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The Transformational Analytical Solution for Nonlinear Convection in the Presence of Two-Way Rotation

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Steady two-dimensional finite amplitude solutions are obtained for the problem of convection in a horizontal fluid layer heated from below and rotated about its vertical axis. The flow is assumed to be laminar and two-dimensional. The density variation is taken into account by the Boussinesq approximation. Different boundaries with prescribed constant temperature are assumed and the solutions are obtained. The transform for summing the variables, which reduce the nonlinear partial differential equation into ordinary differential equation of the high order, is used. The existence of steady subcritical finite amplitude solutions is demonstrated for different Prandtl numbers. A strong reduction in the domain of stable rolls that occurs as the rotation rate is increasing. Convection driven by thermal buoyancy in the presence of the Coriolis force occurs in planetary atmospheres and interiors. Asymptotic expressions for the onset of convection in a horizontal fluid layer of finite extent heated from below and rotating about a vertical axis are derived in the limit of large rotation rates in the case of stress-free upper and lower boundaries. In the presence of the vertical sidewalls the critical Rayleigh number R is much lower than the classical value of the infinity extended layer.

Keywords: convection, rigid, stress-free, mixed boundaries, Boussinesq approximation, nonlinear, analytical solution, Navier-Stocks, PDE, two-way rotation.

1. Introduction

Chandrasekhar (1961) has considered studies of convection motions in a plane horizontal homogenous layer with and without rotation in some detail in the monographs. For the plane fluid layer heated from below it has been shown that the sole stable solution has the form of two-dimensional rolls. These rolls for non-rotating fluid are subject to various types of instabilities depending on the value of the Prandtl number and Rayleigh number.

The problem of the critical conditions for tho onset of conveciion in a rotating fluid layer heated from below has received much attention ever since Rossby (1969). Who noticed in his experiment with a water layer that the onset of convection occurred at a lower critical Rayleigh number; than that predicted by the theory of Chandrasekhar (1961). In later experiments by Lucas, Pfrotenhauer & Donnelly (1987) a similar discrepancy was observed. At about the same time Buell & Catten (1983) found from numerical computations that in a circular layer of a finite radius non-axisymmetric forms of convection could set at Rayleigh numbers considerably below the value R calculated for an infinite layer. This finding appeared to resolve reasonably well the discrepancies between experimentally observed and theoretically predicted values.

More recent detailed observations by Zhong, Ecke & Steinberg (1991) and by Ecke, Zhong & Knobloch (1991) have demonstrated, however, that the onset of convection occurs in the form of drifting waves in contrast to the steady modes assumed in the analysis of Buell & Catton. As has been pointed quite correctly by Ecke *et al.* Time-dependent onset must be regarded as the rule rather than the exception in a rotating system because of the broken left-right symmetry. Detailed computations for the onset of drifting modes by Goldsten *et al.* (1993) show good agreement with the observations of Ecke *et al.* The computations of Buell & Canon (1983) as well as the more resent calculations by Goldstein *et al.* (1993) have demonstrated that the presence of sidewalls supports. The early onset of convection and that insulating sidewalls in particular lead to a substantial decrease in the critical Rayleigh number in comparison to the case of an infinite layer. But the range of the rotation parameter considered in the calculation t as been restricted. Owing to the problems of numerical convergence, and because of the assumption of a circular layer the aspect ratio has entered the analysis as an additional parameter. Since the wall attached convection flow represents basically a boundary-layer phenomenon in terms of the distance from the sidewalls, a finite curvature of the wall is not an essential ingredient of the problem.

In this paper an asymptotic analytical solution as well as a numerical study are presented for the case of a plane sidewalls. The mathematical formulation of the problem is given in Section 3. Both boundaries are Stress-free in Section 4. Both boundaries are rigid in Section 5. The mixed boundary problem is in Section 6. The steady solution is in Section 7. The results are in Section 8. Md a general discussion is given in a concluding Section 9. Figures and references are in the final.

2. Mathematical formulation of the problem

We considered a plane layer with free boundaries and the motion is two-dimensional. This assumption has an experimental justification since the convection starts in the form of two-dimensional rolls. Since $\frac{\partial}{\partial y} \equiv 0$, one can introduce the vorticity as $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ and the stream function ϕ, such that $u = \frac{\partial \phi}{\partial z}$, $w = -\frac{\partial \phi}{\partial x}$, where from $\eta = \nabla^2 \phi$ and the system of momentum and heat transfer equations is transformed into:

$$\frac{\partial V}{\partial t} + V.\nabla V = \nabla^2 V + K\theta - \nabla \Pi - 2\Omega k \wedge V \tag{1}$$

$$\nabla . V = 0 \tag{2}$$

$$p\left(\frac{\partial\theta}{\partial t}+V.\nabla\theta\right)=\nabla^2\theta+RK.V\,. \tag{3}$$

Non-linear effects influence the temperature field through interaction between ϕ and θ, the horizontal velocity component through interaction between ϕ and u. Due to this, in the temperature field a component $\sin 2\Pi z$ type will be generated and in the horizontal velocity field a component will appear. The minimal system describing the finite amplitude convection can be represented.

These equations are normalized in terms of the following physical parameters $d, \frac{d^2}{\mu}$ and $\frac{T_1-T_2}{R}$ as scales for length, time and temperature respectively. μ is the thermal diffusivity and K is the unit vector direction opposite to the gravity vector g. The Rayleigh number R and the Prandtl number P are defined as [1,5]

$$R=\frac{g(T_2-T_1)\gamma^3 d}{\nu\mu}\,,\quad P=\frac{\nu}{\mu}\,, \tag{4}$$

where γ is coefficient of thermal expansion and ν is the kinematics viscosity. The velocity can be expressed as [4,6] in the form

$$V=\left(\frac{\partial\phi}{\partial z},0,-\frac{\partial\phi}{\partial x}\right)\,.$$

By substituting in equations (1)–(3) we get nonlinear P.D.E. Applying the curl operation to equation (1), we assume that the functions $\theta(X,Z)$ and $\phi(X,Z)$ take the form as [10]

$$\begin{aligned}\phi(x,z)&=x+\phi(z)\,,\\ \theta(x,z)&=x+\theta(z)\,.\end{aligned} \tag{5}$$

Applying basic equations and by substituting the assumption (5) into equations (1)

$$\frac{d^4\phi(z)}{dz^4}+\frac{d^3\phi(z)}{dz^3}+2\Omega\frac{d^2\phi(z)}{dz^2}=1 \tag{6}$$

and (3) hence we have

$$\frac{d^2\theta(z)}{dz^2}+P\frac{d\theta(z)}{dz}=R+P\frac{d\phi(z)}{dz}\,. \tag{7}$$

By using (7) into equations (6) and (7) we can easily get analytical solution in the form

$$\phi(z)=\frac{z^2}{4\Omega}-\frac{\mathrm{e}^{-\frac{(1+n)z}{2}}(n-1)c_1}{2\Omega(1+n)}-\frac{\mathrm{e}^{\frac{(n-1)z}{2}}(n+1)c_2}{2\Omega(n-1)}+c_3-zc_4 \tag{8}$$

$$\begin{aligned}\theta(z)\;=\;&\frac{z^2}{4\Omega}-\frac{\mathrm{e}^{-pz}c_1}{p}+c_2+\mathrm{e}^{-\frac{(1+n)z}{2}}\left(-\frac{pc_1(n-1)}{(-1+2p-n)(1+n)\Omega}-\right.\\ &\left.\frac{\mathrm{e}^{nz}p(1+n)c_2}{(n-1)(-1+2p+n)\Omega}\right)+\frac{z(-1+2R\Omega+2p\Omega c_4)}{2p\Omega}\,,\end{aligned} \tag{9}$$

where $n=\sqrt{1-8\Omega}$, see appendix to find the analytical solutions.

2.1. *Both boundaries are stress-free*

The boundary conditions of this case [1,3,5,16] are

$$\begin{aligned} \phi(z) &= \frac{\partial^2\phi(z)}{\partial z^2} = 0, \text{ at } Z = \pm\frac{1}{2}\,, \\ \theta(z) &= 0, \text{ at } Z = \pm\frac{1}{2}\,. \end{aligned} \tag{10}$$

By substituting from these boundary conditions (10) into equations (8,9) hence we will have analytical solutions; see appendix.

$$n = \sqrt{1 - 8\Omega}\,. \tag{11}$$

2.2. *Both boundaries are rigid*

In this case the boundary condition [5,16]

$$\begin{aligned} \phi(z) &= \frac{\partial\phi(z)}{\partial z} = 0, \text{ at } z = \pm\frac{1}{2}\,, \\ \theta(z) &= 0, \text{ at } z = \pm\frac{1}{2}\,. \end{aligned} \tag{12}$$

We use these boundary conditions (12) to express the constants as [3,7,17]. We can also get solution by the same way for determine the constants

2.3. *The mixed boundary problem*

In this problem the boundary conditions [5,16] are

$$\begin{aligned} \phi(z) &= 0, \text{ at } z = \pm\frac{1}{2}\,, \\ \frac{\partial\phi(z)}{\partial z} &= 0, \text{ at } z = -\frac{1}{2}\,, \\ \frac{\partial^2\phi(z)}{\partial z^2} &= 0, \text{ at } z = \ \ \frac{1}{2}\,, \\ \theta(z) &= 0, \text{ at } z = \pm\frac{1}{2}\,. \end{aligned} \tag{13}$$

From equation (8, 9, 13) we can in this case determine the constants and get the exact solution We can also get solution by the same way for determine the constants. Hence the exact solution easily can be found in the form in this method the vertical temperature gradient within the layer is measured in dependence on the temperature difference at the layer boundaries.

3. The Steady Solution

The heat transfer is usually represented in terms of the Nusselt number which describes the ratio between the heat transport which convection and what would be without convection at a given Rayleigh number.

In describing an analytical result for steady convection rolls we shall concentrate on the convective heat transfer. Which not only is the parameter of primary physical interest, but also appears to characterize best the other aspects of convection is viscous dissipation occur at the same rate is the convection heat transport. Number of analytical studies of convection rolls have handed results for the Nusselt number

$$Nu = 1 + \frac{\langle \vee . k\theta(z) \rangle}{R} \tag{14}$$

4. Results

We can see from Fig. 1 that we have three convection cells one of them is located at zero rotation and then this cell splits into two at rotation of 75. Outside these two rotation values almost no cells are formed. That is if we increase the rotation beyond $\Omega = 75$ we do not get convection cells. Nevertheless, if the rotation is reversed we get no convection cells at all.

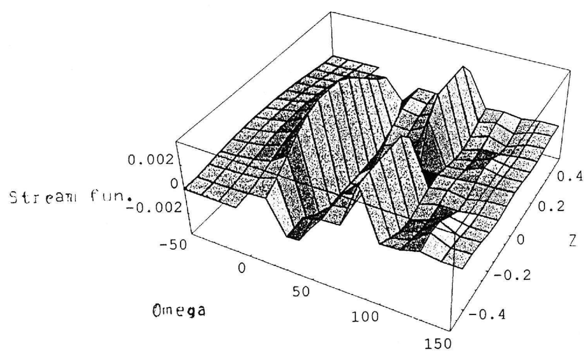

Figure 1 Convection as the relation between stream function ϕ and z-axis with different values of the rotation Ω in the case of rigid boundaries

In Fig. 2 the velocity distribution agrees completely with the convection cell distribution and we find that at zero rotation the velocity is lowering its highest values with a little of negative values. If the rotation is increased, the velocity is perturbed and diminished until it reaches zero. Besides, for negative rotation the velocity is zero-everywhere.

Fig. 3 in the case of stress-free boundary there is only one big cell formed at zero rotation with no other cells elsewhere. Fig. 4 is zooming in at the velocity $u(z)$ axis from -0.01 to 0.01, showing the concentration of velocity lines at the zero rotation.

Fig. 5 – here one prominent convection cell is found at zero rotation, and smaller one at a rotation value of almost 50. Fig. 6 the values of velocity $u(z)$ show a max at zero and 20 units of rotation and are zero elsewhere.

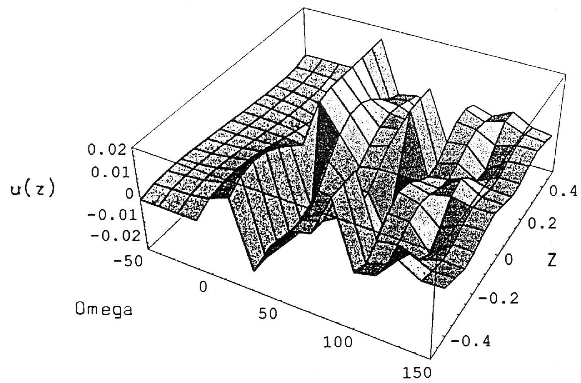

Figure 2 Convection as the relation between the velocity of particles $u(z)$ and z-axis with different values of the rotation Ω in the case of rigid boundaries

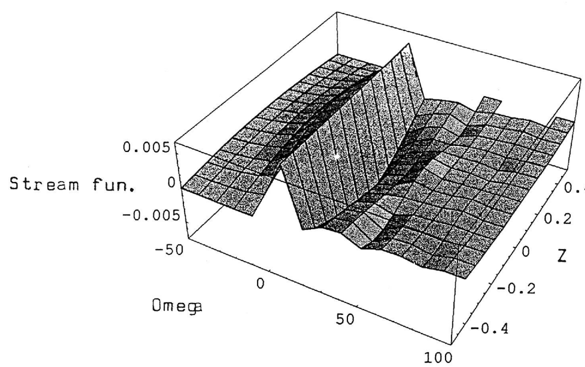

Figure 3 Convection as the relation between stream function ϕ and z-axis with different values of the rotation Ω in the case of stress-free boundaries

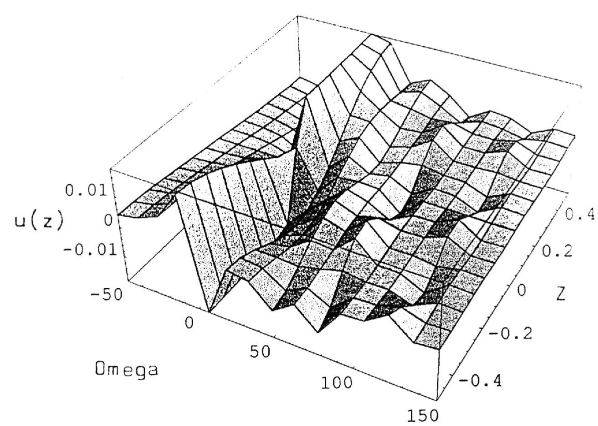

Figure 4 Convection as the relation between the velocity of particles $u(z)$ and z-axis with different values of the rotation Ω in the case of stress-free boundaries

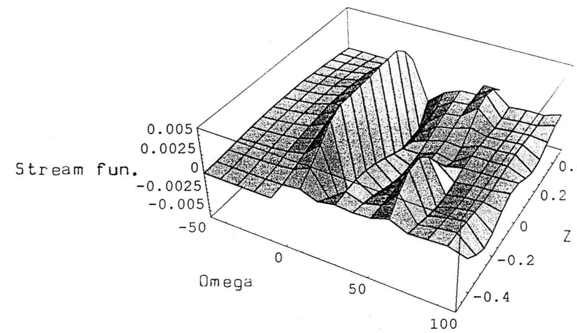

Figure 5 Convection as the relation between stream function ϕ and z-axis with different values of the rotation Ω in the case of mixed boundaries

Fig. 7 represents the convection in the stress-free surfaces under Coriolis forces (rotation forces) from negative to positive value of rotation. The range of negative value is logarithmic. A peak is formed at zero rotation. Positive values of rotation are natural to be found in the opposite direction. We note that the Nusselt number values increase with increasing Prandtl number; i.e. the viscous dissipation is speeded at low Prandtl numbers. These figures have the same shape and size except that they differ in the range the three variables representing the surfaces appear in meaning that the solutions in the cases are identical with different variable-ranges, which is physically acceptable. At the origin we have an inflection line for any of the three surfaces with different Prandtl numbers. Nevertheless. the increase in Prandtl numbers cases an increase in Nusselt numbers.

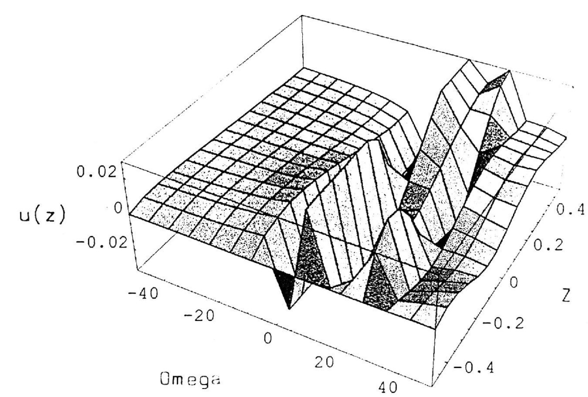

Figure 6 Convection as the relation between the velocity of particles $u(z)$ and z-axis with different values of the rotation Ω in the case of mixed boundaries

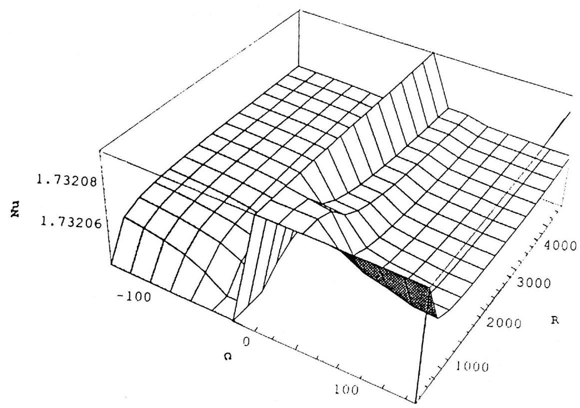

Figure 7 Convection as the relation between the Nusselt number Nu and Rayleigh number R with different values of the rotation Ω in the case of stress-free boundaries at Prandtl number $p = 10$

Fig. 8 represents the convection between two horizontal planes he:tted from below. The surfaces in general have logarithm;c surfaces form. Negative values of rotation are represented, but positive values take imaginary solutions. In these figures we observe that we have different onset convection with Rayleigh numbers. Fig. 9 represents the convection in the case of rigid boundaries for negative rotation they look like the logarithmic curve with Rayleigh number. The convection with no rotation is greater than that with rotation. In these figures show that only the negative rotation in this case can be taken into consideration, for the positive values in this case give imaginary results.

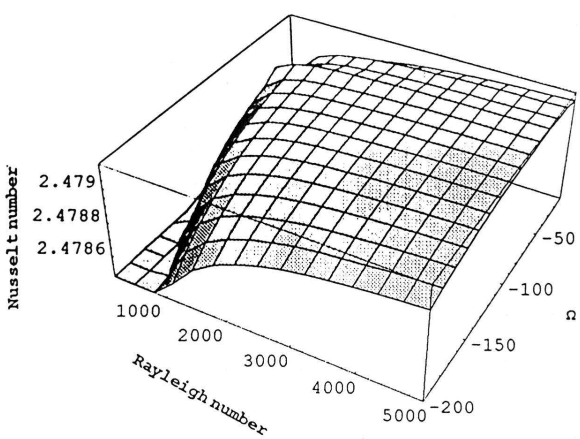

Figure 8 Convection as the relation between the Nusselt number Nu and Rayleigh number R with different values of the rotation Ω in the case of rigid boundaries at Prandtl number $p = 10$

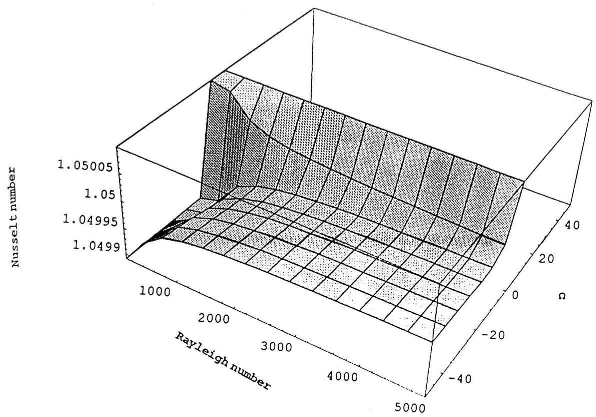

Figure 9 convection as the relation between the Nusselt number Nu and Rayleigh number R with different values of the rotation Ω in the case of mixed boundaries at Prandtl number $p = 10$

5. Conclusions

We note that the case of free boundaries is less stable than with rigid boundaries. Increasing this difference, we proportionally increase the gradient up to the moment the motion will start. The further increase of the temperature difference $\Delta\theta$ leads to a slower increase of the temperature gradient in the central part of the layer and to a faster increase near the boundaries, due to the fact that near the boundaries the thermal boundary layers start forming.

When $Ra < Rac$ the motions are absent in the fluid and the heat flux is formed by purely molecular mechanisms when the Nusselt number $Nu = 1$. For $Ra \geq Rac$, the Nusselt number becomes $Nu \geq 1$ caused by the appearance of the motions in the fluid. Note that at moderate rotation rates for free boundaries, the values of the Nusselt number always decrease with the increase of the angular velocity Ω, while in the case of rigid boundaries, the values of the Nusselt number may increase for fixed Rayleigh number wilt increase. At stronger rotation the Nusselt number does not depend on the type of boundary conditions.

5.1. *Applications of rotation*

The fundamental examples lie in geophysical and astrophysical fluid dynamics. No experimental work seems to have been done with which the theoretical results of the present analysis can be compared especially in the negative direction of rotation. Many self gravitating astronomical bodies including the major planets, the sun, and the Earth's liquid core, correspond to this limit. In the laboratory, an analogous system can be constructed with a very rapidly rotating apparatus, in which the centrifugal force plays the role of self gravitation. The formulation is offered in such a way that both these geophysical system and laboratory analogues are included as special cases. Knowledge of the general behavior of thermal convection in rapidly rotating system is crucial to understanding various geophysical and astrophysical flow; the fluid motion inside the major planets, the Sun and the Earth's core are good examples. A revised asymptotic theory of thermal convection in rapidly rotating systems has been constructed for cases applicable both to geophysical self gravitating systems and their laboratory analogues, where the centrifugal force mimics self gravitating. The paper closes with an outlook on applications to the dynamics in the major planets. Convection driven by thermal buoyancy in the presence of the Coriolis force occurs in planetary atmospheres and interiors.

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Appendix

$$
\begin{aligned}
\varphi[z] = \Big(&-1 + E^{\frac{1}{2}(-1-n)} - E^{\frac{1}{2}(-1+n)} + E^{n} + \\
&2 E^{-\frac{1}{4}(1+n)(-1+2z)} + 2 E^{\frac{1}{4}(-1+n)(1+2z)} - 2 E^{-\frac{1}{4}(1+n)(1+2z)} - \\
&2 E^{\frac{1}{4}(1-2z+n(3+2z))} + n^4 \Big(-1 + E^{\frac{1}{2}(-1-n)} - E^{\frac{1}{2}(-1+n)} + E^{n} + \\
&\quad 2 E^{-\frac{1}{4}(1+n)(-1+2z)} + 2 E^{\frac{1}{4}(-1+n)(1+2z)} - 2 E^{-\frac{1}{4}(1+n)(1+2z)} - \\
&\quad 2 E^{\frac{1}{4}(1-2z+n(3+2z))} + 2 \left(-1 + E^{\frac{1}{2}(-1-n)} - E^{\frac{1}{2}(-1+n)} + E^{n}\right) z\Big) + \\
&4 n^3 \Big(-1 - E^{\frac{1}{2}(-1+n)} + E^{n} + E^{\frac{1+n}{2}} + 2 E^{\frac{1}{4}(-1+n)(1+2z)} - \\
&\quad 2 E^{\frac{1}{4}(1-2z+n(3+2z))} + \left(2 - 2 E^{\frac{1}{2}(-1+n)} + 2 E^{n} - 2 E^{\frac{1+n}{2}}\right) z\Big) - \\
&60 \Omega^2 - 64 E^{\frac{1}{2}(-1-n)} \Omega^2 + \\
&64 E^{\frac{1}{2}(-1+n)} \Omega^2 + 60 E^{n} \Omega^2 - 128 E^{\frac{1}{4}(-1+n(3-2z)-2z)} \Omega^2 + \\
&128 E^{-\frac{1}{4}(1+n)(1+2z)} \Omega^2 + 16 (-1 + E^{n}) z^2 \Omega^2 - \\
&2 E^{-1-\frac{n}{2}} \left(\sqrt{E} - E^{\frac{1}{2}+n} - E^{\frac{2+n}{2}} + E^{1+\frac{3n}{2}}\right) z (-1 + 64 \Omega^2) + \\
&2 n^2 \Big(-3 - E^{\frac{1}{2}(-1-n)} - 3 E^{\frac{1}{2}(-1+n)} + 3 E^{n} + 4 E^{\frac{1+n}{2}} - 2 E^{-\frac{1}{4}(1+n)(-1+2z)} + \\
&\quad 6 E^{\frac{1}{4}(-1+n)(1+2z)} + 2 E^{-\frac{1}{4}(1+n)(1+2z)} - 6 E^{\frac{1}{4}(1-2z+n(3+2z))} + \\
&\quad \left(10 - 2 E^{\frac{1}{2}(-1-n)} - 6 E^{\frac{1}{2}(-1+n)} + 6 E^{n} - 8 E^{\frac{1+n}{2}}\right) z + \\
&\quad 2 \Omega^2 - 2 E^{n} \Omega^2 + 8 (-1 + E^{n}) z^2 \Omega^2\Big) + \\
&4 n \Big(-1 - E^{\frac{1}{2}(-1+n)} + E^{n} + E^{\frac{1+n}{2}} + 2 E^{\frac{1}{4}(-1+n)(1+2z)} - \\
&\quad 2 E^{\frac{1}{4}(1-2z+n(3+2z))} + \left(2 - 2 E^{\frac{1}{2}(-1+n)} + 2 E^{n} - 2 E^{\frac{1+n}{2}}\right) z + \\
&\quad 2 \Omega^2 - 2 E^{n} \Omega^2 + 8 (-1 + E^{n}) z^2 \Omega^2\Big)\Big) \Big/ \\
&(64 (-1 + E^{n}) (1 + n)^2 \Omega^3)
\end{aligned}
$$