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ACTIVE DAMPING OF GEOMETRICALLY NONLINEAR TRANSVERSE BEAM VIBRATIONS

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Abstract

The paper is concerned with the stabilization of an elastic beam nonlinear geometrically subjected to a time-dependent axial forcing. The direct Liapunov method is proposed to establish criteria for the almost sure stochastic stability of the unperturbed (trivial) solution of the structure with closed-loop control. We construct the Liapunov functional as a sum of the modified kinetic energy and the elastic energy of the structure The distributed control is realized by the piezoelectric sensor and actuator, with the changing widths, glued to the upper and lower beam surface. The paper is devoted to the stability analysis of the closed-loop system described by the stochastic partial differential equation without a finite-dimensional approach. The fluctuating axial force is modelled by the physically realizable ergodic process. The rate velocity feedback is applied to stabilize the panel parametric vibrations. Calculations are performed for the Gaussian process with given mean value and variance as well as for the harmonic process with an amplitude A.

Introduction

Piezoelectric materials show great advantages as sensors and actuators in intelligent structures i.e. structures with highly distributed actuators, sensors, and processor networks. Piezoelectric sensors and actuators have been applied successfully in the closed-loop control plates (DIMITRIADIS et al., 1991). A comprehensive static analysis for a piezoelectric actuator glued to a beam was given by CRAWLEY and de LUIS (1987). The relationship between static strains, both in the structure and in the actuator and the applied voltage across the piezoelectric was presented. An extended dynamic model of beam, bonding layers, sensor and piezoactuators with emphasis on active damping was considered (TYLIKOWSKI, 1994). The direct Liapunov method was applied to the stabilization problem of the beam subjected to a wide-band axial time-dependent force (TYLIKOWSKI, 1995) and to a physically realizable force (TYLIKOWSKI, 1999). Tzou and Fu (1992) analysed models of a plate with segmented distributed piezoelectric sensors and actuators, and showed that segmenting improves the observability and the controlability of the system.

The purpose of the present paper is to solve an active control problem of beam parametric vibrations excited by the axial randomly fluctuating force. Assuming the moderate transverse displacement the geometrical nonlinearity is taken into account. The problem is solved using the concept of distributed piezoelectric sensors and actuators with a sufficiently large value of velocity feedback. Real mechanical systems are subjected not only to nontrivial initial conditions but also to permanently acting excitations increasing the structure energy and the active vibration control should be modify in order to balance the supplied energy by external parametric excitation. The applicability of active vibration control is extended to distributed systems with stochastic parametric excitation. The rate velocity feedback is applied to stabilize the beam parametric vibrations. Applying the direct Liapunov method the sufficient almost sure stability conditions for the beam with closed - loop control are derived. A relation betweeen the stabilization effect in nonlinear and linear approach is derived. Almost sure stability domains in terms of the effective retardation time, the feedback constant, and the force mean value and variance are obtained. The fluctuating axial force is modelled by the physically realizable ergodic process. The rate velocity feedback is applied to stabilize the beam parametric vibrations. Calculations are performed for the Gaussian and harmonic processes.

Sensor and Actuator Equations

Let us consider two opposite phase piezoelectric elements bonded to the elastic beam. The normal stress due to the axial u and transverse w beam displacements is given by

$$\sigma_s = E_s\left(\frac{\partial u}{\partial X} + \frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2 - z\frac{\partial^2 w}{\partial X^2}\right) \tag{1}$$

where E_s denotes the sensor Young modulus, and z denotes the distance from the beam neutral axis. Neglecting the influence of the axial displacement small as compared with the transverse one we obtain the formula for the stress in the upper piezolayer

$$\sigma_s^u = E_s\left[\frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2 - \frac{t_b + t_s}{2}\frac{\partial^2 w}{\partial X^2}\right] \tag{2}$$

where t_b and t_s denote the beam and the sensor thickness respectively. In similar way the stress in the lower sensor is given by

$$\sigma_s^d = E_s\left[\frac{1}{2}\left(\frac{\partial w}{\partial X}\right)^2 + \frac{t_b + t_s}{2}\frac{\partial^2 w}{\partial X^2}\right] \tag{3}$$

Subtracting electric displacement in the both layers we eliminate the nonlinear effect in measurements and obtain

$$D_3 = -d_{s31}\left(\sigma_s^g - \sigma_s^d\right) \tag{4}$$

Using the standard equation for capacitance the voltage produced by the measurement sensors is as follows

$$V_s = \frac{1}{C}\int_S D_3 \Phi^s dS \tag{5}$$

where Φ_s is the sensor polarization profile suitably chosen. Usually the polarization profile is determined by a changing width of sensor $b_s(X)$. Using the velocity feedback control the voltage applied to the actuator is given by

$$V_a = \frac{K_a}{C}\frac{dV_s}{dt} \tag{6}$$

where K_a is the feedback gain factor. Using the standard equation for capacitance the voltage applied to the actuator is given by formula

$$V_a = \frac{(t_s + t_p) E_s K_a}{A \in_{33}} \int_S w_{,xxt} b_s(X) dX \tag{7}$$

The normal stresses generated by the applied voltage are as follows

$$\sigma_a = \frac{d_{a31} V_a}{t_a} b_a(X) \tag{8}$$

The piezoelectric constant of actuator is denoted by d_{a31}. The control bending moment can be expressed by the actuator stress, the moment arm $(t_b + t_s)/2$ and the cross section area $t_a b_a(x)$ of actuator in the following way

$$M_X = \sigma_a t_a (t_b + t_a) b_a(X) \tag{9}$$

Substituting the actuator stresses the control moment is related with the time derivative of beam curvature as follows

$$M_X(X) = \frac{K_a (t_b + t_s)(t_b + t_a) t_s d_{s31} d_{a31} E_s E_a}{2 \varepsilon_{33} A_s} b_a(X) \int_0^l b_s(X) W_{,XXT} dX \tag{10}$$

where l is the beam length, and ε_{33} represents the permittivity of sensor material.

Dynamics Equation With Distributed Feedback

A continuous mechanical system (beam or plate in a cylindrical bending) uni-axially loaded in the middle plane by time-dependent force $S = S_o + S(t)$ is considered. The dynamics equation of structure motion includes both an internal passive damping due to viscoelastic properties and an active damping. Finite thickness piezoelectric patches are mounted on opposite sides of the structure. It is assumed that the transverse motion dominates the axial beam vibrations. The sensing and actuating effects of piezoelectric layers are used to extract the mechanical energy and in a final result to stabilize both the free vibration due to initial disturbances and the parametric vibration excited by the time-dependent axial force. We assume a negligible stiffness of the sensor in comparison with that of the structure and reduce the influence of the piezoelectric actuator on the structure to shear forces in bonding layers distributed over the structure surface. Vibration damping of the visco-elastic beam with parametric excitation can be examined differentiating the total energy of the beam with piezoelements (TYLIKOWSKI, 1995). The rate of energy extraction indicates that for the sufficiently large gain factor it is possible to stabilize parametric vibrations. However, the result did not provide an effective quantitive estimation of the minimal active damping coefficients stabilizing the parametric vibration. If the parametric excitation is a realizable ergodic stochastic process dynamic equation should be understood as the partial differential equation with a random parameter.

Consider the Bernoulli-Euler beam axially loaded by a time-dependent force with piezoelectric layers mounted on each of two opposite sides of the beam. The piezoelectric layers are assumed to be bonded on the beam surfaces and the mechanical properties of the bonding material are represented by the effective retardation time of the beam treated as a laminated beam . The effective retardation time is a linear function of both the beam and bonding layer retardation times. It is assumed that the transverse motion dominates the axial beam vibrations.

The thickness of the actuator and the sensor is denoted by t_a and t_s, respectively. The sensing and actuating effects of piezoelectric layers are used to stabilize both the free vibration due to initial disturbances and the parametric vibration excited by the axial force. Assuming a negligible stiffness of the sensor in comparison with that of the beam and reducing the influence of the piezoelectric actuator on the beam to a bending moment distributed along the beam, the dynamics equations for the beam with distributed sensor and actuator layers can be rewritten in the form

$$\rho b t_b W_{,TT} + E_b J\left(W_{,XXXX} + \lambda_b W_{,XXXXT}\right) + \left(S_o + S(T) - \frac{b t_b E_b}{2l}\int_0^l W_{,X}^2 dX\right)W_{,xx} + M_{,XX} = 0 \qquad X \in (0,l) \tag{11}$$

where: T, X denote time and coordinate variable, respectively, W is the transverse displacement, b and l denote the beam width and length, respectively, , E_b is the Young's modulus of the beam, J is the cross-section moment of inertia, λ_b is the effective retardation time, M_X is the distributed moment of piezoelectric origin. An averaged geometrical nonlinearity is represented by the integral in the third component of Eq. (11).

Introduce dimensionless variables: the time, the coordinate and transverse displacement are given by

$$\begin{aligned} t &= T(E_b J / \rho b t_b l^4)^{1/2} \\ x &= X / l \\ w &= W / l \end{aligned} \tag{12}$$

The reduced axial load, internal (passive) damping coefficient λ and the active damping coefficient β_a are given respectively

$$f_o + f(t) = \frac{l^2}{E_b J}\left[S_o + S\left(t\sqrt{\frac{\rho b t_b l^4}{E_b J}}\right)\right] \tag{13}$$

$$\lambda = \lambda_b l^2 /(\rho b t_b E_b J)^{1/2} \tag{14}$$

$$\beta_a = \frac{K_a(t_b + t_s)(t_b + t_a)t_s}{4\in_{33} A_s l(\rho b t_b E_b J)^{1/2}} d_s d_a E_b E_a . \tag{15}$$

The dimensionless bending moment produced by the piezoelectric actuator is as follows

$$m_x(x) = 2\beta_a b_a(x)\int_0^1 b_s(x) w_{,xxt} dx \tag{16}$$

For the case in which the bending moment is equal to zero at the beam ends, the beam can be treated as simply supported. Thus, we assume that the beam transverse displacement satisfies the following boundary conditions

$$w = 0 \qquad w_{,xx} = 0 \qquad x = 0,1 \tag{17}$$

In dimensionless variables Eq. (11) becomes

$$\begin{gathered} w_{,tt} + w_{,xxxx} + \lambda w_{,xxxt} + \\ + \left(f_o + f(t) - \alpha \int_0^1 w_{,x}^2 dx \right) w_{,xx} + m_{x,xx} = 0 \qquad x \in (0,1) \end{gathered} \tag{18}$$

where α is the dimensionless nonlinearity coefficient. Equation (18) with zero initial conditions w(x,0)= w (x,0) = 0 possess a trivial solution w(x,t) = 0, which corresponds to an undeflected beam axis.

Almost Sure Stability Analysis

The purpose of the present paper is to derive criteria for solving the following problem: will the deviations of beam axis from the unperturbed state (trivial solution) be sufficiently small in some mathematical sense in the case when the axial force is the stochastic process. The beam dynamically buckles when the axial force gets so large that the beam with closed-loop control does not oscillate (vibrations of beam with closed- loop control does not decay) and a new increasing mode of oscillations occurs. To estimate a perturbed solution of equation (18) it is necessary to introduce a measure of distance of the solution with nontrivial initial conditions from the trivial one. The most commonly used stability definition used in continuum mechanics, states that an equilibrium state is stable whenever, in the motion following any sufficiently small initial disturbances the displacement w and the velocity w are everywhere arbitrarily small for all t > 0. In order to investigate the behavior of the solutions of stochastic equations a modification of the Liapunov stability definition is needed. The equilibrium state of equation (is said to be almost sure stochastically stable (KOZIN, 1972), if

$$P\left[\lim_{t \to \infty} \| w(.,t) \| = 0 \right] = 1 \tag{19}$$

In the present paper the direct Liapunov method is proposed to establish criteria for the almost sure stability of the unperturbed (trivial) solution of the structure with closed-loop control. The crucial point of the Liapunov method is a construction of a suitable functional, which is positive-definite along any motion of the beam with closed-loop control. We construct the Liapunov functional as a sum of the modified kinetic energy and the elastic energy of the beam

$$\begin{gathered} \mathrm{V} = V_L + \frac{\alpha}{4}\left[\int_0^1 w_{,x}^2 dx \right]^2 = \frac{1}{2}\int_0^1 \left(w_{,t}^2 + 2\lambda w_{,xxxx} + 2\lambda^2 w_{,xxxx}^2 + w_{,xx}^2 - f_o w_{,x}^2 \right) dx + \\ + \frac{\alpha}{4}\left[\int_0^1 w_{,x}^2 dx \right]^2 \end{gathered} \tag{20}$$

where V_L is the Liapunov functional for the linearized equation (with $\alpha=0$). If the classical condition for the static buckling is fulfilled, functional (20) satisfies the desired positive-definiteness condition, and the measure of distance between the perturbed solution and the trivial one can be chosen as the square root of the functional

$$\|w\| = V^{1/2} \tag{21}$$

If realizations of the processes are physically realizable the clasical calculus is applied to calculate the time - derivative of functional (20). Differentiating V_L with respect to time and eliminating $w_{,tt}$ by means of Eq. (18) with $\alpha=0$ we obtain

$$\frac{dV_L}{dt} = -2\lambda V_L + 2U_L \tag{22}$$

where the auxiliary functional U is given by

$$U_L = \int_0^1 \Big[-\lambda w_{,xxt}^2 - \lambda w_{,xxxx}^2 - \lambda(f_o + f(t)) w_{,xxx}^2 + w_{,xx}^2 - f(t) w_{,t} w_{,xx} - m_{x,xx}(w_{,t} + \lambda w_{,xxxx}) + \\ + (w_{,t}^2 + 2\lambda w_{,t} w_{,xxxx} + 2\lambda^2 w_{,xxxx}^2 + w_{,xx}^2 - f_o w_{,x}^2)^2_{,xxxx} \Big] dx \tag{23}$$

Let us focus our attention on the following particular shapes of piezoelectric elements. The sensor and actuator are described by the sinusoidal function with diffrent maximum widths b_s, b_a, respectively

$$b_s(x) = b_s \sin \pi x \qquad b_a(x) = b_a \sin \pi x \tag{24}$$

The beam motion can be expanded into the following sine Fourier series satisfying the boundary conditions (17)

$$w(x,t) = \sum_{i=1}^{\infty} \Psi_i(t) \sin i\pi x \tag{25}$$

We are now in a position to calculate the spatial integral involved in the bending moment (16) with the shape functions (24). We rewrite functional (23) in the form

$$U_L = \int_0^1 \Big[-\lambda w_{,xxt}^2 - \lambda w_{,xxxx}^2 - \lambda(f_o + f(t)) w_{,xxx}^2 + w_{,xx}^2 - f(t) w_{,t} w_{,xx} - \gamma \Psi_{1,t} \sin \pi x (w_{,t} + \lambda w_{,xxxx}) + \\ + (w_{,t}^2 + 2\lambda w_{,t} w_{,xxxx} + 2\lambda^2 w_{,xxxx}^2 + w_{,xx}^2 - f_o w_{,x}^2)^2_{,xxxx} \Big] dx \tag{26}$$

where gain factor γ is calculated from Eq. (24) and Eq. (16). According to Kozin's method we look for a function χ statisfying the following variational inequality

$$U_L \le \chi V_L \tag{27}$$

Solving the associate Euler equation (cf. TYLIKOWSKI, 1999) we obtain the the function λ in explicite form. Thus, if process f(t) is stationary and ergodic the sufficient condition for the almost sure stochastic stability is

$$\langle \chi \rangle \le \lambda \tag{28}$$

where angle brackets denote the mathematical expectation. It can be shown that

$$\chi = \max_{i=1,2,...} \chi_i \tag{29}$$

where sequence $\{\chi_i,\ i=1,2,...\}$ of functions satisfy the following sequence of inequalities

$$U_{Li} \le \chi_i V_{Li} \tag{30}$$

Functionals U_{Li} and V_{Li} are fuctionals U and V calculated at $w(x) = \sin i\pi x$. The auxiliary linearized problem being solved we can now direct our attention to the nonlinear operator equation (18). The time-derivative of funvtional V along arbitrary solution of the nonlinear equation (18) is given as follows

$$\begin{aligned}\frac{dV}{dt} &= -2\lambda\left[V_L + \frac{\alpha}{4}\left[\int_0^1 w_{,x}^2 dx\right]^2\right] + \\ &+ 2\left[U_L - \frac{\alpha\lambda}{2}\left[\int_0^1 w_{,x}^2 dx\right]\left[\int_0^1 w_{,xxx}^2 dx\right] + \frac{\alpha\lambda}{4}\left[\int_0^1 w_{,x}^2 dx\right]^2\right]\end{aligned} \tag{31}$$

We see that the following variational inequality is true for an arbitrary function satisfying the simply supported boundary condition

$$\int_0^1 w_{,xxx}^2 dx \ge \int_0^1 w_{,x}^2 dx \tag{32}$$

It is easy to notice that the following chain of inequalities holds

$$\lambda V \ge \lambda V_L \ge U_L \ge U \tag{33}$$

Therefore the almost sure stability of the linearized problem implies the stability of nonlinear problem.

Inequality (28) gives us a possibility to obtain minimal effective retardation times guaranteeing the almost sure asymptotic stability called critical retardation times. A domain where retardation times are greater than the critical ratardation times is called the stability region. The stability regions as functions of constant component of axial force f_o, axial loading variance , effective retardation time , and gain factor are calculated numerically. First, discrete values of force are chosen and the largest value χ_i corresponding to the given value of force is determined and the expectation is calculated numerically integrating the product of by the probability density function of loading. This is accomplished for various values of parameters by choosing the variance and varying the retardation time until inequality (28) will be satisfied. Numerical calculations are performed for the Gaussian process with the mean F and variance σ and for the harmonic process with an amplitude A. In order to compare both processes the variance of harmonic process $\sigma = A^2 / 2$ is used. The almost sure stability region of beam axially loaded by the zero mean Gaussian process is shown in Fig. 1.

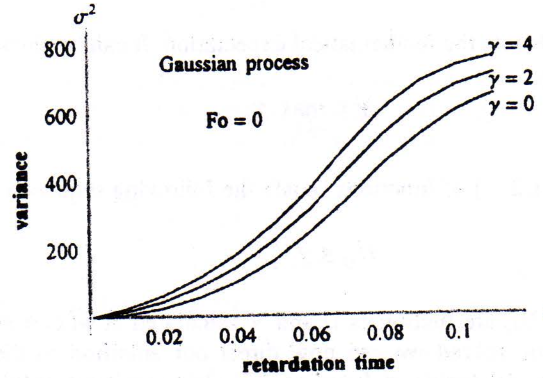

Fig. 1 Stability regions of beam for the zero mean Gaussian force.

It is seen that the stability regions increase as the gain factor increases. Figure 2 compares the stability regions for the beam with the zero mean force loaded by the Gaussian and harmonic processes. It is visible that the influence of the class of excitation is not substantial.

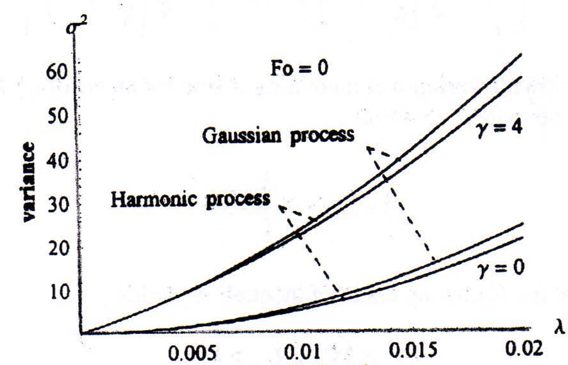

Fig. 2 Comparison of stability regions for the Gaussian and harmonic process

Figure 3 shows the critical variance versus the retardation time for the beam loaded by the constant force close to the static buckling load π^2. The increase of critical variances is much more slower in comparison to the zero mean loading.It is necessary to emphasise that the results are obtained for the particular sinusoidal shapes of distributed sensor and actuator.

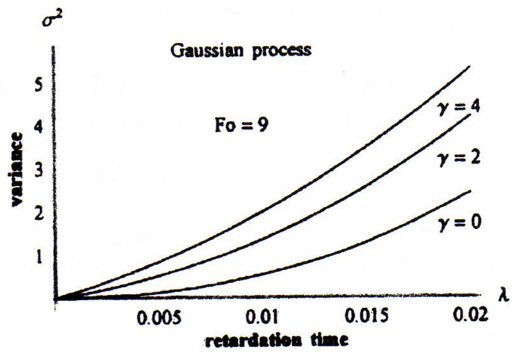

Fig. 3 Influence of constant compresive force on stability regions for the Gaussian loading.

Conclusions

By means of the direct Liapunov method the active stabilization of a vibrating beam with distributed piezoelectric sensor, actuator, and the velocity feedback has been studied. The elastic beam is simply supported and subject to a compressive axial force randomly fluctuating. Without any passive damping and control, the beam motion is unstable due to the parametric excitation. The stabilization of stochastic parametric vibrations needs sufficiently large active damping coefficient proportional to the gain factor. Admissible variances of loading strongly depend on the feedback gain factor. The stability regions do not change qualitatively in going from the Gaussian process to the harmonic one, but the Gaussian loading needs smaller critical retardation time than the harmonic loading. For no axial force, this is the case of free vibration due to the nontrivial initial conditions. As long as the active or passive damping is present, the system is stable and oscillations decay.

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