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A METHOD OF MATERIAL MODELLING WITH THE USE OF STRENGTH HYPOTHESIS OF INNER EQUILIBRIUM STABILITY

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Summary The paper formulates a nonlinear model of physical properties of material, assuming its small strains. Based on strength hypothesis of stability of inner equilibrium the theoretical values of yield point and tensile strength have been determined for the material. The relationships obtained in the work served for determining material constants of the assumed model of the material.

Key words: *material modelling, nonlinear materials, material stability, limit analysis, strength of materials, strength hypothesis*

1. INTRODUCTION

The paper [7] formulates the strength hypothesis of inner equilibrium stability. It consists in assumption that the reason of destruction or failure of material is a loss of inner equilibrium stability of deformed material. According to the hypothesis examination of stability of deformation state of a material characterized by nonlinear physical properties allows theoretical determining the strength properties of the material and calculation the yield point, as well as tensile strength, depending on material parameters. Hence, for an assumed model of physical properties the material parameters may be defined based on experimentally determined values of the yield point and the tensile strength.

Experimental research performed for many materials of important technical meaning, as e.g. steel, cast-iron, non-ferrous metals and their alloys, confirmed linear elasticity of volumetric deformation within large range of the loads. For non-dilatational strain such a property is observed in smaller range. Therefore, a model might be established that would be able to depict the properties of many real materials ascertained in strength tests.

One of contemporary methods of defining physical properties of materials consists in expressing strain energy density as a function of invariants of the state of strain [4]. The method may be used for formulation of nonlinear model of material.

2. NONLINEAR MODEL OF PHYSICAL PROPERTIES OF MATERIAL

Invariants of state of strain in linear theory of elasticity [2] are given by the formulae

$$\begin{aligned} J_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z, \\ J_2 &= \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2), \\ J_3 &= \varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}\gamma_{xy}\gamma_{yz}\gamma_{zx} - \frac{1}{4}(\varepsilon_x\gamma_{yz}^2 + \varepsilon_y\gamma_{zx}^2 + \varepsilon_z\gamma_{xy}^2), \end{aligned} \tag{2.1}$$

where $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are the components of the state of strain.

The invariant J_1 represents relative change in volume of the body. If the body deformation is considered as a sum of pure volumetric strain and pure non-dilatational strain, then the invariants of equivalent pure volumetric strain of strain components in main directions

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon, \tag{2.2}$$

where

$$\varepsilon = \frac{1}{3}J_1, \tag{2.3}$$

have the values

$$J_1^{(v)} = 3\varepsilon = J_1, \quad J_2^{(v)} = 3\varepsilon^2 = \frac{1}{3}J_1^2, \quad J_3^{(v)} = \varepsilon^3 = \frac{1}{27}J_1^3. \tag{2.4}$$

Let $J_1^{(v)}, J_2^{(v)}, J_3^{(v)}$ be the components of generalized displacement related to the change in volume of deformed body. As show the equations (2.4), they depend only on the invariant J_1. Hence, taking into account the assumption

$$J_i = J_i^{(v)} + J_i^{(s)}, \quad i \in \{1, 2, 3\}, \tag{2.5}$$

the components of the generalized displacement related to the change of the form, are given by the formulae

$$J_i^{(s)} = J_i - J_i^{(v)}, \quad i \in \{1, 2, 3\}, \tag{2.6}$$

where the component $J_1^{(s)}$ is equal to zero. The values $J_2^{(s)}, J_3^{(s)}$ may be considered as the components of generalized displacement corresponding to pure change of the form. From this one could conclude that strain energy of linearly elastic material is a function of generalized displacements $J_1, J_2^{(s)}, J_3^{(s)}$, that finds confirmation in Helmholtz formulae, expressing the strain energy of linearly elastic material as a sum

$$W(J_1, J_2^{(s)}) = W^{(v)}(J_1) + W^{(s)}(J_2^{(s)}), \tag{2.7}$$

where the energy of volumetric strain is defined by the formula

$$W^{(v)}(J_1) = \frac{K}{2}J_1^2, \tag{2.8}$$

while the energy of non-dilatational strain is

$$W^{(s)}(J_2^{(s)}) = -2GJ_2^{(s)}. \tag{2.9}$$

The Helmholtz's constant K - the bulk modulus, known as well as volumetric modulus of elasticity, and the Kirchhoff's constant G - the modulus of rigidity, known under the name of shear modulus or Lame's constant μ, may be expressed by Young's modulus E and Poisson's ratio ν by means of the formulae

$$K = \frac{E}{3(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}. \tag{2.10}$$

For nonlinear material it may be assumed that physical properties are determined by the function of strain energy, in the form of a sum

$$W(J_1, J_2^{(s)}) = W^{(v)}(J_1) + \tilde{W}^{(s)}(J_2^{(s)}), \tag{2.11}$$

where the volumetric strain energy is given by the formula (2.8), while the non-dilatational strain energy differs only slightly from the function defined by the formula (2.9). The difference grows with increasing non-dilatational strain energy. Let the non-dilatational strain energy be expressed by the formula

$$\tilde{W}^{(s)}(J_2^{(s)}) = W^{(s)}(J_2^{(s)}) - \beta W_0 \left[\frac{W^{(s)}(J_2^{(s)})}{W_0}\right]^{\alpha}, \tag{2.12}$$

where material constants α and β are numbers, and W_0 is an optional reference value. α is a positive number not equal to 1, while W_0 is a physical value defined with the same units of measure as the function of energy density. Thus, a new model of a material of nonlinear properties is defined, for which the function of strain energy may be written in the shortened form

$$W = W^{(v)} + W^{(s)} - \eta W^{(s)\alpha}, \tag{2.13}$$

where

$$\eta = \beta W_0^{1-\alpha}. \tag{2.14}$$

The function, for β approaching zero, transforms into the function of strain energy of linearly elastic material.

Further part of the paper is devoted to investigation of stability of inner equilibrium of deformed material of physical properties defined by the function of strain energy expressed with the formula (2.12). Energetic definition of the so-called material stability may be found in the book of Jaunzemis [1] – page 505. Nonlinear theory of elasticity [8] applies the definition for searching for the constraints of material functions. On the other hand, the postulate of absolute minimum of total energy of a separated fragment of deformed body serves the purpose of essential justification of Coleman-Noll conditions and strong elipticity. A method of research of stability of the material model with the use of the conditions is presented in the papers [5, 6]. In the present paper the method of examination of stability described in detail in the book [7] will be applied. The method resolves into examination of the sign of second-order variation of the function of energy density. Therefore,

$$\delta W = \delta W^{(v)} + (1 - \alpha \eta W^{(s)\,\alpha-1}) \delta W^{(s)} \tag{2.15}$$

and

$$\delta^2 W = \delta^2 W^{(v)} + \delta^2 W^{(s)} - \alpha \eta W^{(s)\,\alpha-2} [(\alpha - 1)\delta W^{(s)2} + W^{(s)} \delta^2 W^{(s)})]. \tag{2.16}$$

Having determined values of function variation increase for six components of the state of strain and having assumed $\beta \leq 0$, one may find that for $\alpha > 1$ equilibrium state of the deformed material is stable and it does not depend on the state of strain. For the other cases stability of inner equilibrium of deformed material depends on energy of non-dilatational strain and requires an additional analysis. For this reason the value may be considered as a measure of the effort, as, after exceeding a certain level of the energy of non-dilatational strain, the state of material deformation becomes unstable and should be considered as dangerous.

In order to estimate the strength condition resulting from the present analysis and to define the level of energy of the non-dilatational strain leading to unstable state of inner equilibrium one should assume, for the sake of simplicity, that components of the strain are subject to disturbance, in proportion to their current values, i.e. it should be assumed that

$$\frac{\delta \varepsilon_x}{\varepsilon_x} = \frac{\delta \varepsilon_y}{\varepsilon_y} = \ldots = \frac{\delta \gamma_{zx}}{\gamma_{zx}} = \delta . \tag{2.17}$$

Let us examine stability of inner equilibrium with regard to the possibility of appearance a disturbed configuration defined by the condition (2.17). This is equivalent to the question, whether a spontaneous change in system configuration without supply of energy from outside is possible and, if so, under what condition, assuming that the change is subject to the conditions (2.17). Hence

$$\delta W^{(s)} = 2W^{(s)} \delta , \quad \delta^2 W^{(s)} = 2W^{(s)} \delta^2 , \quad \delta^2 W^{(v)} = 2W^{(v)} \delta^2 , \tag{2.18}$$

and from the relationship (2.16)

$$\delta^2 W = 2 [W^{(v)} + W^{(s)} - (2\alpha - 1)\alpha \eta W^{(s)\,\alpha}] \delta^2 . \tag{2.19}$$

Inner equilibrium will be unstable [7] if

$$W^{(v)} + W^{(s)} - (2\alpha - 1)\alpha \eta W^{(s)\,\alpha} \leq 0 , \tag{2.20}$$

or, taking into account the relationship (2.13)

$$W - (\alpha - 1)(2\alpha + 1)\, \eta W^{(s)\,\alpha} \leq 0 . \tag{2.21}$$

The condition is fulfilled for $\beta > 0$ and $\alpha > 1$ or $\beta < 0$ and $\alpha < 1$, if

$$W^{(s)} \geq W_{kr}^{(s)} , \tag{2.22}$$

where critical value

$$W_{kr}^{(s)} = \left[\frac{W}{(\alpha - 1)(2\alpha + 1)\eta} \right]^{\frac{1}{\alpha}} . \tag{2.23}$$

Hence, the state of strain is unstable and material may be damaged, when the energy of non-dilatational strain exceeds a critical value

$$\tilde{W}_{kr}^{(s)} = W_{kr}^{(s)} - \eta\, W_{kr}^{(s)\,\alpha}. \tag{2.24}$$

Attention should be paid to the formulation used. It does not mean that for less values of energy of non-dilatational strain the material is protected from breaking down. As the conclusion has been drawn having reduced the system to one degree of freedom, an experimental verification of the above formula becomes necessary. However, it was shown that a dangerous state of the material defined by the relationship (2.11) depends on the value of the energy of non-dilatational strain.

As it results from the formulae (2.23), (2.24), and (2.14) the value of the energy of non-dilatational strain conducive to instability of the state of strain depends on the coefficients α and β characterizing physical relationship for given material. The values serve as material constants and may be determined by laboratory examination of the material. The method of determination of these values is presented in the next chapter.

It could be supposed that local fulfillment of condition of loss of stability of inner equilibrium in the body of non-homogenous state of strain is conducive only to local change of state of strain and transition to a new stable state of equilibrium in given point and its environment. Nevertheless, such a state should be considered as dangerous from the point of view of the strength. On the other hand, the body of uniform state of strain that does not fulfill the condition of stability of inner equilibrium will be damaged or destroyed.

Defect of the material means temporary loss of its cohesion occurring in result of shift of adjacent material layers during the process of plastic flow, resulting in its durable deformation. The process is characterized with change in state of strain with conserved volume. This feature of the material will be used in the next chapter in order to determine the condition conducive to initiation of the process of plastic flow. In the end of the process the material is hardened that is related to blocking of its ability for further changes of the state of strain in result of sliding between the layers. Destruction of the material means durable loss of cohesion and consists in separation of material layers, resulting in a break or fracture in the material.

3. EXAMINATION OF STABILITY OF STRAIN STATE WITH IMPOSED CONSTRAINTS

In the case of axial tension, the unit elongations in main directions of the strain are given by the formulae

$$\varepsilon_2 = \varepsilon_3 = -\tilde{\nu}\varepsilon_1, \tag{3.1}$$

however, it should be noticed that in the material of nonlinear properties the value $\tilde{\nu}$ is not constant but depends on the elongation ε_1. If the relationships are considered as the constraints imposed on the system, analysis of the triaxial state of strain shows that the conditions (2.17) are met.

Invariants of the state of strain J_1, J_2 and the value $J_2^{(s)}$ are expressed with the formulae

$$J_1 = (1-2\tilde{\nu})\varepsilon_1, \quad J_2 = \tilde{\nu}(\tilde{\nu}-2)\varepsilon_1^2, \quad J_2^{(s)} = -\frac{(\tilde{\nu}+1)^2}{3}\varepsilon_1^2. \tag{3.2}$$

Calculating the functions according to the formulae (2.8) and (2.9) one assumes that ν is determined for small deformation in the material. Then

$$W^{(v)} = \frac{(1-2\nu)E}{6}\varepsilon_1^2, \quad W^{(s)} = \frac{(1+\nu)E}{3}\varepsilon_1^2. \tag{3.3}$$

In accordance to the condition (2.20) inner equilibrium is unstable, when

$$\frac{1}{2}E\varepsilon_1^2 - (2\alpha-1)\alpha\eta\left[\frac{(1+\nu)E}{3}\varepsilon_1^2\right]^{\alpha} \le 0. \tag{3.4}$$

Hence, for $\beta > 0$ and $\alpha > 0.5$ or $\beta < 0$ and $\alpha < 0.5$, theoretical value of the strain is

$$\varepsilon_m = \left\{\frac{\nu_0 E}{2}[(2\alpha-1)\alpha\eta\,\nu_0]^{\frac{1}{\alpha-1}}\right\}^{-\frac{1}{2}}, \tag{3.5}$$

where

$$\nu_0 = \frac{2(1+\nu)}{3}, \tag{3.6}$$

for which the loss of stability of inner equilibrium occurs in uniaxial tension test.

On calculating the derivative of the strain energy function given by the formula (2.13) with regard to the strain component ε_1 the stresses corresponding thereto may be determined as

$$\sigma_1 = \frac{\partial}{\partial\varepsilon_1}(W^{(v)}) + (1-\alpha\eta\,W^{(s)\alpha-1})\frac{\partial}{\partial\varepsilon_1}(W^{(s)}). \tag{3.7}$$

Taking into account the relationship (3.3) one gets

$$\sigma_1(\varepsilon_1) = [1-\alpha\eta\,\nu_0(\frac{\nu_0}{2}E\varepsilon_1^2)^{\alpha-1}]\,E\varepsilon_1. \tag{3.8}$$

Strain component ε_1 corresponding to maximal stress for the state of uniaxial tension may be determined from the necessary condition of the extremum

$$\frac{d\sigma_1}{d\varepsilon_1} = 0, \tag{3.9}$$

that leads to the relationship

$$E - 2(2\alpha-1)\alpha\eta\left[\frac{(1+\nu)E}{3}\right]^{\alpha}\varepsilon_1^{2(\alpha-1)} = 0. \tag{3.10}$$

Comparing the above condition for extremal stress in uniaxial tension to the condition of loss of internal equilibrium stability (3.4) one could remark that critical state shall occur for the strain corresponding to maximal stress R_m that may be determined during static tension test. It results that

$$R_m = \sigma_1(\varepsilon_m) = (1-\frac{1}{2\alpha-1})E\varepsilon_m. \tag{3.11}$$

The assumption expressed by the condition (2.17) is equal to ascertainment that for compressible materials the volume of disturbed system differs compared to its volume for initial state, as the following relationship occurs

$$\delta J_1 = J_1 \delta \neq 0. \tag{3.12}$$

On the other hand, if the condition (2.17) is replaced by

$$\delta J_1 = 0, \tag{3.13}$$

that is equivalent to additional constraints imposed on examined system in the form

$$J_1 = Const, \tag{3.14}$$

and results in reduction of the number of degrees of freedom from 6 to 5, the following relationships occurs

$$\delta J_2^{(s)} = \delta J_2, \quad \delta^2 J_2^{(s)} = \delta^2 J_2 \tag{3.15}$$

and

$$\delta W^{(s)} = -2G\delta J_2, \quad \delta^2 W^{(s)} = -2G\delta^2 J_2, \quad \delta^2 W^{(v)} = 0. \tag{3.16}$$

The most important among the assumptions of the theory of plasticity states that only the form of the body may undergo plastic changes [3]. Therefore, one can assume that plastic flow caused by sliding between material layers is related to deformations under constant volume.

Postulate (3.13) does not mean that the material is incompressible. It means that the subject of investigation is confined only to a particular state in which the material of any properties takes such a disturbed configuration that is characterized by the same volume as for its initial configuration. Such a behaviour of the material corresponds to its plastic flow. The material in this stage of deformation does not change its volume. Hence, it is assumed that the invariant J_1 remains constant, according to (3.14), only for approaching the condition for which the plastic flow of material is initiated.

Taking into account (2.16) one gets

$$\delta^2 W = 2G\{-\delta^2 J_2 - \alpha\eta\, 2G(-2GJ_2^{(s)})^{\alpha-2}\,[(\alpha-1)\delta J_2{}^2 + J_2^{(s)}\delta^2 J_2]\}. \tag{3.17}$$

The analysis will be limited to symmetric state of strain, according to the assumption

$$\varepsilon_2 = \varepsilon_3. \tag{3.18}$$

Then

$$J_2^{(s)} = -\frac{1}{3}(\varepsilon_1 - \varepsilon_2)^2. \tag{3.19}$$

Moreover, if (3.18) is considered as constraints imposed on the system, then, taking into account

$$\delta\varepsilon_2 + \delta\varepsilon_3 = -\delta\varepsilon_1, \tag{3.20}$$

resulting from (3.13), one gets

$$\delta\varepsilon_2 = \delta\varepsilon_3 = -\frac{1}{2}\delta\varepsilon_1, \tag{3.21}$$

and

$$\delta J_2 = -(\varepsilon_1 - \varepsilon_2)\delta\varepsilon_1, \quad \delta^2 J_2 = -\frac{3}{2}\delta\varepsilon_1^2, \tag{3.22}$$

while the relationship (3.17) takes a form

$$\delta^2 W = 3G\{1-(2\alpha-1)\alpha\eta[\tfrac{2}{3}G(\varepsilon_1-\varepsilon_2)^2]^{\alpha-1}\}\delta\varepsilon_1. \tag{3.23}$$

Let the state of strain in the moment of loss of stability be characterized by (3.1), for the coefficient ν approaching the number ν. Then the condition of inner instability may be expressed by inequality

$$1-(2\alpha-1)\alpha\eta\left[\frac{(1+\nu)E}{3}\varepsilon_1^2\right]^{\alpha-1} \le 0. \tag{3.24}$$

From there for $\beta > 0$ and $\alpha > 0.5$ or $\beta < 0$ and $\alpha < 0.5$ the strain amounts to

$$\varepsilon_e = \left\{\frac{\nu_0 E}{2}[(2\alpha-1)\alpha\eta]^{\frac{1}{\alpha-1}}\right\}^{-\frac{1}{2}}, \tag{3.25}$$

for which under uniaxial tension of the material the plasticizing is initiated. Taking into account the formula (3.8) one can determine theoretical value of plasticizing stress for the case of uniaxial tension

$$R_e = \sigma_1(\varepsilon_e) = (1-\frac{\nu_0}{2\alpha-1})E\varepsilon_e. \tag{3.26}$$

The ratio of R_e and R_m is equal to

$$r = \frac{R_e}{R_m} = [1+(1-\nu_0)p]\,\nu_0{}^p, \tag{3.27}$$

where the value ν_0 is given by (3.6), while

$$p = \frac{1}{2(\alpha-1)}. \tag{3.28}$$

Detailed analysis of the formula (3.27) provides many interesting general conclusions, important for practical purposes. It results from this, among others, that in case of ideally incompressible material the value of the ratio r is equal to one. It should be remembered that this conclusion is a result of certain simplification assuming that the material represents idealization of the properties never achieved. Therefore, it cannot be experimentally confirmed or denied, as there are no ideally incompressible material. Let us only confine to quote a particular value of exponent α, defining mathematical model expressed by the formula (2.12), calculated on base of above relation (3.27). Assuming Poisson's ration for aluminium $\nu = 0.34$ and the ratio $r = 0.53$, one gets $\alpha = 1.037$.

Assuming the module E as a reference value W_0 of the formula (2.12) gives

$$\eta = \beta E^{1-\alpha} \tag{3.29}$$

and the formulae (3.25), (3.5), (3.8) may be written in an abridged form

$$\varepsilon_e = (\frac{\nu_0}{2})^{-\frac{1}{2}} [(2\alpha - 1)\alpha\beta]^{-p}, \quad \varepsilon_m = (\frac{\nu_0}{2})^{-\frac{1}{2}} [(2\alpha - 1)\alpha\beta\nu_0]^{-p}, \tag{3.30}$$

$$\sigma_1(\varepsilon_1) = [1 - 2\alpha\beta(\frac{\nu_0}{2})^{\alpha} \varepsilon_1^{2(\alpha-1)}] E\varepsilon_1 . \tag{3.31}$$

Knowing additionally the value of one of characteristic stresses R_e or R_m measured during basic strength tests of the material and Young's modulus E determined for small deformation, it is simple, making use of (3.11) or (3.26), to calculate the value of coefficient β occurring in the formula (2.12). Assuming for aluminium $E = 0.72 \cdot 10^5$ MPa and $R_e = 50$MPa one gets $\beta = 1.39$. Hence, for many materials of technological meaning the values of moduli characterizing their nonlinear physical properties may be determined on the ground of uniaxial tension test. Knowing these moduli and making use of nonlinear model of the material determined by means of the function of inner energy density one can examine stability of any state of strain [7] and indicate dangerous regions of the body, with regard to their state of stress or strain.

4. CONCLUSIONS

Examination of inner equilibrium stability of deformed material is important for the process of mathematical modeling of physical properties of the material.

The paper shows that mathematical model of physical properties of material enables considering its strength properties. In order to achieve the goal should be used the methods of examination of stability of strain state. The material properties described by the model should be chosen with a view to achieving consistency between stable strain areas defined by the model and the ones found during the strength test.

In the nonlinear model proposed in the work only four material constants are assumed. In order to ensure more detailed representation of real properties of material a greater number of the constants might be implemented that would allow fulfillment of additional conditions. However, consideration of the conditions described in the paper and resulting from the analysis of stability of the model is necessary, as they issue from the law of conservation of energy, serving as a basis for investigation of material stability presented in the paper.

REFERENCES

[1] Jaunzemis W.: *Continuum mechanics*. Macmillan, New York 1967

[2] Kozłowski T.: *Outline of elasticity theory*. Arkady Publishers, Warszawa 1968 *(in Polish)*

[3] Walczak J.: *Strength of materials and fundamentals of elasticity and plasticity theory*. Vol II, PWN – Polish Scientific Publishers, Warszawa-Kraków 1973 *(in Polish)*

[4] Wegner T.: *Strain energy method of modelling and determining dynamic characteristics of mechanical elements with high damping*. Publishers of Poznań University of Technology, The Dissertation Series No 323, Poznań 1997 *(in Polish)*

[5] Wegner T.: Limitations of the range of applicability of Mooney's generalised function of strain energy resulting from Coleman-Noll and strong elipticity conditions.

Proceedings of the XXXVIII Symposium „Modelling in mechanics", Wisła 1999, Zeszyty Naukowe Katedry Mechaniki Stosowanej Politechniki Śląskiej – Publishers of Silesia University of Technology, No 10, p. 327-332, Gliwice 1999 *(in Polish)*

[6] Wegner T.: Comparative analysis of scope limitations of non linear models of compressible materials. *Studia i Materiały XLVIII*, The Technology Series, No 1, p. 159-173, Publishers of Tadeusz Kotarbiński Pedagogical College, Zielona Góra 1999 *(in Polish)*

[7] Wegner T.: *Strain energy methods in strength of materials research – The strength hypothesis of inner equilibrium stability.* Publishers of Poznań University of Technology, Poznań 1999 *(in Polish)*

[8] Wesołowski Z.: *Dynamic problems in nonlinear theory of elasticity.* PWN – Polish Scientific Publishers, Warszawa 1974 *(in Polish)*