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MANETODYNAMIC STABILITY OF BOUNDED HOLLOW JET PERVADED BY VARYING MAGNETIC FIELD

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Abstract

The problem is formulated and solved based on the linear perturbation technique. The stability criterion is derived and discussed. The analytical results are confirmed numerically. The uniform magnetic fields interior and exterior the jet have stabilizing influences for all modes of perturbation. The varying transverse magnetic field interior the model is stabilizing or not according to restrictions. The radii ratio of the gas-liquid cylinder has a tendency for stabilizing the model. As the axial magnetic fields intensities interior and exterior the gas jet are so much strong than that of the transverse magnetic field and adapting the cylinders radii ratio the instability of the model is suppressed and stability sets in.

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Tow-stream instability 52.35.Q

1. Introduction

The stability of a hollow jet endowed with surface tension has been indicated by Chandrasekhar (1981) for axisymmetric perturbation. Drazin and Reid (1980) have given its dispersion for all modes of perturbation. Kendall (1986) performed neat experiments for examining the capillary instability of bounded hollow jet. Radwan (1988) and (1997) extended such works by investigating the stability of a hollow jet analytically under the influence of capillary force in addition to other forces. Such a model has interesting applications in several domains of science, cf. Kendall (1986).

Here we intend to investigate the magnetodynamic stability of bounded magnetized hollow jet pervaded internally by oblique varying magnetic field and surrounded by uniform magnetic field.

2. Formulation of the Problem

We consider a gas cylinder of (constant pressure) radius a, immersed in a liquid bounded by cylindrical wall at distance b (with $0 < a < b$).

The matter inside and outside the cylinder are assumed to be penetrated by the magnetic fields

$$\underline{H}_0^g = (0, \beta H_0 r/a, \alpha H_0), \tag{1}$$

$$\underline{H}_0^f = (0, 0, H_0) \tag{2}$$

where α, β are the internal magnetic field parameters while quantities with the superscripts g and f indicate variable inside and outside the cylinder. The components of $\underline{H}_0^g$ and $\underline{H}_0^f$ are taken along the utilizing cylindrical coordinates system (r, φ, z) with the z-axis coinciding with the axis of the cylinder. The liquid is assumed to be incompressible, non-viscous, perfectly conducting and with constant magnetic permeability μ. The liquid matter is acted by the pressure gradient and Lorentz forces, while the gas medium is acted by the electromagnetic force only in addition to the gas constant pressure P_0^g.

The magnetodynamic basic equations for an ideal fluid appropriate to the present model are the following:

In the liquid region:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} + \nabla)\underline{u}\right) = -\nabla p + \mu(\nabla \wedge \underline{H}) \wedge \underline{H}, \tag{3}$$

$$\nabla . \underline{u} = 0, \tag{4}$$

$$\frac{\partial \underline{H}}{\partial t} = (\underline{H} . \nabla)\underline{u} - (\underline{u} . \nabla)\underline{H}, \tag{5}$$

$$\nabla . \underline{H} = 0, \tag{6}$$

In the gas region

$$\nabla . \underline{H}^g = 0, \tag{7}$$

$$\nabla \wedge \underline{H}^g = 0. \tag{8}$$

Here ρ, $\underline{u}$ and p are the fluid mass density, velocity vector and kinetic pressure, $\underline{H}$ is the magnetic field intensity in the liquid medium and idem $\underline{H}^g$ in the gas region. The equation of motion (3) may be written in the form:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} . \nabla)\underline{u} - \frac{\mu}{\rho}(\underline{H} . \nabla)\underline{H} = -\nabla \Pi \tag{9}$$

with

$$\Pi = \frac{P}{\rho} + \frac{\mu}{2\rho}(\underline{H} . \underline{H}) \tag{10}$$

where, $\rho\Pi$ represents the sum of the kinetic and magnetic pressures. One has to refer here also that equations (4) and (6) have been used for obtaining equation (5) from Maxwell's electrodynamic equations. The fundamental equations, (3)-(7) are solved and upon applying appropriate conditions at $r = a$, the unperturbed liquid kinetic pressure distribution p_0 is given by:

$$P_0 = \frac{\mu H_0^2}{2}\left(\alpha^2 + \beta^2 - 1\right) + P_0^g \tag{11}$$

where, P_0 is positive as $\alpha^2 + \beta^2 \geq 1$ as P_0^g is extremely small.

The first term on the right-side of equation (11) is the contribution due to the Lorentz forces, interior and exterior the gas jet influence while the term P_0^g is the gas equilibrium constant pressure.

3. Perturbed state

For small departures from the unperturbed state, every variable quantity $Q(r, \varphi, z; t)$ may be expressed as:

$$Q\,(r, \varphi, z; t) = Q_0\,(r) + \varepsilon_0 Q_1\,(r, \varphi, z; t), |Q_1| << Q_0 \tag{12}$$

Here Q_0 represents the unperturbed part and Q_1 is a small fluctuating part due to perturbation where, Q stands for $\underline{u}$, p, $\underline{H}$ and $\underline{H}_0^g$. By an appeal to the expansion (12) and utilizing the basic equations (3)-(8), the linearized perturbation equations are given as follows.

- *In the liquid region*:

$$\rho\left(\frac{\partial \underline{u}_1}{\partial t}\right) = -\nabla P_1 + \mu\left(\nabla \wedge \underline{H}_0\right) \wedge \underline{H}_1 \tag{13}$$

$$\nabla \bullet \underline{u}_1 = 0, \tag{14}$$

$$\frac{\partial \underline{H}_1}{\partial t} = \nabla \wedge \left(\underline{u}_1 \wedge \underline{H}_0\right) + \nabla \wedge \left(\underline{u}_0 \wedge \underline{H}_1\right), \tag{15}$$

$$\nabla . \underline{H}_1 = 0 \tag{16}$$

- *In the gas region:*

$$\nabla . \underline{H}_1^g = 0 \tag{17}$$

$$\nabla \wedge \underline{H}_1^g = 0 \tag{18}$$

From the viewpoint of the expansion (12) and upon considering a sinusoidal wave, the perturbed radial distance of the gas cylinder is given by:

$$r = R_0 + R_1 \qquad (19)$$

with

$$R_1 = \varepsilon_0 \exp[i(kz + m\varphi) + \sigma t] \qquad (20)$$

Here, R_1 is the elevation of the surface wave normalized with respect to R_0, ε_0 is the initial amplitude with

$$\varepsilon = \varepsilon_0 \exp[\sigma t] \qquad (21)$$

is the amplitude of the surface wave at time t where k and m are the longitudinal and transverse wavenumbers.

By the aid of the expansions (10) and the time-space dependence (20) and from the viewpoint of the stability theory, $Q_1(r, \varphi, z; t)$ could be expressed as:

$$Q_1(r, \varphi, z; t) = Q_1^{*}(r) \exp[i(kz + m\varphi) + \sigma t] \qquad (22)$$

This means that every relevant perturbed quantity could be expressed as an amplitude function of r times exp $[i(kz+m\varphi) + \sigma t]$. Consequently the linearized perturbation equations (13)-(18) are simplified and solved. Apart from the singular solution we have

$$\underline{H}_1 = \frac{H_0}{\sigma} \frac{\partial \underline{u}_1}{\partial z}, \qquad (23)$$

$$\underline{u}_1 = -\frac{\sigma}{\rho\left[\sigma^2 + \Omega_A^2\right]} \nabla \Pi_1 \qquad (24)$$

$$\Pi_1(r, \varphi, z; t) = [AI_m(kr) + Bk_m(kr)]\, \varepsilon_0 \exp[i(kz + m\varphi) + \sigma t] \qquad (25)$$

$$\underline{H}_1^{\varepsilon} = C\, \nabla\left(I_m(kr)\, \varepsilon_0 \exp[i(kz + m\varphi) + \sigma t]\right) \qquad (26)$$

Here A, B and C are constants of integration to be determined, while $I_m(kr)$ and $k_m(kr)$ are the modified Bessel functions of the first and second kind of order m.

4. Dispersion relation

The solution of the basic equations (3)-(8) in the unperturbed and perturbed states represented by equations (11) and (23)-(26) must satisfy appropriate boundary conditions. Under the present circumstances these boundary conditions are given as follows.

4.1 Kinematic boundary conditions

(i) The normal component of the velocity vector u must vanish across the wall at r = b

(ii) The normal component of the velocity vector must be continuous across the gas-liquid interface at r = a. These conditions read:

$$N_0 \cdot \underline{u}_1 + N_1 \cdot \underline{u}_0 = \begin{bmatrix} \dfrac{\partial r}{\partial t} & \text{at } r = a & (27) \\ 0 & \text{at } r = b & (28) \end{bmatrix}$$

from which the constants A and B may be identified at once.

4.2 Magnetodynamic boundary condition

The normal component of the magnetic field must be continuous across the interface (19) at $r = a$, i.e.

$$N_0 \cdot \underline{H}_1 + N_1 \cdot \underline{H}_0 = N_0 \cdot \underline{H}_1^g + N_1 \cdot \underline{H}_0^g \tag{29}$$

This condition gives directly the constant C.

4.3 The pressure balance

The normal component of the total stress tensor must be continuous across the gas liquid interface (19) at r = a. By applying this condition, which is some compatibility condition, the following dispersion relation is established

$$\sigma^2 = \frac{\mu H_0^2}{\rho a^2}\left[-x^2 + M_m(x,y)\left(-\beta^2 + (m\beta + \alpha x)^2 \frac{I_m(x)}{x I'_m(x)}\right)\right] \tag{30}$$

where x (= k a) and y (= q x) are longitudinal dimensionless wavenumbers. The combined Bessel function $M_m(x,y)$ is defined by:

$$M_m(x,y) = x\left(L^m_{x,y}/L^m_y\right) \tag{31}$$

with

$$L^m_{x,y} = I'_m(x) K'_m(y) - I'_m(y) K'_m(x), \tag{32}$$

$$L^m_y = I_m(x) K'_m(y) - I'_m(y) K_m(x) \tag{33}$$

$$q = b/a \tag{34}$$

5. Stability discussion

Equation (30) is the desired dispersion relation of magnetodynamic stability of bounded hollow jet, of negligible motion, ambient with liquid medium acted by the electromagnetic forces. By means of this equation the stability and instability restrictions of the present problem could be identified. The transition points from the oscillation states to those of instability of the model may be obtained from equation (30) such that $\sigma = 0$.

The relation (30) relates the temporal amplification σ or rather oscillation frequency ω as $\sigma(= i\,\omega)$ is imaginary with the fundamental quantity $(\mu H_0^2 / \rho\, a^2)^{-1/2}$ as a unite of time, the modified Bessel functions I_m, K_m and their derivatives, the azimuthal and longitudinal wavenumbers m and x (= ka), the magnetic field parameters α and β and with the parameters ρ, μ, a and H_0 .Now returning to the relation (30), the influence of the axial magnetic field acting interior the bounded hollow jet model is represented by the term $-x^2$ following the natural quantity $\mu H_0^2 / \rho a^2$ in the right side of equation (30) It has always a stabilizing influence and this influence is independent of the perturbation.

The influence of the gas magnetic field is represented by the terms including α and β, i.e. the terms:

$$\left(-\beta^2 + (m\beta + \alpha x)^2 \frac{I_m(x)}{x\, I'_m(x)}\right) M_m(x,y) \tag{35}$$

The influence of the transverse gas magnetic field depends on the sign of the terms in β^2, i.e. the terms in equation (35) as $\alpha = 0$ which are

$$\left(-\beta^2 + m^2\beta^2 \frac{I_m(x)}{x\, I'_m(x)}\right) M_m(x,y) \tag{36}$$

We may prove that $M_m(x, y) > 0$ for all values of $m \geq 0$ and any values of $x > 0$ and $y > 0$, see equation (31). The influence of the axial gas magnetic field is represented by the terms in α^2, i.e. the term in quantity (35) as $\beta = 0$ which are

$$x^2 \left[\alpha^2 I_m(x) / I'_m(x)\right] M_m(x,y) \tag{37}$$

From the recurrence relations (*cf.* Abramowitz and Stegan)

$$2 K'_m(x) = -K_{m-1}(x) - K_{m+1}(x) \tag{38}$$

$$2 I'_m(x) = I_{m-1}(x) + I_{m+1}(x) \tag{39}$$

It is clear that $K'_m(x)$ is always negative and $I'_m(x)$ is never negative for $x > 0$. The function $K_m(x)$ is positive definite and monotonically decreasing while $I_m(x)$ is always definite for $x > 0$ but monotonically increasing.

Therefor. for $0 \leq q < 1$ the inequalities

$$- K'_m(x) < - K'_m(y) \tag{40}$$

$$I'_m(x) > - I'_m(y) \tag{41}$$

are satisfied for $x \neq 0$. It follows for $x \neq 0$ that

$$M_m(x,y) = \frac{x\left[I'_m(x)\,K'_m(y) - I'_m(y)\,K'_m(x)\right]}{\left[I_m(x)\,K'_m(y) - I'_m(y)\,K_m(x)\right]} > 0 \tag{42}$$

By an appeal to the relations (38) and (39) and the inequality (42) for equation (37) we find that the axial gas magnetic field is stabilizing for all values of α, $x > 0$, $y > 0$ not only in the axisymmetric modes $m = 0$ but also in the non-axisymmetric modes $m \geq 1$.

Moreover, in general, concerning the varying gas magnetic field contribution (35): the term $\beta^2 M_m(x,y)$ is always positive as explained before, thus it is always destabilizing. The last term $(m\beta + \alpha x)^2 \left[I_m(x)/x\,I'_m(x)\right]\ M_m(x,y)$ is always negative, so it has stabilizing influence. However, if $m\beta\alpha x$ is negative, then $(m\beta + \alpha x)^2$ is smaller than otherwise and thus less stabilizing. The stabilizing influence of the last term vanishes when $m\beta + \alpha x = 0$, in this case, the growth rate σ is the independent of α and only the destabilizing effect of the azimuthal field remains through the term β^2.

Some of the foregoing analytical results are confirmed numerically. The dispersion relation (30) is formulated in a dimensionless form and inserted in computer for calculation.

First as a simple limiting case when $a = b$, here we have only a gas jet of negligible motion and the dispersion relation is reduced to:

$$\sigma^2 = - \frac{\mu H_o^2}{\rho R_o^2} x^2$$

This indicates that the model will be stable not only for short wavelengths but also for very long wavelengths.

The cases with $1 < q < \infty$ must be calculated numerically to determine exactly the stable and unstable domains for different values of the problem parameters as will be discussed in the next section.

6. Numerical Discussion

The problem is investigated numerically for the most important axisymmetric mode m = o, upon using the stability criterion (30) in the computer.

The values of α and β are taken as $(\alpha, \beta) = (3, 5), (2, 5)$ and $(1,4)$. For each pair values of (α, β), the radii ratio $q(= b/a)$ of the fluid and gas cylinders is considered to be $q = 2, 3$, and 4.

For $(\alpha, \beta) = (3, 5), (2, 5)$ and $(1, 4)$, $q = 2$: it is found that the unstable domains are $0 < x < 1.6589$, $0 < x < 4.5996$ and $0 < x < 7.9518$ while those of stability are $1.6589 \leq x < \infty$, $4.5996 \leq x < \infty$ and $7.9518 \leq x < \infty$ where the equalities are corresponding to marginal stability states (see Figure 1).

For $(\alpha, \beta) = (3, 5)$, $q = 2, 3$ and 4: it is found that the unstable domains are $0 < x < 1.6589$, $0 < x < 1.674$ and $0 < x < 1.6744$ while those of stability are $1.6589 \leq x < \infty$, $1.674 \leq x < \infty$ and $1.6744 \leq x < \infty$. (see Figure 2).

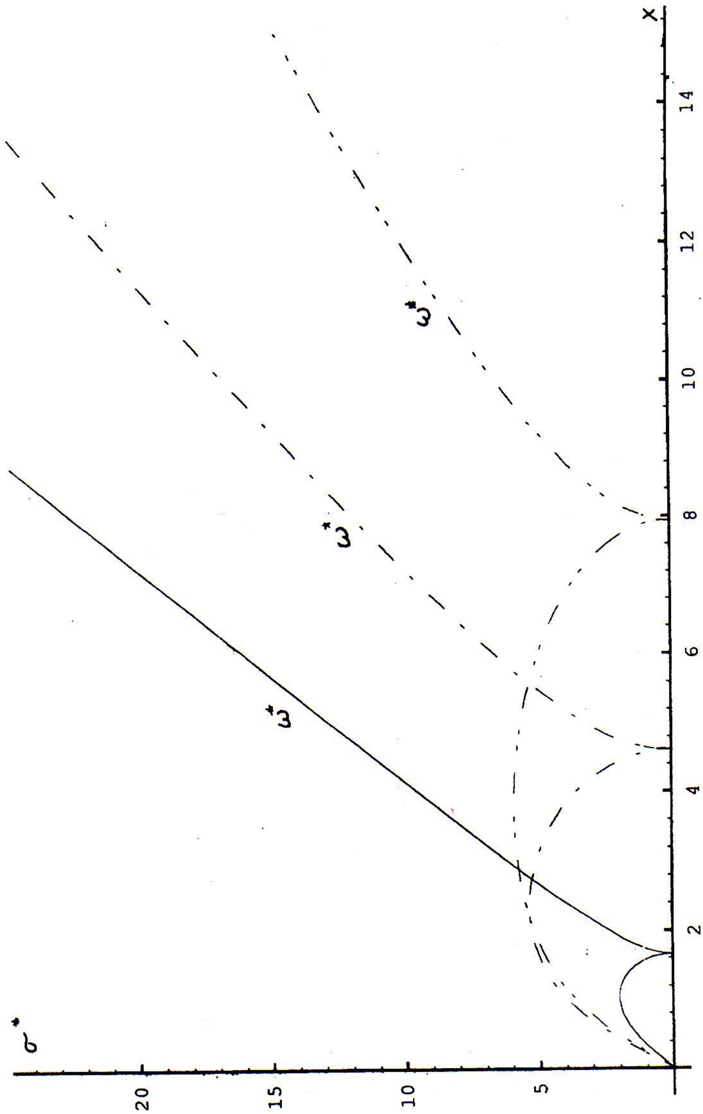

Fig. (1)

Stable and unstable domaines for $q = 2$

——— $(\alpha, \beta) = (3, 5)$

-·-·-·- $(\alpha, \beta) = (2, 5)$

-··-··- $(\alpha, \beta) = (1, 4)$

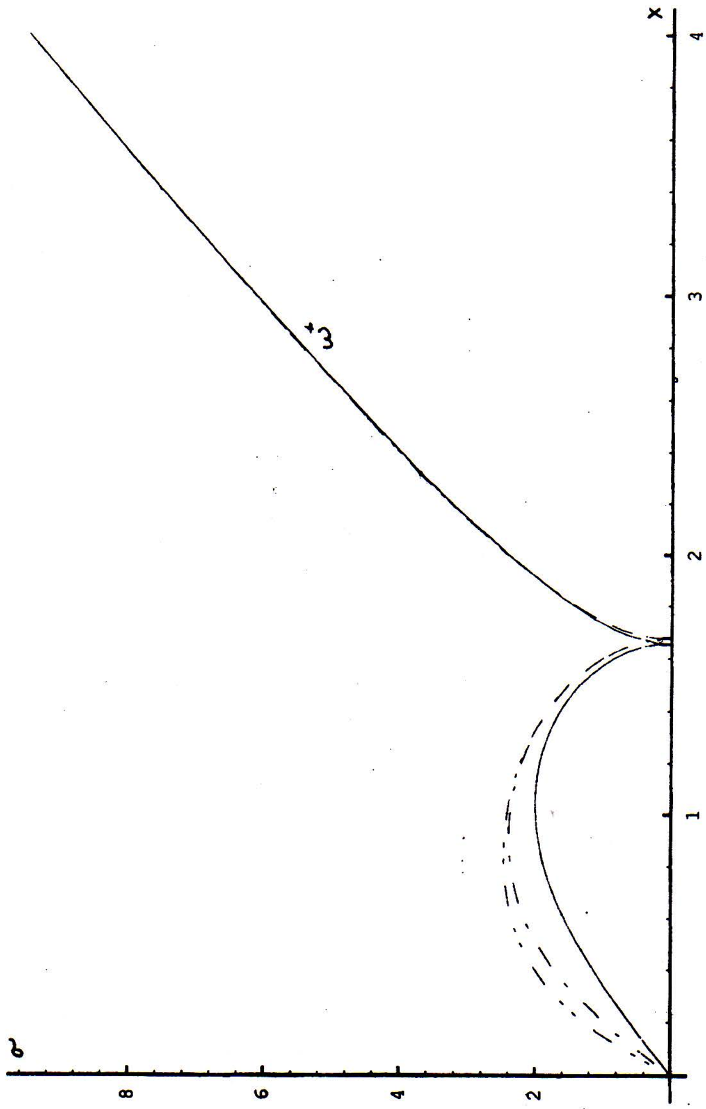

Fig. (2)

Stable and unstable domaines for $(\alpha, \beta) = (3, 5)$

———	q = 2
-·-·-·-·	q = 3
:·-·-·-·	q = 4

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