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# HEAT TRANSFER OF MHD NON-NEWTONIAN CASSON FLUID FLOW BETWEEN TWO ROTATING CYLINDERS

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## Abstract

A theoretical analysis of heat transfer of steady, incompressible and electrically conducting non-Newtonian Casson fluid flow between two rotating cylinders under a radial magnetic field is studied. The problem is considered when the inner cylinder is at rest and the outer cylinder rotating with a constant velocity. In this paper, the velocity distribution, magnetic induction, the temperature distribution, stress, shear rates and rate of heat transfer are obtained analytically by using perturbation technique and shown graphically for various values of aspect ratio, Casson number, Eckert number and magnetic parameter. The critical values of Casson number have been determined.

## Introduction

The model of Casson fluid is concerned as one of the most important applications of theoretical fluid mechanics to problems arising in physiology, mainly in describing the flow of the blood. Casson [1] proposed a model to describe the flow curves of suspensions of pigments in lithographic varnishes used for the preparation of printing inks. Haldar et al. [2] proposed two layered model of blood flow through stenosed arteries, their model was consisted of a peripheral plasma layer free from red cells and core region represented by a Casson fluid. Pham et al. [3] studied numerically the entry and exit flows of Casson fluids and they used Casson constitutive equation recommended for describing blood flow with an appropriate modification proposed by Papanastasiou, which applies everywhere in the flow field in both yielded and unyielded region.

Srivastava et al. [4] investigated the problem of blood flow through an axially nonsymmetrical but radially symmetric stenosed tube when the blood is represented by a Casson fluid and a Newtonian fluid. A mathematical model for solute transfer in blood flow through cylindrical tube with permeable wall has been proposed by Indira et al. [5], the blood is represented by the Casson fluid model with constant viscosity. The solute transfer within and across the tube wall is considered to be both by diffusion and convection. Yan et al. [6] discussed the problem of yield surface of viscoelastic and plastic fluids in a vane viscometer the shear stress was determined by assuming that the material is held in space between the vane blades so that it behaves like a rigid cylinder, the finite element method has been used to model the behavior of Herschel-Bulkely (Bingham), Casson and viscoelastic (Maxwell type) fluids. Dash et al. [7] investigated the effect of yield stress on the flow characteristics of a Casson fluid in a homogeneous porous medium bounded by a circular tube and they employed the Brinkman model to account for the Darcy resistance offered by the porous medium. Dash et al. [8] studied the changed flow pattern in a narrow catheterized artery and an estimate of the increased flow resistance is made. The anomalous behavior of blood in small blood vessels has been taken into account by modeling blood as a Casson fluid. The importance of Casson fluid in describing blood is introduced by Mazumdar [9].

Bigyani et al. [10] studied the flow of Casson fluid between two rotating cylinders they, also, defined Casson fluid as a shear thinning fluid which has an infinite viscosity at zero shear rates, a yield stress below which no flow occurs and a zero viscosity at infinite rates of flow. Various experiments preformed on blood with varying hematocrit, anticoagulants, temperature, etc., strongly suggest its non-Newtonian behaviour which has been shown to obey stress-strain relation for Casson liquid. [11-14] verified this finding. Jones [15] investigated that the flow of the blood in a circular tube can be considered as a Casson fluid.

Shul'man [16] studied the flow of Casson fluid through an infinite annular channel under the action of a constant pressure gradient. The numerical solution of the entrance flow for a Casson fluid obtained by Shah et al. [17] and for annular tube by Liu et al. [18]. Victor et al. [19] presented steady state heat transfer to blood flow in the entrance region of a tube and they considered it as a Casson fluid. The problem of blood flow considered as a Casson fluid through non-circular ducts has been investigated by Batra [20]. Godbole et al. [21] studied a finite element analysis of blood flow obeying Casson's relation. Casson's constitutive equation has been found to be accurately applicable by [22-23]. Sundstrom et al. [24] studied laminar free covection in inclined rectangular enclosures. Nield et al. [25] have been investigated forced convection in a fluid-saturated porous medium channel with isothermal or isoflux boundaries. Viscoplastic flow between approaching or parallel circular plates presented by Klimov et al. [26]. Kashevarov [27] obtained an exact solution of the problem of convection heat exchang for an elliptic cylinder and a plate in a fluid with small Prandtl number. Michio Yoneya et al. [28] studied a forced convection heat transfer from circular cylinder with constant heat flux in saturated porous medium.

Eldabe and Oaf [29] analyzed the flow and heat transfer of dilute polymer solutions through porous medium between two horizontal plates, and according to the Denn model, which represents the rheological behaviors of dilute polymer solutions, they obtained the analytical expressions for the velocity and temperature fields. The non-Newtonian flow formation in Coutte motion in magnetohydrodynamics (MHD) with time varying suction, was investigated by Eldabe and Ahmed [30], the viscoelastic fluid used in this research was suggested by Walters [31]. Eldabe and Elmohandis [32] extended the problem of Eldabe and Ahmed for a pulsatile viscoelastic flow through a channel bounded two permeable parallel plates. Eldabe et al. [33] studied unsteady magnetic boundary-layer flow of power-law non-Newtonian conducting fluid through porous medium past an infinite porous flat plate.

The main idea of this work is to study the mathematical analysis of the magnetohydrodynamic non-Newtonian Casson fluid flow between two rotating cylinders, in the presence of a radial magnetic field, and showing the relation between the different parameters of the motion and external forces, in order to investigate how to control the motion of the fluid by changing these parameters and external forces. Some of the applications concerning this idea are the flow of oil under ground where there is a natural magnetic field. The other example is the motion of the blood through the arteries.

## Basic Equations

The rheological equation of state for an isotropic, incompressible flow of a Casson fluid can be written as [34],

$$\tau_{ij} = -p\delta_{ij} + 2\mu(j_2)e_{ij}, \tag{1}$$

where

$\mu(j_2) = \left(K_c\, j_2^{1/4} + 2^{-3/4}\tau_y^{1/2}\right)^2 j_2^{-1/2}$ is the apparent viscosity. $p$, $\tau_y$, and $K_c^2$ are pressure, yield stress and Casson's coefficient of viscosity respectively. $\delta_{ij}$ is the Kronecker delta and $e_{ij}$, $\tau_{ij}$ are the rate of strain tensor and stress components respectively. $j_2 = \frac{1}{4}e_{ij}\,e_{ij}$ is the rate of strain tensor invariant. The flow conditions are given by [33]

$$e_{ij} = 0 \qquad \text{if } j_2' < \tau_y^2,$$

$$= \frac{\tau_{ij}}{2\mu(j_2)} \qquad \text{if } j_2' \geq \tau_y^2, \tag{2}$$

where $j_2' = \frac{1}{4}\tau_{ij}'\tau_{ij}'$ is the second invariant of deviatoric stress tensor and $\tau_{ij}' = \tau_{ij} + p\delta_{ij}$ is deviatoric stress component.

The basic equations of motion MHD neglecting displacement current and free charges are

Maxwell's equations

$$\frac{\partial H_i}{\partial x_i} = 0, \tag{3}$$

$$J_i = \varepsilon_{ijk}\frac{\partial H_k}{\partial x_j}, \tag{4}$$

$$\frac{\partial H_i}{\partial t} = -\mu_e\,\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j}, \tag{5}$$

Ohm's equation,

$$J_i = \sigma\left(E_i + \mu_e\,\varepsilon_{ijk}\,\upsilon_j H_k\right). \tag{6}$$

Equations of motion, continuity and energy are respectively:

$$\rho\frac{D\upsilon_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} + \mu_e\,\varepsilon_{ijk}\,J_j\,H_k, \tag{7}$$

$$\frac{D\rho}{Dt} + \rho e_{ii} = 0, \tag{8}$$

$$\rho c\left[\frac{\partial T}{\partial t} + \upsilon_j\frac{\partial T}{\partial x_j}\right] = k\frac{\partial^2 T}{\partial x_i\,\partial x_i} + \tau_{ij}\frac{\partial \upsilon_i}{\partial x_j} + \frac{1}{\sigma}\delta_{ij}\,J_i\,J_j, \tag{9}$$

where $J_i$, $\sigma$, $\mu_e$, $H_i$, $E_i$ and $\upsilon_i$ represent current density, electrical conductivity, magnetic permeability, magnetic field, electric field and the velocity respectively, $\rho$ is the density, $c$ is the heat capacity, $k$ is the thermal conductivity, $T$ is the temperature, and the rate of strain tensor can be written as

$$e_{ij} = \frac{1}{2}\left(\upsilon_{i,j} + \upsilon_{j,i}\right), \tag{10}$$

## Mathematical Analysis

Let us consider an electrically conducting, non-Newtonian Casson incompressible fluid flow between two rotating cylinders of radii $a$ and $b$ $(b > a)$. The system is stressed by an external radial magnetic field of strength $H_0\,a/r$. The cylindrical polar coordinates system $(r,\theta,z)$ are used with origin at the center of the cross section of the cylinders. With these assumption for steady, laminar, axisymmetrical, peripheral the velocity is $\vec{\upsilon}(0,u(r),0)$ and the magnetic field is $\vec{H}(H_0\,a/r, h(r), 0)$. From eq. (1) the corresponding stress tensor component is given by

$$\tau_{r\theta} = \left(\tau_y^{1/2} + K_c\left(2e_{r\theta}\right)^{1/2}\right)^2, \tag{11}$$

where,

$$e_{r\theta}=\frac{1}{2}\left(\frac{du}{dr}-\frac{u}{r}\right). \tag{12}$$

According to our considerations eqs. (7-9) can be written as

$$\frac{\rho u^2}{r}=\frac{\partial p}{\partial r}+\mu_e\frac{h}{r}\frac{d}{dr}(rh), \tag{13}$$

$$\frac{d\tau_{r\theta}}{dr}+\frac{2\tau_{r\theta}}{r}+\frac{\mu_e H_0 a}{r^2}\frac{d}{dr}(rh)=0, \tag{14}$$

$$\frac{\partial p}{\partial z}=0, \tag{15}$$

$$k\left[\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}\right]+2\tau_{r\theta}e_{r\theta}+\frac{1}{\sigma r^2}\left(\frac{d}{dr}(rh)\right)^2=0. \tag{16}$$

Also, Ohm's low takes the form,

$$J_z=\sigma\left(E_z-\frac{\mu_e H_0 a}{r}u\right), \tag{17}$$

where $J_z=\frac{1}{r}\frac{d}{dr}(rh)$.

The appropriate boundary conditions of this problem are

$$\left.\begin{aligned}&\tau_{r\theta}=\frac{Y_c K_c^2 U}{b}\rightarrow(u=0), && h=0,\quad T=T_1,\quad p=p_1\rho U^2, \quad \text{at} \quad r=a\\ &\tau_{r\theta}=Y_c K_c^2 U\left(\frac{1}{b}+Y_c\right)\rightarrow(u=\beta U), && T=T_2, \quad \text{at} \quad r=b\end{aligned}\right\} \tag{18}$$

where $Y_c=\frac{b\tau_y}{UK_c^2}$ is the Casson number.

Let us introduce the following non-dimensional variables as:

$$\left.\begin{aligned}&r^*=\frac{r}{b}, \quad \tau^*_{r\theta}=\frac{\tau_{r\theta}}{K_c^2 U}, \quad e^*_{r\theta}=\frac{e_{r\theta}}{U/b}, \quad J^*_z=\frac{J_z}{\sigma\mu_e H_0 U}, \quad u^*=\frac{u}{U}\\ &\Theta=\frac{T-T_1}{T_2-T_1}, \quad h^*=\frac{h}{\sigma\mu_e H_0 U}, \quad p^*=\frac{p}{\rho U^2}, \quad E_z=\sigma\mu_e H_0 U E\end{aligned}\right\} \tag{19}$$

The equations (13-17) in dimensionless form after dropping star mark may be written as

$$\frac{u^2}{r}=\frac{dp}{dr}+\frac{K_1 M^2 h}{r}\frac{d}{dr}(rh) \tag{20}$$

$$\frac{d\tau_{r\theta}}{dr}+\frac{2\tau_{r\theta}}{r}+\frac{M^2\alpha}{r^2}\frac{d}{dr}(rh)=0, \tag{21}$$

$$\frac{1}{r}\frac{d}{dr}(rh)=b\left(E-\frac{\alpha u}{r}\right), \tag{22}$$

$$r^2 \frac{d^2\Theta}{dr^2} + r\frac{d\Theta}{dr} + c_1 E_c\left(2b\, r^2 \tau_{r\theta}\, e_{r\theta} + M^2\left(\frac{d}{dr}(r\,h)\right)^2\right) = 0, \tag{23}$$

where $K_1 = \dfrac{\sigma \mu_e K_c^2}{\rho}$, $c_1 = \dfrac{K_c^2}{k}$ and $E_c = \dfrac{U^2}{T_2 - T_1}$ is the Eckert number.

Subjected to the dimensionless boundary conditions

$$\left.\begin{array}{llll} \tau_{r\theta} = \dfrac{Y_c}{b} \to (u=0) & \Theta = 0, & h = 0, \quad p = p_1, & \text{at} \quad r = \alpha \\ \tau_{r\theta} = Y_c\left(\dfrac{1}{b} + Y_c\right) \to (u = \beta) & \Theta = 1, & & \text{at} \quad r = 1 \end{array}\right\} \tag{24}$$

where $\alpha = a/b$ is the aspect ratio.

Substitute from (11), (12) and (22) in (21), we obtain the following non-linear differential equation

$$r^2 \frac{d^2\tau_{r\theta}}{dr^2} + 3r\frac{d\tau_{r\theta}}{dr} - \alpha^2 b^2 M^2 \tau_{r\theta} + 2\alpha^2 b\sqrt{bY_c}\, M^2 \tau_{r\theta}^{1/2} - \alpha^2 b M^2 Y_c = 0. \tag{25}$$

In order to solve the equation (25) according to the boundary conditions (24) let,

$$r = e^{\xi}, \tag{26}$$

$$\tau_{r\theta} = \frac{1}{b}Y_c + g_0(\xi)Y_c^2 + g_1(\xi)Y_c^3 + g_2(\xi)Y_c^4 + O(Y_c^5), \tag{27}$$

substituting from (26), (27) in (25), comparing the coefficients of $Y_c^2$, $Y_c^3$, $Y_c^4$ and neglecting higher order of $Y_c^4$, we obtain the following differential equations

$$\frac{d^2 g_0}{d\xi^2} + 2\frac{dg_0}{d\xi} + \alpha^2 M^2 b^2 g_0 = 0, \tag{28}$$

$$\frac{d^2 g_1}{d\xi^2} + 2\frac{dg_1}{d\xi} + \alpha^2 M^2 b^2 g_1 = -\frac{3}{4}\alpha^2 M^2 b^3 g_0^2, \tag{29}$$

$$\frac{d^2 g_2}{d\xi^2} + 2\frac{dg_2}{d\xi} + \alpha^2 M^2 b^2 g_2 = -\frac{3}{2}\alpha^2 M^2 b^3 g_0\, g_1, \tag{30}$$

according to (24) the boundary conditions on $g_0$, $g_1$, and $g_2$ are

$$\left.\begin{array}{lll} g_0 = g_1 = g_2 = 0 & \text{at} & \xi = Log\,\alpha, \\ g_0 = 1,\ g_1 = g_2 = 0 & \text{at} & \xi = 0, \end{array}\right\} \tag{31}$$

the solutions of equations (28-30) subjected to the boundary conditions (31), can be obtained as,

$$g_0(\xi) = \frac{\alpha^{\lambda_2} e^{\lambda_1 \xi} - \alpha^{\lambda_1} e^{\lambda_2 \xi}}{\alpha^{\lambda_2} - \alpha^{\lambda_1}}, \tag{32}$$

$$g_1(\xi)=C_1(\alpha,\lambda_1,..,\lambda_5)e^{\lambda_1\xi}+C_2(\alpha,\lambda_1,..,\lambda_5)e^{\lambda_2\xi}+\lambda_3 e^{(\lambda_1+\lambda_2)\xi}-\lambda_4 e^{2\lambda_1\xi}-\lambda_5 e^{2\lambda_2\xi}, \quad (33)$$

$$g_2(\xi)=\lambda_6 e^{3\lambda_1\xi}-\lambda_7 e^{3\lambda_2\xi}+\lambda_8 e^{2\lambda_1\xi}-\lambda_9 e^{2\lambda_2\xi}+C_3(\alpha,\lambda_1,..,\lambda_{11})e^{\lambda_1\xi}+C_4(\alpha,\lambda_1,..,\lambda_{11})e^{\lambda_2\xi} -\lambda_{10}e^{(2\lambda_1+\lambda_2)\xi}+\lambda_{11}e^{(\lambda_1+2\lambda_2)\xi}, \quad (34)$$

where $\lambda_1,..,\lambda_{11}$ and the functions $C_1(\alpha,\lambda_1,..,\lambda_5)$, $C_2(\alpha,\lambda_1,..,\lambda_5)$, $C_3(\alpha,\lambda_1,..,\lambda_{11})$, $C_4(\alpha,\lambda_1,..,\lambda_{11})$ are defined in the appendix.

For real values of $\lambda_1$, $\lambda_2$ it's necessary that $Y_c < Y_1$, where $Y_1=\dfrac{\tau_y}{\alpha^2 b\,U\,\sigma\,\mu_e^2 H_0^2}$ is the critical Casson number.

## Magnetic Induction:

Substituting from (26), (27) in (21), the magnetic induction can be represented as,

$$h=\frac{e^{\xi}}{M^2\alpha}\left[\frac{Y_c}{b}\left(\frac{\alpha^2-e^{2\xi}}{e^{2\xi}}\right)+\left(\frac{\alpha^2}{e^{2\xi}}-g_0(\xi)\right)Y_c^2-Y_c^3\,g_1(\xi)-Y_c^4\,g_2(\xi)\right]. \quad (35)$$

## Rate of Strain

From (11) the rate of strain takes the form

$$e_{r\theta}=-\frac{b}{2}\left[g_0(\xi)Y_c^2+\left(g_1(\xi)+\frac{3b}{4}g_0^2(\xi)\right)Y_c^3+\left(g_2(\xi)+\frac{3b}{2}g_0(\xi)g_1(\xi)\right)Y_c^4\right]. \quad (36)$$

## The Velocity of The fluid

Substituting from equations (36), (26) in (12) the velocity distribution can be written as

$$u=e^{\xi}\left[F_1(\xi)-F_1(Log\,\alpha)\right]. \quad (37)$$

## The Pressure of the Fluid

In order to obtain the pressure of the fluid substitute from (35), (37) in (20), so it takes the following formula

$$p=F_2(\xi)-F_2(Log\,\alpha)+p_1. \quad (39)$$

## The Temperature Distribution

Substituting from (26), (27), (35) and (36) in (23), the temperature distribution can be obtained as,

$$\Theta=\frac{c_1 E_c\left[F_3(Log\alpha)-F_3(0)\right]}{Log\alpha}\xi-\frac{\xi}{Log\alpha}+c_1 E_c\left[F_3(0)-F_3(\xi)\right]+1, \quad (42)$$

where $F_1(\xi)$, $F_2(\xi)$ and $F_3(\xi)$ are defined in the appendix.

## Rate of Heat Transfer

Non dimensional form of the rate of heat transfer on the outer cylinder can be written as:

$$Q = \Theta'\big|_{\xi=0} \quad , \tag{43}$$

$$Q = \frac{c_1 E_c}{Log\alpha}\left[F_3(Log\alpha) - F_3(0) - c_1 E_c - F_3'(0) Log\alpha\right] \tag{44}$$

## Discussions

In this paper we proposed the problem of heat transfer of Casson fluid between two rotating cylinders, under the action of radial magnetic field. The non linear partial differential equations have been calculated approximately by using perturbation theory for small Casson number.

Figure (1) shows that the velocity distribution increases as Casson number increases in the core of flow region, when the value of aspect ratio $\alpha = 0.5$, magnetic parameter $M = 6$. While in figure (2) the velocity distribution decreases as magnetic parameter increases, for Casson number $Y_c = 0.58$, aspect ratio $\alpha = 0.5$. From figure (3) it's clear that the magnetic induction increases as Casson number increases for magnetic parameter $M = 7$ and aspect ratio $\alpha = 0.2$.

The effect of aspect ratio on the magnetic induction is presented in figure (3), for $\xi \approx -1.7$ to $\xi \approx -1.26$ it is clear that the magnetic induction increases with increasing of values of aspect ratio, while it decreasing for $\xi > -1.26$. The pressure of the fluid in the flow region increases as Casson number increases and this is obviously clear from figure (5), for magnetic parameter $M = 4.5$ and aspect ratio $\alpha = 0.1$. On the other hand, figure (6) shows that the pressure distribution decreases as magnetic parameter increases in the case of Casson number $Y_c = 0.63$ and aspect ratio $\alpha = 0.1$.

Figure (7) illustrates the effect of Casson number on the temperature distribution where it's increases as Casson number increases in the case of aspect ratio $\alpha = 0.2$ and Eckert number $E_c = 6$. Also the temperature increases as Eckert number increases for aspect ratio $\alpha = 0.2$ and Casson number $Y_c = 0.65$, this is clear in the figure (8). Figure (9) shows the relation between rate of heat transfer and Casson number, it is clear that the rate of heat transfer decreases as Casson number increases.

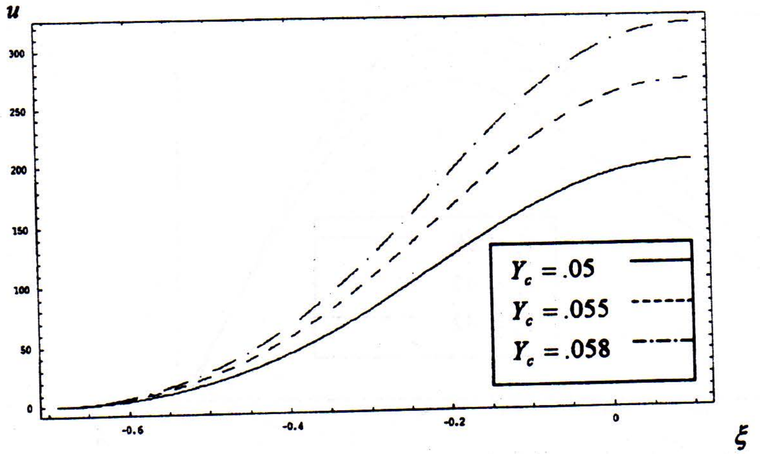


**Fig. (1). Velocity distribution plotted against $\xi$.**

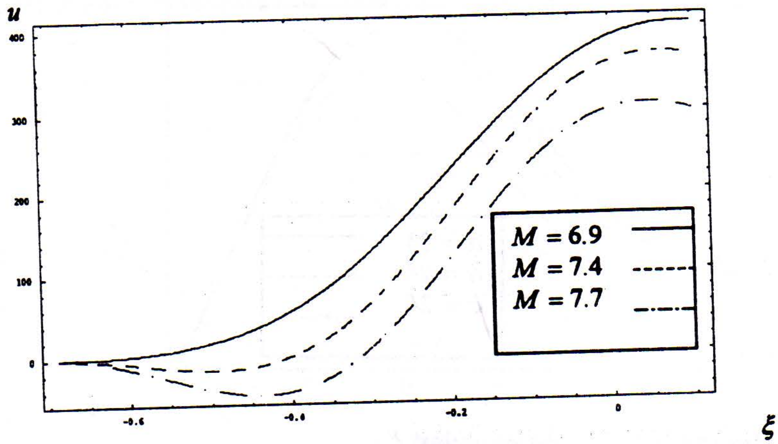


**Fig. (2). Velocity distribution plotted against $\xi$.**

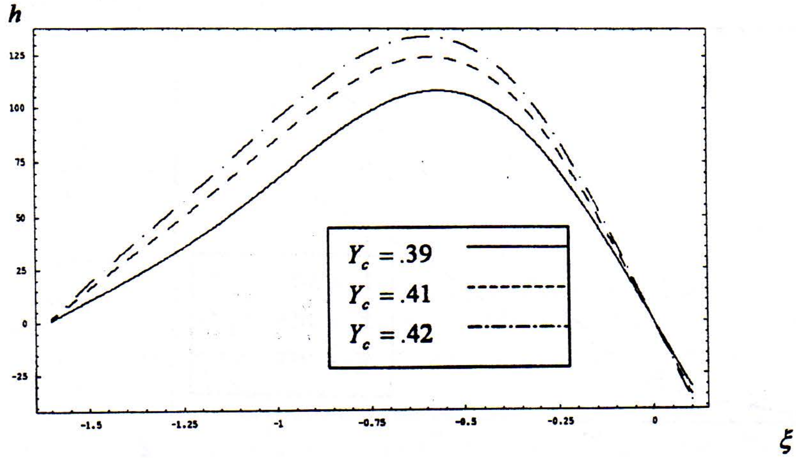


**Fig. (3). Magnetic induction plotted against $\xi$ .**

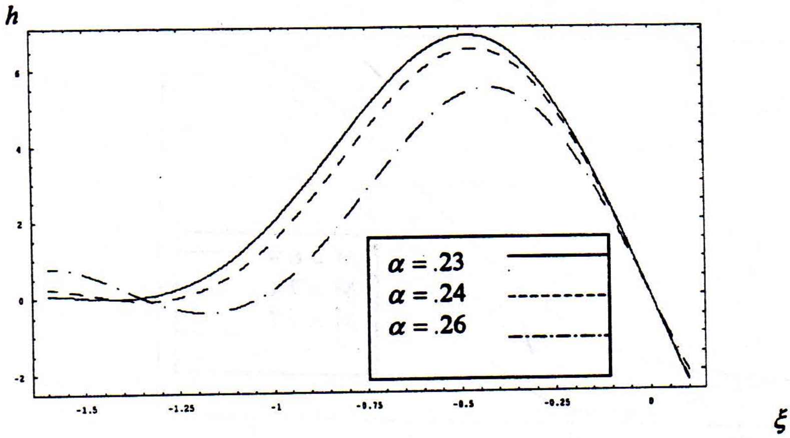


**Fig. (4). Magnetic induction plotted against $\xi$ .**

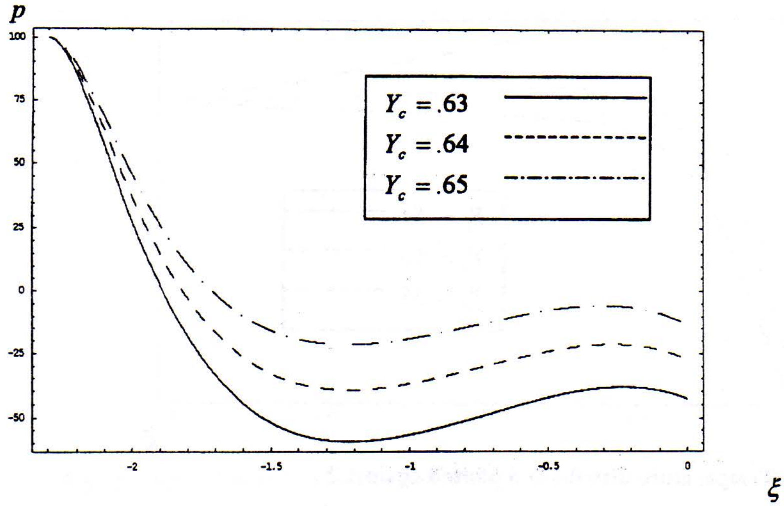


**Fig. (5). Pressure of the fluid plotted against $\xi$.**

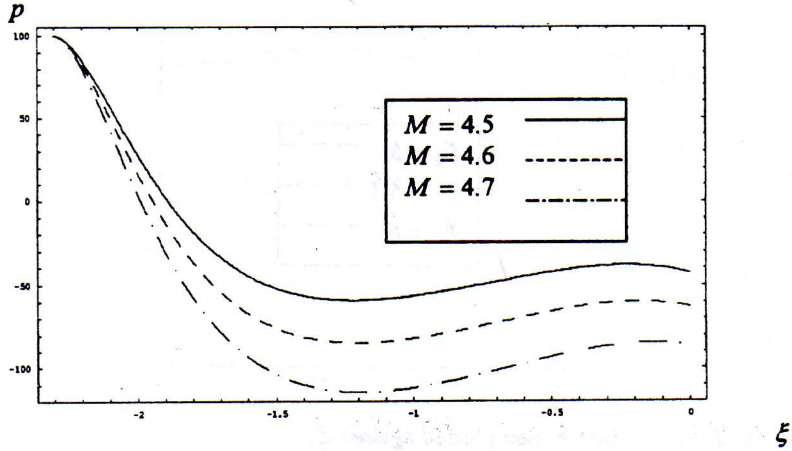


**Fig. (6). Pressure of the fluid plotted against $\xi$.**

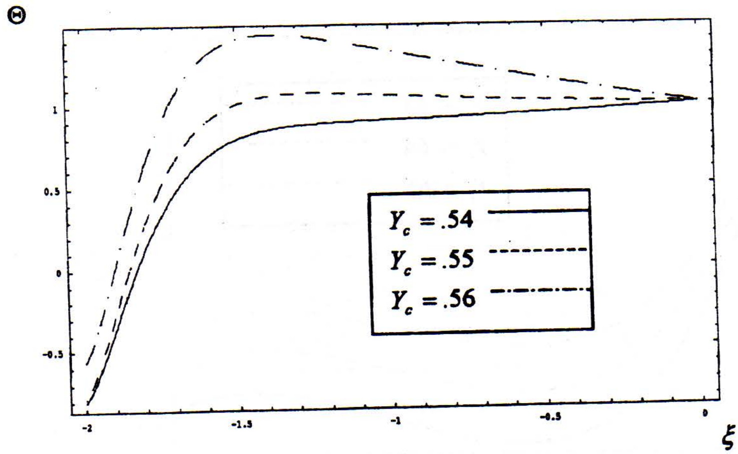


**Fig. (7). Temperature distribution plotted against $\xi$.**

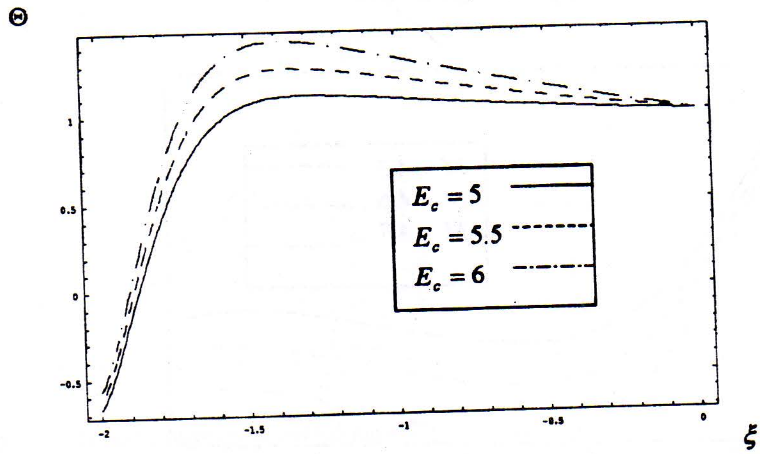


**Fig. (8). Temperature distribution plotted against $\xi$.**

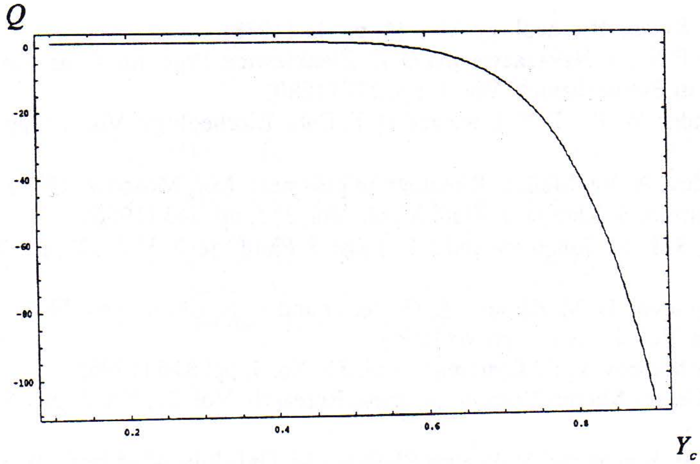


**Fig. (9). Rate of heat transfer plotted against $Y_c$.**

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## Appendix

$$\lambda_1 = -1 + \sqrt{1-\alpha^2 M^2 b^2}, \qquad \lambda_2 = -1 - \sqrt{1-\alpha^2 M^2 b^2},$$

$$\lambda_3 = \frac{3\alpha^{\lambda_1+\lambda_2+2} M^2 b^3}{2\left(\alpha^{\lambda_2} - \alpha^{\lambda_1}\right)^2 \left[(\lambda_1+\lambda_2)^2 + 2(\lambda_1+\lambda_2) + \alpha^2 M^2 b^2\right]},$$

$$\lambda_4 = \frac{3\alpha^{2(\lambda_2+1)} M^2 b^3}{4\left(\alpha^{\lambda_2} - \alpha^{\lambda_1}\right)^2 \left[4\lambda_1^2 + 4\lambda_1 + \alpha^2 M^2 b^2\right]},$$

$$\lambda_5 = \frac{3\alpha^{2(\lambda_1+1)} M^2 b^3}{4\left(\alpha^{\lambda_2} - \alpha^{\lambda_1}\right)^2 \left[4\lambda_2^2 + 4\lambda_2 + \alpha^2 M^2 b^2\right]},$$

$$\lambda_6 = \frac{3\alpha^{\lambda_2+2} M^2 b^3 \lambda_4}{2\left(\alpha^{\lambda_2} - \alpha^{\lambda_1}\right) \left[9\lambda_1^2 + 6\lambda_1 + \alpha^2 M^2 b^2\right]},$$

$$\lambda_7 = \frac{3\,\alpha^{\lambda_1+2} M^2 b^3 \lambda_5}{2\left(\alpha^{\lambda_2} - \alpha^{\lambda_1}\right) \left[9\lambda_2^2 + 6\lambda_2 + \alpha^2 M^2 b^2\right]},$$

$$\lambda_8 = \frac{3\alpha^{\lambda_2+1} M^2 b^3 \left[\lambda_3 \alpha^{\lambda_2}\left(1-\alpha^{\lambda_1}\right)+\lambda_4\left(\alpha^{2\lambda_1}-\alpha^{\lambda_2}\right)+\lambda_5 \alpha^{\lambda_2}\left(\alpha^{\lambda_2}-1\right)\right]}{2\left(\alpha^{\lambda_2}-\alpha^{\lambda_1}\right)^2\left[4\lambda_1^2+4\lambda_1+\alpha^2 M^2 b^2\right]},$$

$$\lambda_9 = \frac{3\alpha^{\lambda_1+2} M^2 b^3 \left[\lambda_3 \alpha^{\lambda_1}\left(\alpha^{\lambda_2}-1\right)+\lambda_4 \alpha^{\lambda_1}\left(1-\alpha^{\lambda_1}\right)+\lambda_5\left(\alpha^{\lambda_1}-\alpha^{2\lambda_2}\right)\right]}{4\left(\alpha^{\lambda_2}-\alpha^{\lambda_1}\right)^2\left[4\lambda_2^2+4\lambda_2+\alpha^2 M^2 b^2\right]},$$

$$\lambda_{10} = \frac{3\,\alpha^2 M^2 b^3\left(\alpha^{\lambda_2}\lambda_3+\alpha^{\lambda_1}\lambda_4\right)}{\left(\alpha^{\lambda_2}-\alpha^{\lambda_1}\right)\left[\left(2\lambda_1+\lambda_2\right)^2+2\left(2\lambda_1+\lambda_2\right)+\alpha^2 M^2 b^2\right]},$$

$$\lambda_{11} = \frac{3\,\alpha^2 M^2 b^3\left(\alpha^{\lambda_1}\lambda_3+\alpha^{\lambda_2}\lambda_5\right)}{\left(\alpha^{\lambda_2}-\alpha^{\lambda_1}\right)\left[\left(\lambda_1+2\lambda_2\right)^2+2\left(\lambda_1+2\lambda_2\right)+\alpha^2 M^2 b^2\right]},$$

$$C_1(\alpha,\lambda_1,..,\lambda_5) = \frac{\lambda_3 \alpha^{\lambda_2}\left(1-\alpha^{\lambda_1}\right)+\lambda_4\left(\alpha^{2\lambda_1}-\alpha^{\lambda_2}\right)+\lambda_5 \alpha^{\lambda_2}\left(\alpha^{\lambda_2}-1\right)}{\alpha^{\lambda_1}-\alpha^{\lambda_2}},$$

$$C_2(\alpha,\lambda_1,..,\lambda_5) = \frac{\lambda_3 \alpha^{\lambda_1}\left(\alpha^{\lambda_2}-1\right)+\lambda_4 \alpha^{\lambda_1}\left(1-\alpha^{\lambda_1}\right)+\lambda_5\left(\alpha^{\lambda_1}-\alpha^{2\lambda_2}\right)}{\alpha^{\lambda_1}-\alpha^{\lambda_2}},$$

$$C_3(\alpha,\lambda_1,..,\lambda_{11}) = \left[\lambda_6\left(\alpha^{3\lambda_1}-\alpha^{\lambda_2}\right)+\lambda_7 \alpha^{\lambda_2}\left(1-\alpha^{2\lambda_2}\right)+\lambda_8\left(\alpha^{2\lambda_1}-\alpha^{\lambda_2}\right)+\right.$$
$$\left.\lambda_9 \alpha^{\lambda_2}\left(1-\alpha^{\lambda_2}\right)+\lambda_{10}\alpha^{\lambda_2}\left(1-\alpha^{2\lambda_1}\right)+\lambda_{11}\alpha^{\lambda_2}\left(\alpha^{\lambda_1+\lambda_2}-1\right)\right] / \alpha^{\lambda_2}-\alpha^{\lambda_1}$$

$$C_4(\alpha,\lambda_1,..,\lambda_{11}) = \left[\lambda_6 \alpha^{\lambda_1}\left(1-\alpha^{2\lambda_1}\right)+\lambda_7\left(\alpha^{3\lambda_2}-\alpha^{\lambda_1}\right)+\lambda_8 \alpha^{\lambda_1}\left(1-\alpha^{\lambda_1}\right)+\right.$$
$$\left.\lambda_9\left(\alpha^{2\lambda_2}-\alpha^{\lambda_1}\right)+\lambda_{10}\alpha^{\lambda_1}\left(\alpha^{\lambda_1+\lambda_2}-1\right)+\lambda_{11}\alpha^{\lambda_1}\left(1-\alpha^{2\lambda_2}\right)\right] / \alpha^{\lambda_2}-\alpha^{\lambda_1}$$

$$F_1(\xi) = 2\int e_{r\theta}\, d\xi, \quad F_2(\xi) = 2\int\left(u^2 - K_1 M^2 \frac{h}{e^{\xi}}\frac{d}{d\xi}\left(e^{\xi} h\right)\right)d\xi,$$

$$F_3(\xi) = \int\left(2b\,\tau_{r\theta}\, e_{r\theta}\, e^{2\xi} + \frac{M^2}{e^{2\xi}}\left(\frac{d}{d\xi}\left(e^{\xi} h\right)\right)^2\right)d\xi,$$

$$F_3(\xi) = \int\left(2b\,\tau_{r\theta}\, e_{r\theta}\, e^{2\xi} + \frac{M^2}{e^{2\xi}}\left(\frac{d}{d\xi}\left(e^{\xi} h\right)\right)^2\right)d\xi,$$