [17]. We follow the procedure adopted there. Thus, we assume that equation (45) admits the following time - dependent solution :

$$A = m\exp[-i(\sigma_1\varepsilon^{-1} - R_1 - Q_1 m^2)]\tau \tag{46}$$

Substituting from (46) into (45), we obtain

$$\sigma_1\varepsilon^{-1} + iQ_2 m^2 + F_1\exp[-2i(R_1 + Q_1 m^2)\tau] = 0 \tag{47}$$

By separating real and imaginary parts of equation (47), we get

$$\sigma_1\varepsilon^{-1} + F_1\cos 2(R_1 + Q_1 m^2)\tau = 0 \tag{48}$$

$$Q_2 m^2 - F_1\sin 2(R_1 + Q_1 m^2)\tau = 0 \tag{49}$$

Squaring equations (48) and (49) and adding, we obtain

$$m^4 = [F_1^2 - (\sigma_1^2/\varepsilon^2)]/Q_2^2 \tag{50}$$

m^2 is real when

$$F_1^2 - (\sigma_1^2/\varepsilon^2) > 0 \tag{51}$$

or

$$m^2 = [F_1^2 - (\sigma_1^2/\varepsilon^2)]^{(1/2)}/Q_2 \tag{52}$$

The solution (46) must be bounded, this requires that

$$Q_2[F_1^2 - (\sigma_1^2/\varepsilon^2)] > 0 \tag{53}$$

To examine the stability criteria, we perturb the solution (46) according to

$$A = (m + \alpha + i\beta)\exp[-i(\sigma_1\varepsilon^{-1} - R_1 - Q_1 m^2)]\tau \tag{54}$$

where α and β are real. Substituting (54) into (45) and neglecting nonlinear terms in α and β, we get

$$-\frac{\partial\beta}{\partial\tau} + P_1\frac{\partial^2\alpha}{\partial\xi^2} - P_2\frac{\partial^2\beta}{\partial\xi^2} + 2Q_1 m^2\alpha - (1/F_1)[\sigma_1\varepsilon^{-1}(S_1\frac{\partial\alpha}{\partial\xi} + S_2\frac{\partial\beta}{\partial\xi}) - Q_2 m^2(-2\beta F_1$$
$$-S_1\frac{\partial\beta}{\partial\xi} + S_2\frac{\partial\alpha}{\partial\xi})] = 0 \tag{55}$$

and

$$\frac{\partial\alpha}{\partial\tau} + P_1\frac{\partial^2\beta}{\partial\xi^2} + P_2\frac{\partial^2\alpha}{\partial\xi^2} + 2Q_2 m^2\alpha - (1/F_1)[\sigma_1\varepsilon^{-1}(-2\beta F_1 - S_1\frac{\partial\beta}{\partial\xi} + S_2\frac{\partial\alpha}{\partial\xi})$$
$$+Q_2 m^2(S_1\frac{\partial\alpha}{\partial\xi} + S_2\frac{\partial\beta}{\partial\xi})] = 0. \tag{56}$$

Equations (55) and (56) are linear equations, their solutions can be taken the form

$$\alpha(\xi,\tau) = A\exp[iq\xi + \Omega\tau] \tag{57}$$

$$\beta(\xi,\tau) = B\exp[iq\xi + \Omega\tau] \tag{58}$$

where q and Ω are the wave number and frequency respectively.

Substituting (57) and (58) into (55) and (56), we find that q and Ω satisfy the following dispersion relation

$$\Omega^2 + C_1\Omega + (C_2 + iC_3) = 0, \tag{59}$$

where C_1, C_2 and C_3 are given in the appendix.

The necessary and sufficient conditions for stability require that [15]

$$(2\sqrt{F_1^2 - (\sigma_1^2/\varepsilon^2)} - q^2P_2) > 0 \tag{60}$$

$$q^8 + b_1q^6 + b_2q^4 + b_3q^2 + b_4 > 0 \tag{61}$$

The transition curves separating stable region from unstable corresponding to

$$q^* = (2/P_2)\sqrt{F_1^2 - (\sigma_1^2/\varepsilon^2)}, \tag{62}$$

$$q^{*4} + b_1q^{*3} + b_2q^{*2} + b_3q^* + b_4 = 0 \tag{63}$$

where $q^2 = q^*$

In what follows, we shall give numerical discussions for stability of the system under consideration by drawing the transition curves. The transition curves are represented by equations (62) and (63) in the q^* - k plane. In the following figures, the letter S denotes stable regions while the unstable ones are characterized by the symbols U1 and U2.

The values of q^*, as described by equations (62) and (63), are the critical values of the disturbance. These critical values, which are known as the transition curves, separate the stable from the unstable regions.

In figure (1) q^* is plotted versus k for a system having the particulars $\rho^{(1)} = 0.978$ gm/cm^3 , $\rho^{(2)} = 5.542E\text{-}04$ gm/cm^3 , $\tilde{\varepsilon}^{(1)} = 61.03$, $\tilde{\varepsilon}^{(2)} = 1.078$, T = 58.8 dynes/cm , g = 980 cm/sec^2 , $\omega_0 =$ 309 HZ , $E_0 = 4.0$ dynes/esu , $\varepsilon = 0.1$ and $\hat{E} = 129$ dynes/esu. The regions in the (q^* - k) plane where every point in it satisfies the inequalities (60) and (61) are labeled by S. The unstable regions are indicated by the symbol U. In figure (1) , the unstable regions are characterized by the symbols U1 and U2 and the stable region is indicated by S.

Figure (2) represents the same system in figure (1) , but the field frequency has the value $\omega_0 = 317$ HZ. The comparison between figure (1) and (2) show that the increase of the frequency ω_0 increases the unstable region U1 , while the unstable region U2 is decreased .Thus , the field frequency ω_0 plays a dual role in the stability analysis .El - Dib [17] found that the field frequency has a destabilizing influence in the absence of mass and heat transfer.

Figure (3) represents the same system in figure (1) , but $E_0 = 4.3$ dynes/esu. From figures (1) and (3) , it is shown that the increase of the electric field decreases the unstable region U1 while the unstable but the region U2 is increased .Therefore , One can say that the electric field plays a dual role the stability . El-Dib [17] show that the electric field plays a stabilizing role in the absence of mass and heat transfer.

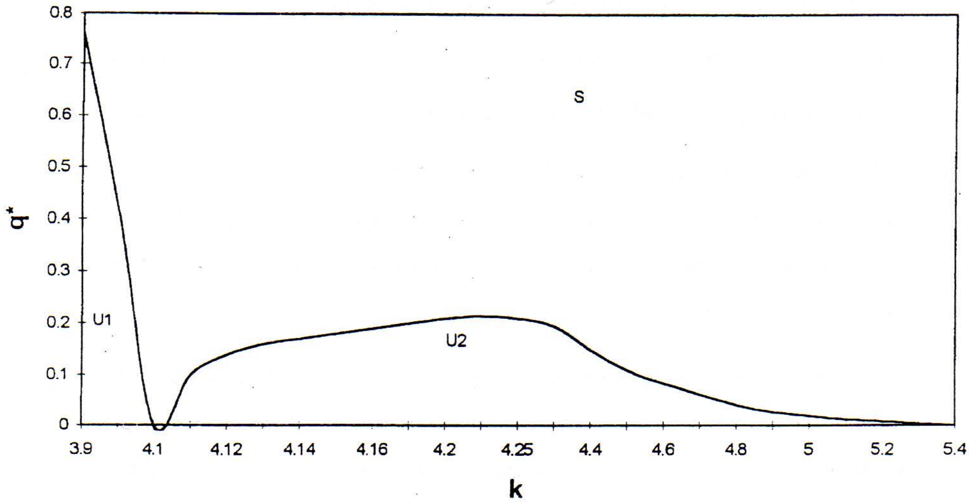
(5.2.2) The case of ω_0 near ω_r :

We express the nearness of ω_0 to ω_r by introducing a detuning parameter σ that defined according to

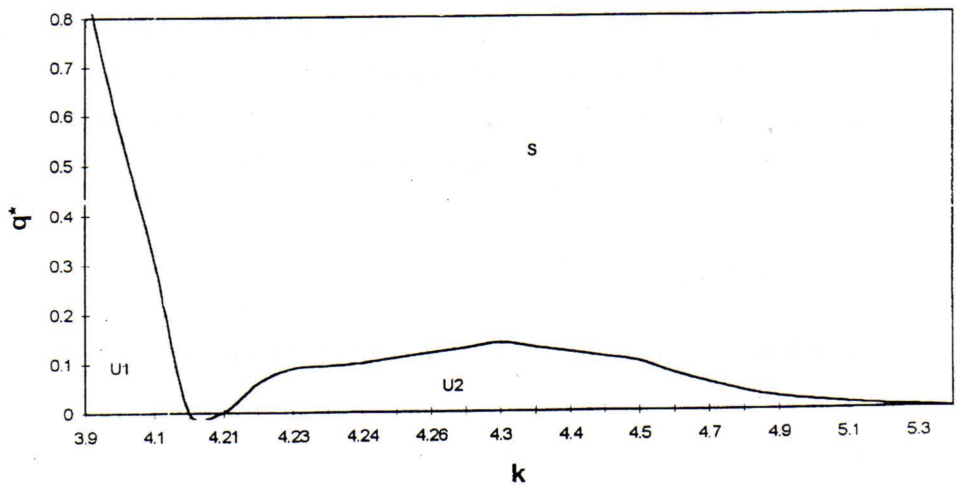
$$\omega_0 = \omega_r + \varepsilon^2 \sigma \tag{64}$$

Therefore , the solvability condition in this case is obtained as :

$$i\frac{\partial A}{\partial \tau} + (P_1 + iP_2)\frac{\partial^2 A}{\partial \xi^2} + M_1 A + (Q_1 + iQ_2)A^2\bar{A} + G_1 \bar{A} e^{-2i\sigma\tau} = 0 \tag{65}$$



Figure(1) : Represents a system for $\rho^{(1)} = 0.977$ gm/cm^3 , $\rho^{(2)}$ =5.542E-04 gm/cm^3 , $\tilde{\varepsilon}^{(1)} = 61.03$, $\tilde{\varepsilon}^{(2)} = 1.0078$, $\omega_0 = 309$ HZ, T=58.8 dynes/cm g = 980cm/sec^2 , E_0 =4.0 dynes/esu , $\varepsilon = 0.1$ and $\hat{E}$ =129 dynes/esu.



Figure(2) : Represents the same system considered in Figure (1) , but $\omega_0 = 317$ HZ.

where G_1 is given in the appendix.

Equation (65) is a parametric nonlinear Schrödinger equation with complex coefficients.

In the second resonance case of ω_0 approaches ω_r, the stability conditions are given by [16]

$$-q^{*}P_2 + 2Q_2 m_1 > 0 \tag{66}$$

$$q^{*2} + S_3 q^{*} + S_4 > 0 \tag{67}$$

where

$$m_1^2 = \sqrt{(G_1^2 - \sigma)} / Q_2,$$

$$S_3 = [1/(P_1^2 + P_2^2)](-2P_1\sigma - 2m_1^2 Q_1 P_1 - 4P_2 Q_2 m_1^2),$$

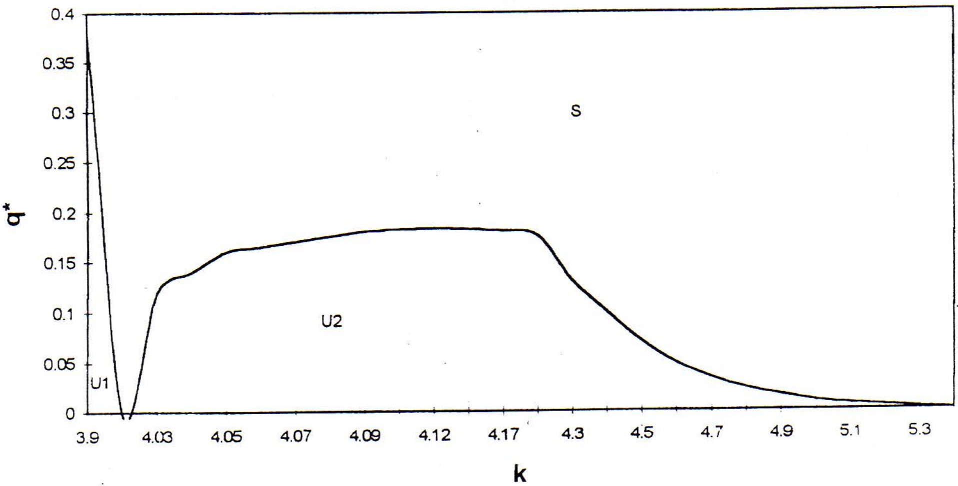
$$S_4 = [1/(P_1^2 + P_2^2)](4m_1^2 Q_1 \sigma + 4Q_2^2 m_1^4),$$

The transition curves separating stable regions from unstable regions corresponding to

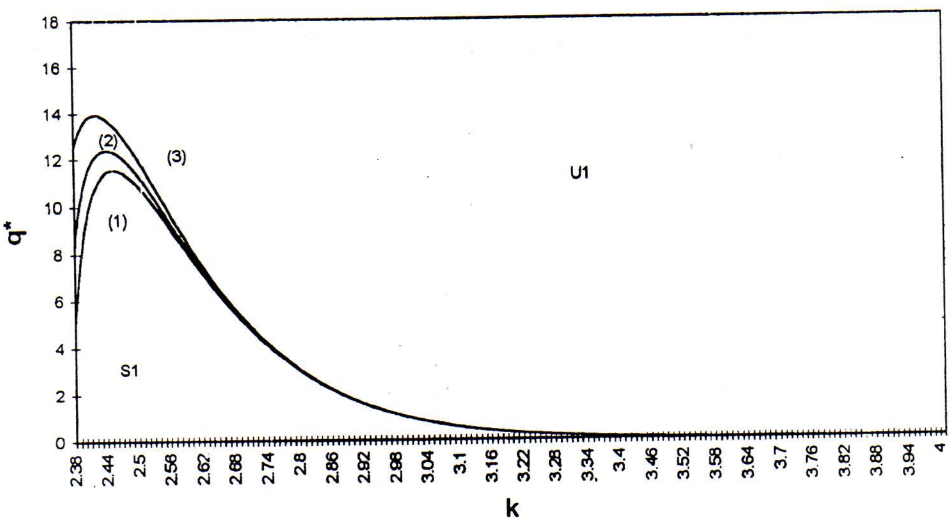
$$q^{*} = (2/P_2)\sqrt{G_1^2 - \sigma^2}, \tag{68}$$

$$q^{*2} + S_3 q^{*} + S_4 = 0 \tag{69}$$

Figure (4) represents a system for $\rho^{(1)} = 0.978$ gm/cm^3, $\rho^{(2)} = 5.542E\text{-}04$ gm/cm^3, $\tilde{\varepsilon}^{(1)} = 61.03$, $\tilde{\varepsilon}^{(2)} = 1.078$, $T = 58.8$ dynes/cm, $g = 980$ cm/sec^2, $\omega_0 = 2312$ HZ, $E_0 = 100$ dynes/esu and $\hat{E} = 80$ dynes/esu. For curve (1), $\omega_0 = 2312$ HZ. The region in the $(q^{*}$ - k) plane which every point in it satisfies the inequalities (66) and (67) are indicated by symbol S. In figure (4) the stable region is characterized by the symbol S1 and unstable region is indicated by U. In figure (4), curves (2) and (3) represent the same system for curve (1) but for curve (2), $\omega_0 = 2300$ HZ and for curve (3), $\omega_0 = 2280$ HZ. The comparison among the three curves, show that the stable



Figure(3) : Represents the same system considered in Figure (1) , but E_0 = 4.3 dynes / esu.



Figure(4) : Represents the same system considered in Figure (1), but E_0 = 100 dynes/esu and $\hat{E}$ = 80 dynes/esu., for curve (1) , ω_0 = 2312 HZ. For curves (2) and (3) , ω_0 = 2300 , 2280 HZ respectively .

region S1 is increased as the field frequency ω_0 is decreased. Thus, the filed frequency has a destabilizing effect.

Figure (5) consists of three curves, the first curve (1) is the same as curve (1) in figure (4). But for curve (2), $E_0 = 102$ dynes/esu and curve (3), $E_0 = 105$ dynes/esu. From figure (5), it is shown that as electric field increases the unstable region is decreased. One can say that the electric field has a stabilizing influence.

6. Conclusions :

For our study of the nonlinear interfacial instability of two electrified miscible fluids, we come to the following conclusions :

(1) The method of multiple scales is used to obtain two parametric nonlinear Schrödinger equations in the resonance cases and a classical nonlinear Schrödinger equation in the non - resonance.

(2) The stability criterion is affected by the amplitude m of the temporal solution (46) which depends on the parameter of the parametric Schrödinger equation.

(3) The necessary and sufficient conditions for stability are obtained.

(4) In the resonance case of ω_0 near $2\omega_r$, the field frequency and the electric field play a dual role in the stability analysis.

(5) In the second resonance case of ω_0 near ω_r, the field frequency has a destabilizing effect but the electric field has a stabilizing influence.

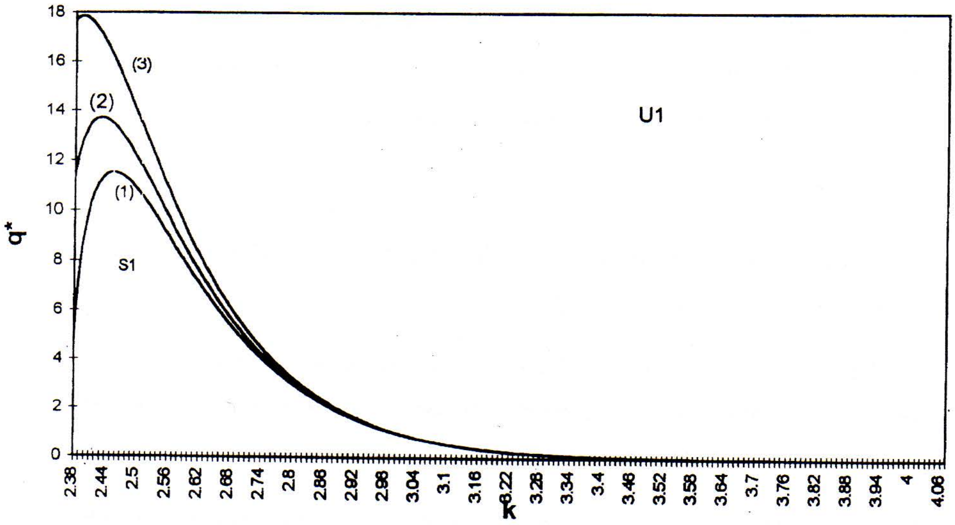


Figure (5) : Represents the same system considered in Figure (4) for curve (1) E_0 = 100 dynes /esu., but for curves (2) and (3) E_0=102 and 105 dynes/esu. respectively.

Appendix

$$\frac{d\omega}{dk}=(1/2L_1\omega_r)\Big[(\omega_r^2-\omega_i^2)\Big(\frac{\rho^{(1)}h_1}{\sinh^2 kh_1}+\frac{\rho^{(2)}h_2}{\sinh^2 kh_2}+\frac{L_1}{k}\Big]-\omega_i\Big[\frac{\alpha h_2}{\sinh^2 kh_2}+\frac{\alpha h_1}{\sinh^2 kh_1}$$

$$+\frac{L_2}{k}\Big]\Big[\frac{E_0^2}{\varepsilon^{*2}(k)}\Big]k^2(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2(h_1\tilde\varepsilon^{(1)}\sinh^2 kh_2+h_2\tilde\varepsilon^{(2)}\sinh^2 kh_1)+2k^2\sigma+$$

$$+k(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sinh kh_2\sinh kh_1\Big(\frac{E_0^2}{\varepsilon^{*2}(k)}\Big]\Big]+i(1/2L_1)\Big[\frac{h_1}{\sinh^2 kh_1}(2\omega_r\rho^{(1)}+\alpha)$$

$$+\frac{h_2}{\sinh^2 kh_2}(2\omega_r\rho^{(2)}+\alpha)\Big]$$

$$R_1=\left[\frac{-\hat E^2E_0^2k^4(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^4\sinh^2 kh_1\sinh^2 kh_2}{8L_1\omega_r^3\omega_0(\omega_0^2-4\omega_r^2)\varepsilon^{*2}(k)}\right][4\omega_r(\omega_0-\omega_r)(\omega_0+2\omega_r)+\omega_0$$

$$\times(\omega_0^2+4\omega_r^2)]-\left[\frac{-E_0^2k^2(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sinh kh_1\sinh kh_2}{4L_1\omega_r\varepsilon^*(k)}\right]$$

$$F_1=\left[\frac{-\hat E E_0k^2(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sigma\sinh kh_1\sinh kh_2}{2L_1\omega_r^2\varepsilon^*(k)}\right]$$

$$S_1=-\left[\frac{\hat E E_0k^3(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sinh kh_1\sinh kh_2}{4L_1^2\omega_r^2\omega_0(\omega_0-2\omega_r)\varepsilon^*(k)}\right]\Big[[2\omega_0^2\Big\{\frac{\rho^{(2)}h_2}{k\sinh^2 kh_2}+\frac{\rho^{(1)}h_1}{k\sinh^2 kh_1}\Big\}$$

$$\times\frac{\alpha}{k}\Big\{\frac{h_2}{\sinh^2 kh_2}+\frac{h_1}{\sinh^2 kh_1}\Big\}][2\omega_r(\omega_0-2\omega_r)+\omega_0^2]\Big]$$

$$S_2=-\left[\frac{\hat E E_0k^3(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sinh kh_1\sinh kh_2}{4L_1^2\omega_0\omega_r^3\varepsilon^*(k)}\right]\Big[\Big[\frac{L_1}{k^2}+\frac{\rho^{(2)}h_2}{k\sinh^2 kh_2}+\frac{\rho^{(1)}h_1}{k\sinh^2 kh_1}\Big]$$

$$\times[(\omega_r^2-\omega_i^2)(\omega_0+2\omega_r)-2\omega_0\omega_r^2]-\omega_i(\omega_0+2\omega_r)][\frac{L_2}{k^2}+\frac{\alpha}{k}\Big\{\frac{h_2}{\sinh^2 kh_2}$$

$$+\frac{h_1}{\sinh^2 kh_1}\Big\}]+(\omega_0+2\omega_r)[2k\sigma+\{kE_0/\varepsilon^{*2}(k)\}(h_1\tilde\varepsilon^{(1)}\sinh^2 kh_2+h_2\tilde\varepsilon^{(2)}$$

$$\times\sinh^2 kh_1)+\frac{E_0}{\varepsilon^*(k)}(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2\sinh kh_1\sinh kh_2]\Big]+\left[\frac{\hat E E_0k(\tilde\varepsilon^{(2)}-\tilde\varepsilon^{(1)})^2}{2\omega_r\varepsilon^*(k)L_1}\right]$$

$$\Big[\frac{k}{2\varepsilon^*(k)}(h_1\tilde\varepsilon^{(1)}\sinh^2 kh_2+h_2\tilde\varepsilon^{(2)}\sinh^2 kh_1)+\sinh kh_1\sinh kh_2[\frac{3}{2}+\frac{k^2}{\omega_rL_1}\times$$

$$\times\Big\{\frac{\rho^{(1)}h_1}{k\sinh^2 kh_1}+\frac{\rho^{(2)}h_2}{k\sinh^2 kh_2}\Big\}]\Big]$$

$$C_1 = 4Q_2m^2 - 2P_2q^2$$

$$C_2 = q^4(P_1^2 + P_2^2) + q^2[-2Q_1m^2P_1 + (1/F_1^2)(S_1\sigma_1\varepsilon^{-1} - S_2Q_2m^2)^2 + (1/F_1^2)x$$

$$x(S_2\sigma_1\varepsilon^{-1} + Q_2m^2S_1)^2 - 2P_1\sigma_1\varepsilon^{-1} - 4Q_2m^2P_2] + 4m^2(Q_1\sigma_1\varepsilon^{-1} + Q_2^2m^2)$$

$$C_3 = (2q/F_1)(\sigma_1\varepsilon^{-1}S_1 - S_2Q_2m^2)(Q_1m^2 - \sigma_1\varepsilon^{-1})$$

$$G_1 = \left[\frac{-\hat{E}^2k^2(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 \sinh kh_1 \sinh kh_2}{8L_1\omega_r\varepsilon^*(k)}\right][1 + \{4k^2E_0^2(\tilde{\varepsilon}^{(2)} - \tilde{\varepsilon}^{(1)})^2 \sinh kh_1$$

$$x \sinh kh_2 / \omega_0(\omega_0 - 2\omega_r) + \varepsilon^*(k)L_1\}]$$

$$b_1 = [1/P_2^2(P_1^2 + P_2^2)][-4m^2Q_2P_2(P_1^2 + P_2^2) + P_2^2\{-2m^2(Q_1P_1 + 2Q_2P_2)$$

$$-2P_1\sigma_1\varepsilon^{-1} + (S_1^2 + S_2^2)\}]$$

$$b_2 = [1/P_2^2(P_1^2 + P_2^2)][4m^2Q_2^2(P_1^2 + P_2^2) - P_2^2\{-2m^2(Q_1P_1 + 2Q_2P_2) - 4m^2Q_2P_2\}$$

$$x\{-2m^2(Q_1P_1 + 2Q_2P_2) - 2P_1\sigma_1\varepsilon^{-1} + (S_1^2 + S_2^2)\} + 4P_2^2(Q_1m^2\sigma_1\varepsilon^{-1} + F_1^2 - \sigma_1^2\varepsilon^{-2})]$$

$$b_3 = [1/P_2^2(P_1^2 + P_2^2)][(4m^2Q_2^2 + P_2^2)\{-2m^2(Q_1P_1 + 2Q_2P_2) - 2P_1\sigma_1\varepsilon^{-1} + (S_1^2 + S_2^2)\}$$

$$-(1/F_1^2)(\sigma_1\varepsilon^{-1}S_1 - m^2Q_2S_2)^2(m^2Q_1 - \sigma_1\varepsilon^{-1})^2]$$

and

$$b_4 = [1/P_2^2(P_1^2 + P_2^2)][16m^2Q_2(m^2Q_2 - P_2)(m^2Q_1\sigma_1\varepsilon^{-1} + F_1^2 - \sigma_1^2\varepsilon^{-2})]$$

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