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# NONLINEAR MHD ANALYSIS FOR FORCE-FREE MAGNETIC FIELD INSTABILITY OF MAGNETIZED FLUID CYLINDER

Ahmed E. Radwan
Mathematics Department, Faculty of Science
Qatar University
Doha, Qatar

**Abstract**

The linear and nonlinear instabilities of force-free magnetic fields of perfectly conducting fluid cylinder have been analyzed. Indeed, the instability of force-free magnetized flows poses a challenge to our physical understanding and mathematical skills. Equilibrium state is studied by taking Lundquist field (1951) and it is found that the fluid kinetic pressure is non-uniform. Normal mode analysis is utilized for investigating the linear perturbation technique while in nonlinear ones an iteration procedure (see Radwan (1989)) is used. The stability criterion is discussed analytically and numerically and the unstable domains are identified. Surprising results are obtained due to Lundquist Bessel model effect. The second order perturbation equations are derived and solved. Upon applying appropriate conditions, the coefficients control nonlinear theory are identified in general and in the case in which the force-free tenuous field is unperturbed in the linear analysis. Second order analysis does not alter in a direct way the unstable domains. However, it gives corrections and domain of validity for the linear analysis, since no linear theory can predict its domain of validity.

## 1. Introduction

The instability of a full liquid cylindrical jet endowed with surface tension has been a subject of research for more than a century. This is not only academic viewpoint but also for its wide applications in several domains of science. Plateau [1] was the first to obtain the instability critical wavelength experimentally and theoretically with primary methods. The decisive breakthrough came with Rayleigh [2] who developed an elegant mathematical technique for the break-up of the cylindrical jet; and he laid the theoretical foundation and also the concept of maximum mode of instability for such and similar problems. Several extensions, including dissipation and rotation, and all the early reported works have been summarized by Chandrasekhar [3]. A third of century has passed since the investigations of nonlinearities of capillary instability of a fluid jet by Yuen [4] , Wang [5] , Nayfeh [6] , and, finally, a complete analysis by Katukani et al [7]. Chandrasekhar [3] has investigated the influence of a constant magnetic field on the fluid capillary instability (see [3] p.542) by using the method of representing the physical variables in terms of poloidal and toroidal quantities. The stability of a full liquid jet with varying magnetic fields and other different cylindrical models has been investigated by Radwan [8, 9, 10, 11, 12].

Nowadays, in particular in the last decade, the scientific province and attention has been turned to more complicated problems than the naive one of Rayleigh [2] (cf. Uberoi et al [13] , Drazin et al [14] , Cheng [15] , Kendall [16] , Lin & Lain [17] and pioneering works of Radwan [11] and [12] ). All this work has slowly built up a subject of understanding the instability of pure hydrodynamic or/and MHD (with uniform magnetic field) models, but much remains to be discovered.

To the best of our knowledge, the analytical and numerical stability studies of a liquid cylinder under combined effect of surface tension and force-free magnetic field (Lundquist model fields [18] ) in the nonlinear version, have not been attempted so far, and this will be a scope of our study here.

## 2. Formulation of the problem

Force-free magnetic fields play an essential role in magnetohydrodynamic studies. They are characterized by the vanishing of the electromagnetic force.

In view of their practical applications in the astrophysical domain as well as more recently in plasma physics, they are worthwhile to be investigated and to find out their effects on the capillary stability of a full fluid cylinder. In the astrophysical domains: force-free magnetic fields are of interest for understanding the dynamical behavior of the spiral arms of galaxies and structure of the sun's corona cf. Boyd and Sanderson [19]. For some practical applications in Plasma physics: the force-free field is an important tool for several reasons. (i) Force-free fields are the most general equilibrium fields in the pressureless regions, see equations (12)- -(14). (ii) Force-free fields in which the current density is parallel to field, see equation (10) , have been the subject of considerable attention in a large scale of science. (iii) Force-free fields still appear to be reasonably good approximation for real magnetic fields in Laboratory experiments (Cambridge, Trieste, Garching, Culham,...etc). Furthermore, they can be studied in a rather detailed way from the mathematical point of view. Lundquist [18] was the first to study the equilibrium state of the force-free fields. The force-free magnetic field (14) is that of Lundquist [18].

The main purpose of the present work is to investigate the linear and nonlinear MHD stability of a fluid cylinder under combined influence of the capillary, electromagnetic force-free field and pressure gradient forces. The fluid cylinder is pervaded by homogeneous magnetic field (15) while the pressureless tenuous region surrounding the field cylinder is pervaded by the Lundquist force-free field (14). The capillary force acts along the fluid cylinder boundary surface. The fluid cylinder of radius $R_0$ in the equilibrium state is considered to be incompressible, non-viscous and perfectly conducting.

## 3. Basic equations

The basic MHD equations express the coupling between the electromagnetism Maxwell's equations and pure hydrodynamic ones (equations of surface tension and motion with Lorentz force) together with Ohm's law. For the present problem in which the fluid cylinder medium is free from resultant free charges and characterized by slowly varying field, these equations are

$$\rho(\frac{\partial}{\partial t} + (\underline{u}.\nabla))\underline{u} = -\nabla p + \mu(\underline{J} \wedge \underline{H}) \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla . \underline{u}) = 0 \qquad (2)$$

$$p_s = T(\nabla . \underline{\hat{n}}_s) = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad (3)$$

$$\nabla \wedge \underline{E} + \mu \frac{\partial \underline{H}}{\partial t} = 0 \qquad (4)$$

$$\nabla . \underline{H} = 0 \qquad (5)$$

$$\nabla . \underline{E} = 0 \qquad (6)$$

$$\nabla \wedge \underline{H} = \mu \underline{J} \qquad (7)$$

$$\underline{J} = S\,(\underline{E} + \mu(\underline{u} \wedge \underline{H})) \qquad (8)$$

Here $p$ is the fluid kinetic pressure, $\underline{u}$ the velocity vector, $T$ the surface tension coefficient, $p_s$ the pressure surface due to the capillary force, $R_1$ and $R_2$ the principle radii of cylindrical curvature, $\underline{\hat{n}}_s$ the unit outward normal vector to the boundary surface of the cylinder $f(r, \theta, z; t) = 0$ given by

$$\underline{\hat{n}}_s = \nabla f(r, \theta, z; t) / \left|\nabla f(r, \theta, z; t)\right| \qquad (9)$$

$\mu$ the magnetic permeability, $\underline{J}$ the electric current density, $\underline{H}$ the magnetic field intensity, $\underline{E}$ the electric field intensity and $S$ the electric conduction.

In the tenuous medium region surrounding the fluid cylinder, the force-free magnetic field exists, this is characterized by the vanishing of the electromagnetic (Lorentz) force

$$\mu(\underline{J} \wedge \underline{H}) = 0 \qquad (10)$$

Equation (10) may be rewritten as

$$(\nabla \wedge \underline{H}) \wedge \underline{H} = 0 \qquad (11)$$

This means that in those regions (where a force-free field exists), neither the magnetic field nor the current $\underline{J}$ vanishes but the currents must flow parallel to the magnetic field. Equation (11) may then be replaced by the relation

$$(\nabla \wedge \underline{H}^f) = \eta \underline{H}^f \qquad (12)$$

where the superscript f over $\underline{H}$ signifies the force-free magnetic field and $\eta$ is a force-free parameter having dimension of reciprocal of length, see equation (14), and satisfies some restrictions, see equation (17). Equation (12) with the conservation of flux

$$\nabla . \underline{H}^f = 0 \tag{13}$$

constitute the basic equations of the force-free field where the pressureless tenuous regions exist.

## 4. Equilibrium state

In investigating the problem at hand, we use cylindrical polar coordinates $(r, \theta, z)$ system with z-axis coinciding with the axis of the cylinder. In the present case of equilibrium state the different fluid variables are independent of $\theta, z$ and $t$. Equations (12) and (13) are solved for the tenuous medium surrounding the fluid cylinder. The solution with cylindrical and longitudinal symmetries $\partial/\partial\theta = 0$ and $\partial/\partial z = 0$, is given by

$$\underline{H}_0^f = (0, Y_1(\eta r), Y_0(\eta r))\, H_0 \tag{14}$$

where the subscript o denotes equilibrium quantities. $Y_1(\eta r)$ and $Y_0(\eta r)$ are the ordinary second-kind Bessel functions of orders one and zero, respectively, and $H_0$ is the intensity of the magnetic field vector

$$\underline{H}_0 = (0, 0, H_0) \tag{15}$$

pervaded in the interior of the cylindrical liquid jet. The magnetic field (14) represents the so-called Bessel function model, see Lundquist [18]. It is worthwhile to mention here that the magnetic field which is considered to be penetrated in an ordinary vacuum surrounding the fluid cylinder (see Chandrasekhar [3] and Radwan [12]), can be obtained from (14) by inserting $\eta = 0$ into equation (12). Moreover, with appropriate choices the solution (14) may be replaced by different vacuum magnetic fields which have been utilized by Radwan [19]- -[12]. Upon applying the pressure balance condition across the fluid-force regions interface at $r = R_0$ , the pressure distribution $p_0$ of the fluid in the unperturbed state is identified. It is given by

$$p_0 = \frac{T}{R_0} + (\frac{\mu H_0^2}{2})(Y_1^2(\beta) + Y_0^2(\beta) - 1) \tag{16}$$

where $\beta(=\eta R_0)$ is a dimensionless force-free wave number. The first term $(\frac{T}{R_0})$ on the right side of (16) represents the contribution of the capillary force. The term $(-\mu\frac{H_0^2}{2})$ represents the contribution of the acting electromagnetic force in the interior of the fluid, while the terms including $Y_1^2$ and $Y_0^2$ are due to the electromagnetic force-free magnetic field. Physically speaking, $p_0$ must be positive, so in postulating that the body forces are predominant and overcoming the capillary force, $\beta$ must satisfy the restrictions

$$\left(Y_1^2(\beta)+Y_0^2(\beta)\right) \geqslant 1 \tag{17}$$

where the equality holds for a limiting case of zero fluid pressure.

## 5. Perturbation analysis

For small departures from the equilibrium state, every fluid variable $Q(r,\theta,z;t)$ describing the motion of the fluid can be expressed by a series (cf.Radwan [24])

$$Q(r,\theta,z;t)=\sum_{n=0}\epsilon^n Q_n(r,\theta,z;t) \tag{18}$$

where $Q$ stands for $\underline{u}$, $p$, $\underline{H}$, $\underline{n}_s$, $p_s$ and $\underline{H}^f$ and the perturbed radial distance of the cylinder. Here $\varepsilon$ is the amplitude of the linear terms at time $t$, expressed as (cf. Chandrasekhar [3]):

$$\varepsilon=\varepsilon_0\exp(\sigma t) \tag{19}$$

where $\varepsilon_0=\varepsilon(0)$ is the initial amplitude and $\sigma$ is the temporal amplification. The deformed (fluid-force free regions) interface, based on the linear perturbation technique, could be described by

$$r=R_0+\varepsilon\Re_1 \tag{20}$$

with

$$\Re_1=R_0\cos(kz)\cos(m\theta) \tag{21}$$

is the elevation of the surface wave measured from the unperturbed position where $k$ and $m$ are the longitudinal and azimuthal wave numbers..

Substitution of the series expansion (18) into the basic equations (1)- -(9) yields, by identification of the powers of $\varepsilon$, hierarchy of coupled systems of lower order. The system of zero order corresponds to equilibrium already studied, the first order is pertaining to the linearized system and the second and higher to nonlinear systems. By taking into account that the fluid is perfectly conducting, inviscid and incompressible, the relevant linear perturbation equations of motion of the fluid are given by

$$\frac{\partial \underline{u}_1}{\partial t} = \frac{\mu}{\rho}(\underline{H}_0.\nabla)\underline{H}_1 - \nabla\xi_1 \tag{22}$$

$$\xi_1 = \frac{p_1}{\rho} + (\frac{\mu}{2\rho})(\underline{H}.\underline{H})_1 \tag{23}$$

$$\nabla.\underline{u}_1 = 0 \tag{24}$$

$$\frac{\partial \underline{H}_1}{\partial t} = \nabla \wedge (\underline{u}_1 \wedge \underline{H}_0) + \nabla \wedge (\underline{u}_0 \wedge \underline{H}_1) \tag{25}$$

$$\nabla.\underline{H}_1 = 0 \tag{26}$$

$$p_{1s} = -\frac{T}{R_0^2}\left(\Re_1 + \frac{\partial^2 \Re_1}{\partial \theta^2} + R_0^2 \frac{\partial^2 \Re_1}{\partial z^2}\right) \tag{27}$$

where equation (25) is originally obtained upon combining equations (4)- -(8), $\rho\xi_1$is the total MHD pressure which is the sum of the fluid kinetic pressure and magnetic pressure acting in the interior of the fluid cylinder.

From the point view of the space-time dependence (19) and (21) and based on the linear perturbation technique (see Radwan's [12] recent work say), the perturbation equations (22)- -(27) are solved. As a result, apart from singular solutions, their finite solutions as $r$ tends to zero for axisymmetric perturbation ($\frac{\partial}{\partial\theta} = 0$) are given by

$$\underline{u}_1 = \frac{-\sigma A}{(\sigma^2 + \Omega_A^2)} \nabla(I_0(kr)\cos kz) \tag{28}$$

$$\underline{H}_1 = \frac{kH_0 A}{(\sigma^2 + \Omega_A^2)} \nabla(I_0(kr)\cos(\frac{\pi}{2} - kz)) \tag{29}$$

$$\xi_1 = AI_0(kr)\cos kz \tag{30}$$

$$p_{1s} = \frac{-T}{R_0^2}(1 - k^2 R_0^2)\cos kz \tag{31}$$

Here $A$ is an unspecified constant of integration, $I_0(kr)$ is the modified Bessel function of the first kind of order zero and

$$\Omega_A = (\frac{\mu H_0^2 k^2}{\rho})^{\frac{1}{2}} \tag{32}$$

is the Alfven wave frequency defined in terms of $H_0$.

Using the series expansions (18) for the force-free magnetic field equations (12) and (13), the relevant perturbation equations for the tenuous region surrounding the fluid cylinder are

$$\nabla \wedge \underline{H}_1^f = \eta \underline{H}_1^f \tag{33}$$

$$\nabla . \underline{H}_1^f = 0 \tag{34}$$

These equations, based on the linear perturbation technique, are solved in a general case then on a physical basis the singular solutions as $r$ tends to infinity are excluded.

The components of $\underline{H}_1^f$ are given by

$$H_{1r}^f = H_{1r}(r) \sin kz \tag{35}$$

$$H_{1\theta}^f = H_{1\theta}(r) \cos kz \tag{36}$$

$$H_{1z}^f = H_{1z}(r) \cos kz \tag{37}$$

Here, the amplitudes $H_{1r}(r)$, $H_{1\theta}(r)$ and $H_{1z}(r)$ are

$$H_{1r}(r) = B\frac{kK_0'(\xi)}{\sqrt{k^2 - \eta^2}} \tag{38}$$

$$H_{1\theta}(r) = B\frac{K_0'(\xi)}{\sqrt{k^2 - \eta^2}} \tag{39}$$

$$H_{1z}(r) = BK_0(\xi) \tag{40}$$

in terms of

$$\xi^2 = (k^2 - \eta^2)r^2 \tag{41}$$

where $B$ is an arbitrary constant, $K_0(\xi)$ is the modified Bessel function of the second kind of order zero and the prime over it denotes derivative with respect to the argument.

The constants $A$ and $B$ can be determined by applying the compatibility condition of the normal component of the velocity and the continuity condition of the normal component of the magnetic field across the interface of the fluid-force free regions at $r = R_0$. Doing this, we find

$$A = -R_0^2 \frac{(\sigma^2 + \Omega_A^2)}{x\, I_0'(x)} \tag{42}$$

$$B = -H_0\, y\, (\frac{Y_0(\beta)}{K_0'(y)}) \tag{43}$$

Here $y(= (x^2 - \beta^2)^{1/2})$ is given by (41) at $r = R_0$ and

$$x = kR_0 \tag{44}$$

where x and y are the dimensionless ordinary and force-free longitudinal wave numbers, respectively.

Moreover, we have to apply the boundary condition that the normal component of the total stress tensor due to electromagnetic forces in the interior and exterior of the fluid cylinder must be discontinuous due to the capillary force across the fluid boundary surface at $r = R_0$. Consequently, we end up with the eigenvalue relation

$$\sigma^2 = \sigma^2(T) + \sigma^2(H_0) \tag{45}$$

with

$$\sigma^2(T) = \frac{T}{\rho R_0^3} \frac{x\, I_0'(x)}{I_0(x)} (1 - x^2) \tag{46}$$

$$\sigma^2(H_0) = \frac{\mu H_0^2}{\rho R_0^2} \left\{ \frac{x\, I_0'(x)}{I_0(x)} \left[ \frac{yK_0(y)}{K_0'(y)} Y_0^2(\beta) + 3\beta\,(Y_0(\beta)Y_1(\beta)) - Y_1^2(\beta) \right] - x^2 \right\} \tag{47}$$

### 6. Discussions

#### 6.1. General comments

As a general statement the dispersion relation (45) with (46) and (47) tells us whether the model is stable or not and eventually determines the critical value of the wave number that separates the stable states from those of instability. The marginal stability states can be obtained such that $\sigma = 0$. The dispersion relation (45) includes the most of informations about stability. It contains the entity of $(\frac{\rho R_0^3}{T})^{1/2}$ as well as $(\frac{\rho R_0^2}{\mu H_0^2})^{1/2}$ as a unit of time, a dimensionless wave number $x$, dimensionless force-free longitudinal wave number $y$ and the parameter $\eta$ which is the current-field ratio normalized with respect to $R_0$. The dispersion relation (45) contains also like the other problems of other cylindrical configurations subject to different forces, see Radwan [11] and [12]: the modified Bessel functions of the first and second kinds of order zero and their derivatives but here with argument $x$ or $y$ instead of $x$ only in those problems. Moreover (45) contains here the ordinary Bessel functions $Y_0(\beta)$ and $Y_1(\beta)$ of first kind of orders zero and unity. Careful and extensive investigations for the stability criterion (45) are needed to identify stability and instability states

### 6.2. Magnetodynamic stability discussions

The results of discussions of the present flow are new. Such results come out in investigating the dispersion relation (45) at $T = 0$ or more precisely in investigating the relation (47).

Here it is found more plausible that we analyze the dispersion relation (47) for various cases of interest. We make discussions for three different categories, where in each one the fluid cylinder is pervaded by homogeneous magnetic field.

Category (i)

We consider here that an axial force-free magnetic field only, pervades in the tenuous region surrounding the fluid cylinder, i.e, we put $Y_1 = 0$ in (47).

The influence of the magnetic field in the interior of the fluid cylinder is represented by the term $(-x^2)$. It has always a stabilizing effect. The influence of the axial force-free magnetic field in this case is represented by the term $xy\left[I_0'(x)K_0(y)Y_0^2(\beta)/(I_0(x)K_0'(y))\right]$

We can show, by the aid of the recurrence relations of the modified Bessel functions (see Abramowitz and Stegun [20]), that it has always a stabilizing effect.

Therefore, we conclude that a fluid cylinder pervaded by a homogeneous magnetic field and surrounded by an axial force-free magnetic field, is stable for all non-zero real values of $x$, $y$ and $\beta$.

Category (ii)

We consider here that the fluid cylinder surrounded by transverse force-free magnetic field , i.e, we put $Y_0 = 0$ in (47). Following the same steps as have been done for category (i), one can show that the model in such a case with $Y_0 = 0$ is also stable for all non-zero real values of $x$, $y$ and $\beta$.

Category (iii)

Here, we consider the general case in which the fluid cylinder is pervaded by the magnetic field $(0, 0, H_0)$ and surrounded by the force-free magnetic field $(0, Y_1(\eta r), Y_0(\eta r))H_0$. The dispersion relation of the present case is given by the relation (47) including all the terms concerning axial and transverse force-free fields.

The influence of the homogeneous magnetic field in the interior of the fluid cylinder is still represented by the term $(-x^2)$ and it has a stabilizing effect. This effect is valuable for all short and long wavelengths. The influence of the azimuthal force-free field is represented by the terms in $Y_1$. It has also a stabilizing influence via the term $Y_1^2(\beta)$, while the effect of the mixed term containing $Y_0(\beta)$ , $Y_1(\beta)$ is found to be stabilizing in regions not neighboring and idem for the destabilizing regions. So the principle of exchange of instability is valid in contrary to the problems in which uniform or even nonuniform magnetic fields are pervading into the fluid where ordinary stable and unstable domains only are found. In general, we can see that the Lundquist force-free magnetic field (see equation (14)) has a stabilizing effect if the following restrictions are satisfied

$$(Y_1^2(\beta) + \frac{yK_0(y)}{K_1(y)}Y_0^2(\beta)) \geq 3\beta\,(Y_0(\beta)Y_1(\beta)) \qquad (48)$$

otherwise it is destabilizing. Consequently a fluid cylinder under the combined effect of the pressure gradient force, electromagnetic forces in the interior and exterior of the fluid cylinder, is stable for each non-zero real values of $x$ and $y$ if, and only if

$$\left(x^2 + \frac{yK_0(y)Y_0^2(\beta)}{K_1(y)} + Y_1^2(\beta)\right) \geq 3\beta\,(Y_0(\beta)Y_1(\beta)) \qquad (49)$$

$$\beta > 0 \qquad (50)$$

are satisfied and vice versa.

### 6.3. Hydrodynamic stability

The hydrodynamic oscillation and instability of the present fluid cylinder, as only capillary force is acting on it, could be identified through the investigation of the dispersion relation (45) with $H_0 = 0$. The discussions reveal that the fluid jet is unstable in the domain $x < 1$ while it is stable in all other neighboring domains $x \geq 1$, where $x = 1$ represents the marginal stability $\sigma = 0$. This means that the model is stable as long as the perturbed wavelength $\lambda (= 2\pi/k)$ is shorter than the circumference $2\pi R_0$ of the perturbed cylinder and vice versa.

### 6.4. Hydrodynamic discussions

In such a general case the fluid cylinder model is under the combined effect of the capillary force, electromagnetic force-free field outside the fluid cylinder and the ordinary Lorentz body force in the interior of the cylinder, in addition to the fluid pressure gradient force.

The instability could be identified upon investigating equation (45) in its general form. The instability results of this case could be obtained by combining the results of the particular (6.2 ) and (6.3). In order to clarify and confirm the analytical results and to examine the effect of the force-free magnetic field on the capillary instability, the non-dimensional eigenvalue relation.

$$\begin{aligned} \frac{\sigma^2}{T/\rho R_0^3} &= \frac{x\, I_0'(x)}{I_0(x)}(1 - x^2) \\ &+ \left(\frac{H_0}{H_s}\right)^2 \left\{ -x^2 + \frac{x\, I_0'(x)}{I_0(x)} \left[ Y_1^2(\beta) + 3\beta (Y_0(\beta) Y_1(\beta) + \frac{y K_0(y) Y_0^2(\beta)}{K_0'(y)} \right] \right\} \end{aligned} \tag{51}$$

has been computed on a computer, where $H_s$ is defined by

$$H_s^2 = \frac{T}{\mu R_0} \tag{52}$$

The calculation have been already performed for different values of $(H_0/H_s)$ and $\beta (= \eta R_0)$. The numerical data are collected and presented graphically. There are many features and properties in these numerical illustrations.

For $(H_0/H_s) = 0$ , it is found for all values of $\beta$ that the model is unstable only in the domain $0 < x < 1$ and stable otherwise.

$$\rho\xi_2 = p_2 + \frac{\mu}{2}(\underline{H}.\underline{H})_2 \tag{54}$$

$$\nabla.\underline{u}_2 = 0 \tag{55}$$

$$2\sigma\underline{H}_2 = \nabla \wedge (\underline{u}_2 \wedge \underline{H}_0) + \nabla \wedge (\underline{u}_1 \wedge \underline{H}_1) + \nabla \wedge (\underline{u}_0 \wedge \underline{H}_1) \tag{56}$$

$$\nabla.\underline{H}_2 = 0 \tag{57}$$

$$2\sigma\Re_2 = u_{2r} + \Re_1\frac{\partial u_{1r}}{\partial r} - \frac{u_{1\theta}}{R_0}\frac{\partial \Re_1}{\partial \theta} - u_{1z}\frac{\partial \Re_1}{\partial z} \tag{58}$$

where the influence of the capillary force is neglected here in the present study of second order for simplicity.

Exterior of the fluid cylinder

$$\nabla \wedge \underline{H}_2^f = 2\eta\underline{H}_2^f \tag{59}$$

$$\nabla.\underline{H}_2^f = 0 \tag{60}$$

The first order velocity vector and the magnetic field are gradients of scalar functions, see equations (28) and (29), moreover the driving force is a gradient. By the use of the Lagrangian extension theory, see Callebaut [21] and Radwan [24], [25], we are able to show that the velocity vector $\underline{u}_2$ is given by

$$\underline{u}_2 = \nabla\Psi_2 \tag{61}$$

such that

$$\Psi_2 = \frac{\sigma\, R_0^2\, a}{k\, I_0'(2x)}\, I_0(2kr)\cos(2kz) \tag{62}$$

where $a$ is a dimensionless second order coefficient. Substitution of (21), (28), (30), (42), (61) and (62) into the equation of motion of the deformed fluid-force free regions interface (58), the second order elevation of the surface wave $\Re_2$ is given by

$$\Re_2 = \varepsilon^2 R_0(A_2\cos(2kz) - \frac{1}{4}) \tag{63}$$

where

$$A = a - \frac{1}{4} + \frac{1}{2}\frac{xI_0(x)}{I_0'(x)} \tag{64}$$

For $(H_0/H_s) = 0.1$ , it is found that the model is MHD stable for $\beta$-values : $0.01, 0.03, 0.05$ and $0.07$. In such case, for $\beta = 0.09$, it is found that the model is MHD unstable in the domain $0 < x < 0.7$ and stable in the other domain $0.7 \leq x < \infty$. This means that the unstable domains are little bit increasing with increasing $\beta$ values as $(H_0/H_s) = 0.1$ .

For $(H_0/H_s) = 0.2$ , it is found that the model is stable for $\beta$-values : $0.01, 0.03, 0.05, 0.07$ and $0.09$ , and there is no unstable domain. However the stability domains are decreasing with increasing values of $\beta$ in the range $0 < \beta < 1.0$ . This mean that $\beta$ plays an important role in shrinking the stable domains .

For $(H_0/H_s) = 1.0$ , i.e., strong magnetic field is pervaded in this case which is more than the previous cases, it is found that the model is MHD stable for $\beta$-values : $0.01, 0.03, 0.05, 0.07$ and $0.09$. However the MHD stability character is very strong and no expected unstable domains (whatever is the large value of $\beta$) occur and the stability is never suppressed.

From the foregoing discussions we conclude the following.
In the absence of the magnetic field, i.e. , $H_0 = 0$, $\beta = 0$, the capillary force has stabilizing effect in the domain $1 \leq x < \infty$ and destabilizing effect in the domain $0 < x < 1$. The magnetic field is strongly stabilizing while the force-free magnetic field is destabilizing the fluid cylinder model, however as the magnetic field is very strong, the (force free) $\beta$-destabilizing effect will never be overcoming the stabilizing influence of the ordinary magnetic field penetrated in the fluid medium.

## 7. Nonlinear perturbation analysis

We carry out a kind of nonlinear perturbation analysis introduced earlier by Callebaut [21] and utilized by Radwan [24], [25] for investigating perturbation of different models. Inserting the series development (18) with (19) into the basic equations (1)- -(9) for the interior of the fluid cylinder and equations (12), (13) for the tenuous medium in the exterior of the fluid cylinder, one obtain the second order perturbation equations as follows.

Interior of the fluid cylinder

$$2\sigma \underline{u}_2 + (\underline{u}_1.\nabla)\underline{u}_1 - \frac{\mu}{\rho}\left((H_0.\nabla)\underline{H}_2 + (\underline{H}_1.\nabla)\underline{H}_1\right) = -\nabla\xi_2 \qquad (53)$$

Appropriate substitution of (56) and the results of the first order perturbation into (53) and then integration of the resulting expression yields

$$-\xi_2 = 2\sigma\Psi_2\left(1+\frac{\Omega_A^2}{\sigma^2}\right)+\frac{1}{2}(\underline{u}_1.\underline{u}_1)-\frac{\mu}{2\rho}(\underline{H}_1.\underline{H}_1)-C_{2\xi} \tag{65}$$

Alternatively, the second order magnetodynamic MHD pressure $\rho\xi_2$ for the interior of the fluid cylinder is given by

$$\begin{aligned}-\xi_2 &= R_0^2(\sigma^2+\Omega_A^2)\left\{\tfrac{2a}{xI_0'(2x)}I_0(2kr)+\tfrac{1}{4}\left[\tfrac{(I_0'(kr))^2}{I_0^2(kr)}-1\right]\right\}\cos(2kz)\\ &+\tfrac{R_0^2}{4(I_0'(x))^2}(\sigma^2-\Omega_A^2)\left((I_0'(kr)^2+I_0^2(kr)\right)-C_{2\xi}\end{aligned} \tag{66}$$

where $C_{2\xi}$ is an arbitrary constant of integration. Moreover, the second order magnetic field in the interior of the fluid cylinder, on utilizing (61), (62) and the explicit forms for $\underline{u}_1$ and $\underline{H}_1$ , is given in the form

$$\begin{aligned}\underline{H}_2 &= \left\{\tfrac{-2H_0\,xa}{I_0'(2x)}I_0'(2kr)\sin(2kz)\right\}\underline{e}_r+\{0\}\,\underline{e}_\theta\\ &+\left\{\tfrac{-2H_0\,xa}{I_0'(2x)}I_0(2kr)\cos(2kz)+\tfrac{R_0H_0\,x}{2(I_0'(x))^2}(kr\left((I_0'(x))^2+I_0^2(kr)\right)\right\}\underline{e}_z\end{aligned} \tag{67}$$

Keeping in view the space-dependence, the second order perturbation equations (59) and (60) for the force-free tenuous region surrounding the fluid cylinder are solved. The components of the second order perturbed force-free magnetic field $\underline{H}_2^f$ are given by

$$H_{2r}^f(r,z) = H_{2r}(r)\sin(2kz) \tag{68}$$

$$H_{2\theta}^f(r,z) = H_{2\theta}(r)\cos(2kz) \tag{69}$$

$$H_{2z}^f(r,z) = H_{2z}(r)\cos(2kz) \tag{70}$$

such that

$$H_{2r}(r) = \frac{C}{(k^2-\eta^2)^{\frac{1}{2}}}\,kK_0'(2\xi) \tag{71}$$

$$H_{2\theta}(r) = \frac{C}{(k^2-\eta^2)^{\frac{1}{2}}}\,K_0'(2\xi) \tag{72}$$

$$H_{2z}(r) = CK_0(2\xi) \tag{73}$$

here $C$ is an arbitrary function of integration and will be a complicted function of the ordinary and modified Bessel functions, $\xi$ (defined by (41)), $K_0(2\xi)$ is the second kind modified Bessel function of order zero and the prime over it means derivative with respect to argument. Noting that the first kind modified Bessel function is omitted, since the solutions (68)- -(73) must be finite as $r$ tends to infinity. The solution (68)- -(73) are totally different and also incisive and tremendous from those obtained recently by Radwan [22]- -[25] and from the naive ones of Callebaut [21].

To determine $C$ , we have to apply the continuity of the normal component of the magnetic field across the fluid-force free regions interface at $r = R_0$. After some lengthy and cumbersome calculations we get

$$C = \frac{R_0 y H_0 Y_1(\eta R_0)}{2K_0'(2y)} + \frac{y H_0 Y_0(\eta R_0)}{K_0'(2y)} - \frac{y K_0(y)}{K_0'(2y)} - 2a - \frac{x I_0(x)}{I_0'(x)} \tag{74}$$

Now applying the boundary condition that the normal component of the total stress tensor must be continuous across the fluid-force free regions interface at the unperturbed position $r = R_0$ , we get

$$(\rho\xi)_2 = \left(\frac{\mu}{2}(\underline{H}^f.\underline{H}^f)\right)_2 \tag{75}$$

taking into account that the interface is displaced with respect to the unperturbed boundary surface $r = R_0$ , additional terms should be added if we take the value at $r = R_0$. Using Taylor expansion, any physical quantity $\zeta_2$ in second order is expressed as

$$(\zeta)_2 = \zeta_2(R_0) + \Re_1 \frac{\partial \zeta_1}{\partial r} + \Re_2 \frac{\partial \zeta_0}{\partial r} + \frac{1}{2}\Re_1^2 \frac{\partial^2 \zeta_0}{\partial r^2} \tag{76}$$

where $\Re_1$ and $\Re_2$ are given by (21) and (63) together with (64). Note that the left side of (75) could be expressed as

$$(\rho\xi)_2 = (p)_2 + \left(\frac{\mu}{2}(\underline{H}.\underline{H})\right)_2 \tag{77}$$

with

$$(p)_2 = p_2(R_0) + \Re_1 \frac{\partial p_1}{\partial r} + \Re_2 \frac{\partial p_0}{\partial r} + \frac{1}{2}\Re_1^2 \frac{\partial^2 p_0}{\partial r^2} \tag{78}$$

$$(\underline{H}.\underline{H})_2 = 2(\underline{H}_0.\underline{H}_2)+2\Re_1\frac{\partial(\underline{H}_0.\underline{H}_1)}{\partial r}+\underline{H}_1.\underline{H}_1+\frac{1}{2}\Re_1\frac{\partial^2 H_0^2}{\partial r^2}+\Re_2\frac{\partial(\underline{H}_0.\underline{H}_0)}{\partial r}\ , \text{ at } r=R_0 \tag{79}$$

Substituting $\underline{H}_i^f$ , $H_i$ (with $i = 0, 1, 2$) , $\Re_1$ and $\Re_2$ into equation (75), taking into account (76) and (77), and then equating the quantities in $\cos(2kz)$ and those which are free from $\cos(2kz)$, the second order perturbation coefficients $a$ and $C_{2\xi}$ are determined. These coefficients in their general forms are very lengthy and too cumbersome and are therefore not given here. However, on using the recurrence relations and second order differential Bessel equations for the ordinary and modified Bessel functions, the coefficients $a$ and $C_{2\xi}$ are simplified considerably and are given by

$$\begin{aligned}\frac{4\rho}{\mu H_0^2}C_{2\xi} &= -\frac{x^2+L_0^2(x)}{2} + Y_1(\beta)Y_0(\beta)\left[-4\beta + (4x + xL_0(x) + 3x^2L_0^{-1}(x))K_0'(y)\right]\\ &+Y_0^2(\beta)\left[3\beta^2 + x^2 + xy(L_0(x) + 3x^2L_0^{-1}(x))\frac{K_0(y)}{\beta} - \frac{2x}{\beta}(\beta^2+y^2)K_0'(y)\right]\\ &+Y_1^2(\beta)\left(4 + L_0(x) + 3x^2L_0^{-1}(x)\right)\end{aligned} \tag{80}$$

$$\begin{aligned}&4ayY_0^2(\beta)\left(\frac{2K_0(2y)}{K_0'(2y)} - \frac{L_0(2x)K_0(y)}{L_0(x)K_0'(y)}\right) + 4a\beta Y_0(\beta)Y_1(\beta)\left[2 - L_0(2x)L_0^{-1}(x)\right]\\ &+4aY_1^2(\beta)\left(1 - L_0(2x)L_0^{-1}(x)\right)\\ &= Y_0^2(\beta)\beta^2 + x^2 - 2y^2 + M_0(y)\left[M_0(y) + 3x^2L_0^{-1}(x) - L_0(x)\right]\\ &-\tfrac{1}{2}M_0(2y)\left(L_0(x) - M_0(y)\right) + \beta Y_0(\beta)Y_1(\beta)\left[3x^2L_0^{-1}(x) - 5L_0(x) + 4M_0(x) + M_0(2y)\right]\\ &+Y_1^2(\beta)\left(4 + \tfrac{1}{2}\beta^2 + 3x^2L_0^{-1}(x) - 3L_0(x)\right)\end{aligned} \tag{81}$$

where

$$L_0(x) = \frac{x\ I_0(x)}{I_0'(x)} \tag{82}$$

$$M_0(x) = \frac{x\ K_0(x)}{K_0'(x)} \tag{83}$$

Moreover, if we assume that the force-free magnetic field is not perturbed in the linear state and this could be achieved if $B = 0$ (see equation (43)), the foregoing expressions (80) and (81) are simplified considerably. Taking into account that as $\underline{H}_1^f = 0$ is not necessary $\underline{H}_2^f = 0$.

As we can see, the second order coefficients are functions of the ordinary and force-free wave numbers $x$ and $y$ , the force-free dimensionless parameter $\beta$ and ordinary and modified Bessel functions. Their discussions which required lengthy and accurate calculations, reveal that the stable or/and unstable perturbation modes do not alter directly under this technique. In discussing their behavior the second order coefficients are found to be singular as $x$ tends to zero. Therefore it might be proper to study second order perturbation for $m \neq 0$ or/and taking into account the surface tension effect or we may take other kind of flows e.g. swirling jet (see Shtern and Hussain [26]). However, the latter seems to be a cumbersome which we propose to do in future.

It is important to refer here also that some of the main features of the non-linear theory are : (i) the higher order terms yield corrections to the linear theory and (ii) it delimits the domain of validity of the linear theory since no linear theory can predict its domain of validity.

## 8. References


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