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# NUMERICAL ANALYSIS OF DUFFING OSCILLATOR WITH DRY FRICTION DAMPER

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**Abstract.**

*Mathematical modelling and numerical analysis of the classical Duffing oscillator with four different dry friction dampers is presented. Then the influence of the different types of friction characteristics on the system dynamic responses is systematically examined pointing out a necessity of proposing a new dry friction model, which is constructed based on the non-reversible approach [4,8]. A comparison between the dynamic responses for all models including the new one is provided..*

## 1. Introduction.

Dry friction appears in many mechanical systems met in practice by most engineers. Usually, this is unnecessary effect and most unwanted one. For many years the topic of dry friction has been actively researched with many attempts to identify the causes of unwanted behaviour such as squel of car brakes, extensive wear of the cutting tools, and others. From the mathematical point of view dry friction problems are also cubersome as the inclusion of the dynamics of dry fiction implicates appearance of the discontinuous differential equations, where the character of this discontinuity depends upon the friction character adopted. In general there many different types of dry friction models and it is crucial to appropriately chose one which suits best to the modelled problem. For example if one considers dynamics of the system where the relative velocity practically remains constant, there is no need for a sophisticated dry friction models and even the simplest one described by the Coulomb law will suffice. However in many cases the variation of the relative velocity is large and often the velocity changes its sign. In such situations the chosen model must account for the transition ftom static to dynamic friction and must provide a means of guiding the system through zero relative velocity. These types of mathematical models should be able to predict both phases of stick with a higher friction coefficient and slip where this coefficient is smaller. These are the reasons why systems with dry friction possess many different types of dynamical behaviour, such as periodical, non-periodic, chaotic and sometimes even static responses [2, 6, 7, 9, 11].

A practical engineering approaçh, endebted to Coulomb simplifies the friction force to constant value directed opposite to the relative velocity of the contacting bodies. Such force can take two values with identical level butopposite sign only. Newer experiments show non-linear dependence on the contact velocity rather than the constant one. That was why most efforts were directed to built non-linear friction models and to determine differences in maximal values of the static and dynamic friction forces – see [3]. Another attempts to determine different types of friction characteristics – showing dependencies on the relative acceleration on the contacting surface – so called non-reversible friction characteristics and association between friction characteristics and system motion character are shown in [1, 4, 8, 10].

In this paper we present a comparison between the dynamical responses of the classical Duffing oscillator equipped with additional dry friction dampers described by the following friction models:

i. classical Coulomb,
ii. model proposed by Popp – Stelter,
iii. non-reversible friction characteristics,
iv. novel model of dry friction (where the relative acceleration value is taken under consideration).

This comparison is facillated by a systematic bifurcational analysis of the above-mentioned Duffing oscillators.

## 2. Mathematical Modelling and Numerical Analysis.

The subject of the current studies is a classical Duffing oscillator to which a dry friction damper has been added, which is depicted in Fig.1 The dynamics of the systems can be described by the following second order differential equation

$$m\ddot{x} + kx^3 + c\dot{x} + Nfsign(\dot{x}) = F\cos(\omega t) \quad (1)$$

where: $m$ – oscillator mass, $kx^2$ – non-linear stiffness, $c$ – damping coefficient,

$\omega$ – excitation frequency, $F$ – amplitude of the excitation force, $N$ – normal pressure force, $f$ – friction function. Relative velocity of the friction surfaces is equal to the momentary velocity of the system $\dot{x}$.

After dividing both sides of the equation (1) by the value of the mass and introducing: $x=x_1$, $\dot{x}=x_2$, $a=k/m$, $h=c/m$, $p=F/m$, $\varepsilon=N/m$, the analysed system can be transformed to a set of two first order differential equations given below

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p\cos(\omega t) - ax_1^3 - hx_2 - \varepsilon f sign(x_2), \end{aligned} \tag{2}$$

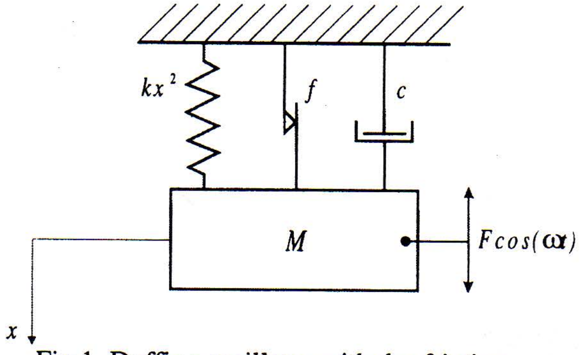


Fig.1. Duffing oscillator with dry friction.

For the purpose of constructing the bifurcation diagrams through a means of numerical simulation constant values of the parameters were assumed – $a$=1.00, $h$=0.10, $p$=10.0, $\omega$=1.00, while the value of the normal pressure $\varepsilon$ was selected as a the branching parameter. The friction force function was described by four different models which will be introduced and explained below. Each friction modelled is followed by a bifurcation diagram correcponding to it.

### *2.1 Coulomb Model (Model I).*

This model bases on the classical Coulomb approach, where the friction function – Eq.(3) takes two different constant values for the stick ($f_s$) and slip ($f_d$) phases.

$$f = \begin{cases} f_s, & \dot{x} = 0 \\ f_d, & |\dot{x}| > 0 \end{cases} \tag{3}$$

A graphical portrait of the Coulomb friction as a function of the relative velocity for the positive part of the abscissa is shown in Fig.2, where Fig. 3 presents a corresponding bifurcation diagram.

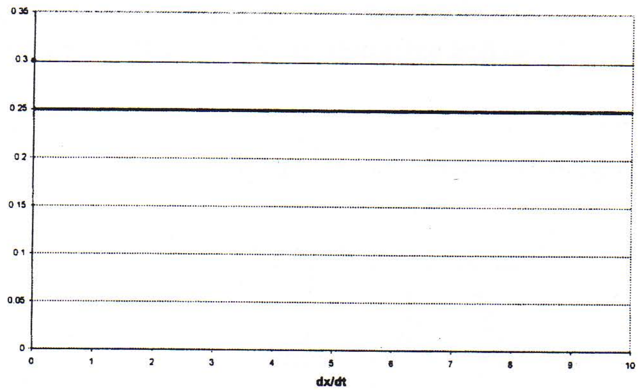


Fig.2. Coulomb characteristic of friction; $f_s$=0.30, $f_d$=0.25.

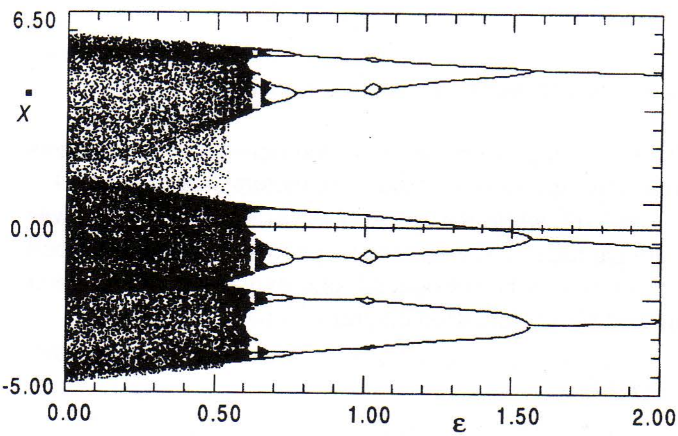


Fig.3. Bifurcation diagram showing the dynamics of the Duffing oscillator with dry friction descibed with the Coulomb model (Fig.2) while normal pressure increases.

### *2.2 Popp – Stelter Model (Model II).*

In this model the dependence of the friction coefficient on the relative velocity has a non-linear character as shown in Fig.4 described by the following formula [3]

$$f = \left( \frac{f_s - f_d}{1 + \eta_1 |\dot{x}|} + f_d + \eta_2 \dot{x}^2 \right) \tag{4}$$

Similarly to the previous model a bifurcation diagram has been prepared. The differences between them will be outlined in the Discussion and Conslusion section.

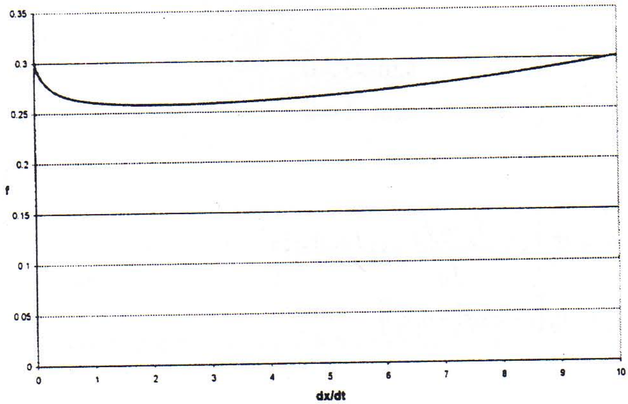


Fig.4. Friction characteristics for the Popp – Stelter model; $\eta_1$=4.00, $\eta_2$=0.0005, $f_s$=0.30, $f_d$=0.25.

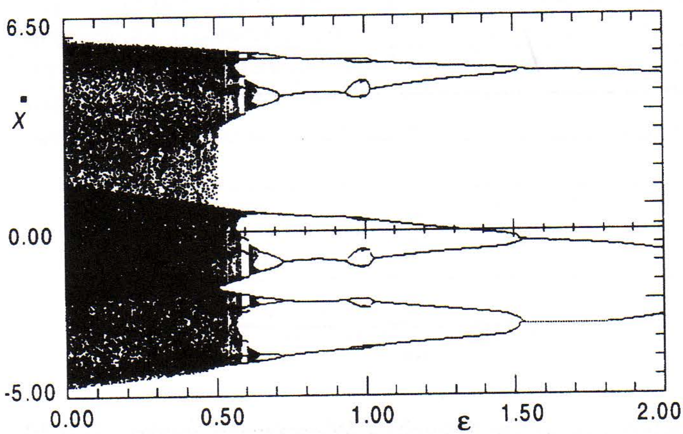


Fig.5. Bifurcation diagram showing the dynamics of the Duffing oscillator with dry friction described with the Popp – Stelter model (Fig.4) while normal force increases.

### *2.3 Non-reversible friction characteristics (Powell & Wiercigroch, Model III).*

This model assumes a non-reversible characteristics of the friction force (Fig.6) [4, 8, 10]. Here the friction force dependence on the relative velocity are described with two functions – first in the relative acceleration increase phase ($f_u$), and then in the phase when the relative acceleration decreases ($f_l$), respectively – Eq.(5)

$$f = \begin{cases} f_u, & \operatorname{sgn}(\ddot{x}) > 0 \\ f_l, & \operatorname{sgn}(\ddot{x}) < 0 \end{cases}$$

$$f_u = f_d\left[1 + \frac{f_s - f_d}{f_d}\exp\left(-a|\dot{x}|\right)\right]$$
$$f_l = f_d\left[1 - \exp\left(-b|\dot{x}|\right)\right] \tag{5}$$

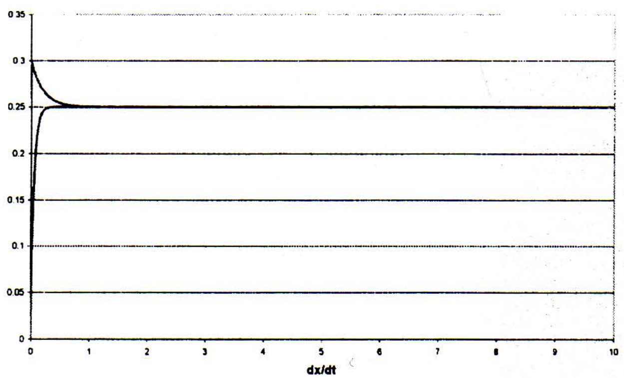


Fig. 6. Friction characteristics in the Powell – Wiercigroch model; $f_s$=0.30, $f_d$=0.25, $a$=5, $b$=20.

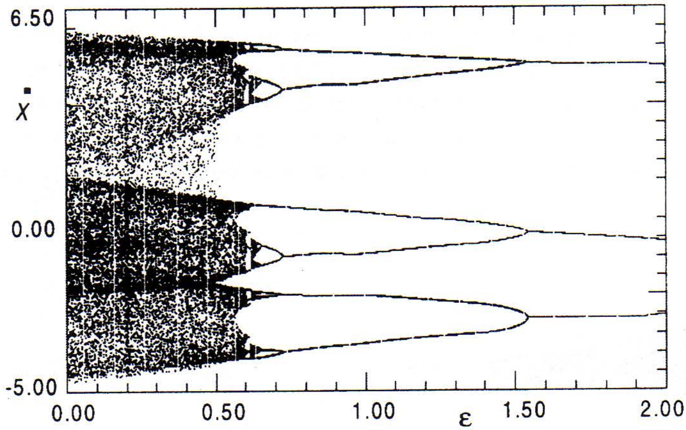


Fig.7. Bifurcation diagram presenting the dynamics of the Duffing oscillator with dry friction for the Powell – Wiercigroch model (Fig.6) while normal pressure increases.

### *2.4. Novel model of dry friction (Model IV).*

This model arises from the non-reversible model III, preserving the non-reversibility aspect with an addition saying that the friction characteristics is dependent not only on the relative acceleration sign but also on its value. Such dependence has been observed in the real experiment [6]. The proposed novel model consists of a symmetrical non-reversible characteristics with an auxiliary assumption that the parameter in the exponent is a function of the relative acceleration. The model is described with the following equations

$$f = \begin{cases} f_u, & \text{sgn}(\dot{x}) > 0 \\ f_l, & \text{sgn}(\dot{x}) < 0 \end{cases}$$

$$f_u = f_d\left[1 + \frac{f_s - f_d}{f_d}\exp\left(-a(\ddot{x})|\dot{x}|\right)\right] \qquad a(\ddot{x}) = \frac{a_1}{|\ddot{x}| + a_2}$$

$$f_l = f_d\left[1 - \frac{f_s - f_d}{f_d}\exp\left(-a(\ddot{x})|\dot{x}|\right)\right] \tag{6}$$

where $a_1$, $a_2 > 0$.

As the consequence of the introducing of the relative acceleration into the dry friction model a closer relation between the friction characteristics and the system dynamics occurs, and this is shown in Figure 8. Friction characteristics for different values of the $\varepsilon$ parameter and respective phase portraits are shown and clearly presents that friction function given by Eq.(6) is a picture of the system attractor [this sentence is unclear and requires a futher explanation]

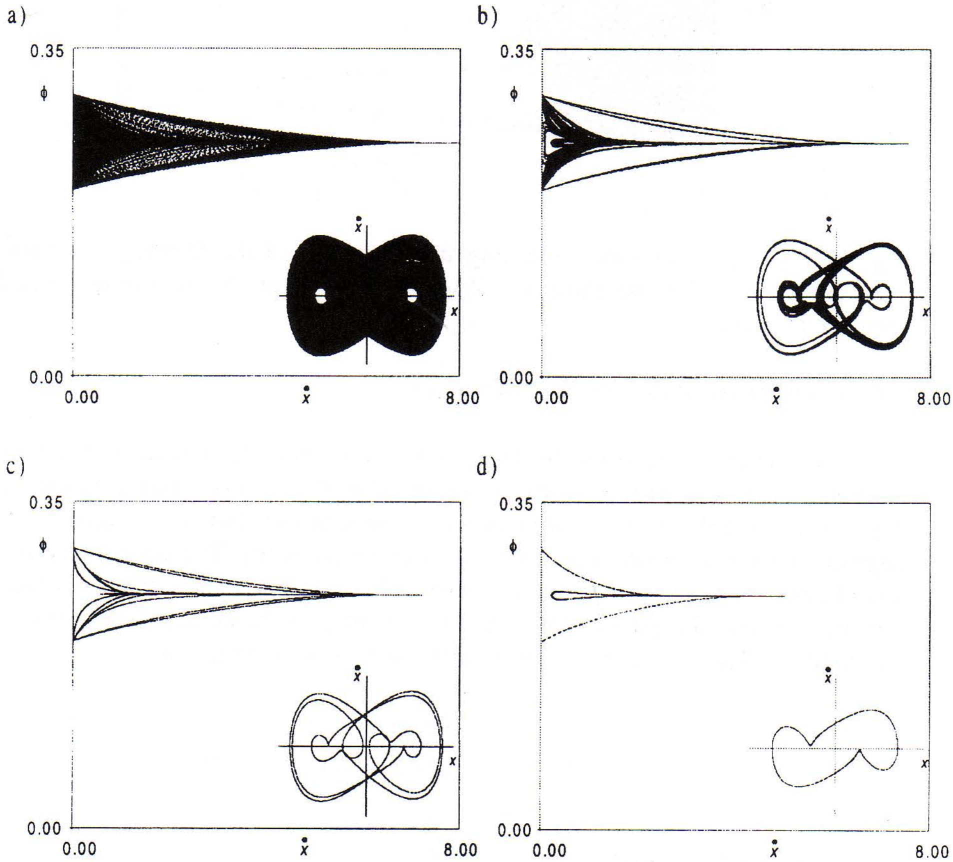


Fig.8. Friction characteristics for the Novel friction model and the phase portraits of the attractors for the assumed different ε parameter values; a) $\varepsilon$=0.1, b) $\varepsilon$=0.6, c) $\varepsilon$=2.0, d) $\varepsilon$=20.0; $f_s$=0.30, $f_d$=0.25, $a_1$=12, $a_2$=0.10.

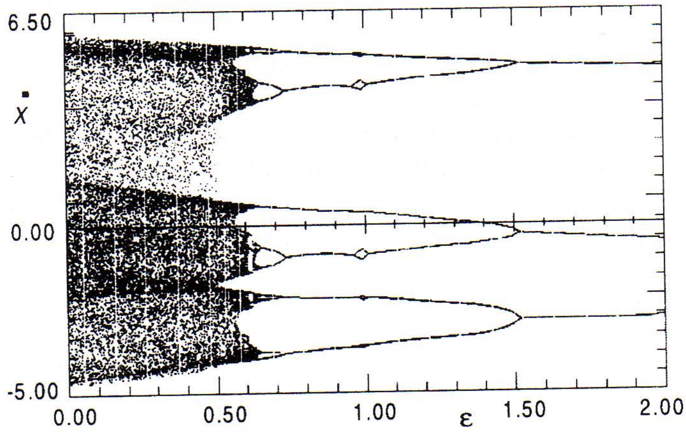


Fig.9. Bifurcation diagram presenting the dynamics of the Duffing oscillator with dry friction described by the Novel friction model (Fig.8) while normal pressure increases.

**3. Discussion and Conclusions.**

The basic aim of this study was the investigation of the influence of different friction models on the dynamical behaviour of a nonlinear mechanical system (Duffing oscillator). Four different dry friction models, which differ in a level of complication of the friction characteristics description are introduced and analysed by a means of bifurcation diagrams. The analysis was started with the simplest Coulomb model (Model I) and concluded with proposing a new model (Model IV), which in the authors' view might better reflect the dynamics of the system for the cases when the system oscillated around zero relative velocity. This is a subject of a separate study and will be reported later.

In the described numerical experiment the range of the friction coefficient changes is close to each other what makes the comparison of the results possible. It allows to draw a conclusion that any choice of the model results in a similar long-term dynamics of the system – the shapes of the bifurcation diagrams are almost the same (Figs 3, 5, 7, 9). All diagrams show the loss of stability of the chaotic solution into the periodic one takes place as a return period-doubling bifurcation. The most important factor for the system behaviour is the normal pressure parameter ε. A larger normal pressure means the larger friction force and higher energy dissipation level. The numerical experiment carried out shows that the increase of friction damping leads to the destruction of chaotic dynamics of the system and then the periodic solution appears.

The above analysis allows for the general conclusion – a friction model does not decide on the long-term dynamical behaviour of the Duffing oscillator with a dry friction, as the most important for modelling is a mean friction force level. Therefore, we can assume that for engineering applications the classical Coulomb approach is a good enough approximation of the real friction function if one is interested in examining the long-term dynamic responses.

**Acknowlegments**

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