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Unsteady Emptying of a Pressure Vessel

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In this paper the problem of the unsteady emptying of a pressure vessel is solved by integrating the equations representing conservation of mass and energy in adiabatic expansion numerically for air where the adiabatic index $\gamma = 1.4$, two unknown functions are determined namely; the pressure and temperature inside the bottle. The ambient pressure and temperature are assumed constants and are used for non-dimensionalization. The area of the orifice is considered a constant throughout the whole expansion.

Keywords: pressure vessel, unsteady process.

1. Introduction

In compressed air and gas technology, pressure vessels are commonly used. Precautions are made that the valves of the vessels are tight enough to prevent escape of the content gas to the environment. However, corrosion rust and imperfect closure may result in the escape of the gas to the ambient air. The area of the orifice will determine the rate of expansion. We shall consider the case; this rate is rapid enough so that there is no time for heat to flow to the vessel i.e., the process will be adiabatic. The following elementary thermodynamic analysis [1] will show:

$$\text{(Enthalpy - intemal energy)/unit mass and unit temperature } = \frac{P}{\rho T}\,.$$

For an ideal gas $\frac{P}{\rho T} = R$, the gas constant, enthalpy=$c_p T$, internal energy= $c_\nu T$, i.e.

$$c_p - c_\nu = R\,, \tag{1}$$

together with the definition

$$\frac{c_p}{c_\nu} = \gamma\,. \tag{2}$$

The first law $dQ = dW + dE$, where value dQ is heat added to the system (either positive or negative), dW is the work done on the system (either positive or negative) and dE the change in the internal energy.

For adiabatic process $dQ = 0$, therefore

$$\begin{aligned} 0 &= Pd\frac{1}{\rho} + c_\nu dT\,, \text{ written for unit mass,} \\ &= Pd\frac{1}{\rho} + \frac{c_\nu}{R} d\left(\frac{P}{\rho}\right) = -Pd\rho + \frac{c_\nu}{R}(\rho dP - Pd\rho)\,, \\ &= -Pd\rho + \frac{1}{\gamma-1}(\rho dP - Pd\rho) = -(\gamma-1)Pd\rho + (\rho dP - Pd\rho)\,, \end{aligned}$$

which implies

$$\gamma P d\rho = \rho dP \text{ therefore } \frac{dP}{P} = \gamma\frac{D\rho}{\rho}\,, \quad \text{i.e.,} \quad \frac{P}{\rho^\gamma} = \text{constant}\,.$$

2. Statement of the problem

Consider a pressure vessel of constant volume V and initial pressure $P^* > P_a$ and temperature T_a =ambient temperature and P_a the ambient pressure. The area of the orifice is a and the gas will start escape at $t = 0$; the following conservation laws are:

1. Conservation of mass:

$$\frac{dm}{dt} = \frac{d}{dt}\left(\frac{PV}{RT}\right) = \frac{V}{R}\frac{d}{dt}\left(\frac{P}{T}\right) = -a\rho\nu\,,$$

ρ is the exit gas density equal to $\frac{P_a}{RT}$

$$\frac{1}{2}\nu^2 = \int_{P,\frac{P}{RT}}^{P_a,\frac{P_a}{RT}} \frac{dP}{\rho} \text{ with } \frac{P}{\rho^\gamma} = \frac{P_a}{\rho_a^\gamma}$$

therefore

$$\nu = \sqrt{\frac{2\gamma}{\gamma-1}}\sqrt{RT_a}\frac{1}{\bar{T}^\gamma}\left(\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right)^{\frac{1}{2}}$$

where

$$\bar{P} = \frac{P}{P_a} \text{ and } \bar{T} = \frac{T}{T_a}\,.$$

The conservation of mass gives

$$\frac{d}{d\tau}\left[\frac{\bar{P}}{\bar{T}}\right] = -\bar{T}^{-\frac{\gamma+1}{2}}\left[\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right]^{\frac{1}{2}}\,, \tag{3}$$

where

$$\tau = \frac{a}{V}\sqrt{\frac{2\gamma RT_a}{\gamma-1}}t\,.$$

2. The conservation of energy:
   The rate of energy drop in the vessel = the rate of energy escape from orifice

$$\frac{d}{dt}\left[PV + mc_\nu T\right] = \dot{m}\left[c_\nu T + \frac{1}{2}\nu^2\right]$$

$$\text{L.H.S.} = P_a V \frac{\gamma}{\gamma - 1}\frac{d\bar{P}}{dt}$$

$$\text{L.H.S.} = \frac{P_a}{T_a}\frac{V}{R}\frac{d}{dt}\left(\frac{\bar{P}}{\bar{T}}\right)\left[c_\nu \bar{T}_a \bar{T} + \frac{\gamma}{\gamma - 1} R T_a \bar{T}^{(1-\gamma)}\left(\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right)\right].$$

Equating both sides

$$\frac{d}{d\bar{P}}\left(\frac{\bar{P}}{\bar{T}}\right) = \frac{\gamma}{\bar{T}\left[1 + \gamma \bar{T}^{-\gamma}\left(\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right)\right]} \tag{4}$$

Equations (3) and (4) are written in the form

$$\frac{d}{d\tau}\left[\frac{\bar{P}}{\bar{T}}\right] = -F(\bar{P}, \bar{T}), \tag{5}$$

$$\frac{d}{d\bar{P}}\left[\frac{\bar{P}}{\bar{T}}\right] = -G(\bar{P}, \bar{T}). \tag{6}$$

The set (5), (6) reduces to

$$\frac{d\bar{P}}{d\tau} = -\frac{F(\bar{P}, \bar{T})}{G(\bar{P}, \bar{T})}, \tag{7}$$

$$\frac{d\bar{T}}{d\tau} = -\frac{F(\bar{P}, \bar{T})}{G(\bar{P}, \bar{T})}\frac{\bar{T}}{\bar{P}}\left(1 - \bar{T}\, G(\bar{P}, \bar{T})\right). \tag{8}$$

The system (7) and (8) has the initial conditions

$$\bar{P}(0) = \bar{P}^* \gg 1,$$

$$\bar{T}(0) = 1.$$

and

$$F(\bar{P}, \bar{T}) = \bar{T}^{-\frac{\gamma+1}{2}}\left[\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right]^{\frac{1}{2}},$$

$$G(\bar{P}, \bar{T}) = \frac{\gamma}{\bar{T}\left[1 + \gamma \bar{T}^{-\gamma}\left(\bar{P}^{\frac{\gamma-1}{\gamma}} - 1\right)\right]}$$

3. Comment

The results show that for $P^* = 100$ the pressure drops to approximately 1 after a time $\tau = 234 \times 0.05 = 11.7$ which is equivalent to

$$11.7 = \frac{a}{V}\sqrt{\frac{2\gamma RT_a}{\gamma - 1}}t = 348\frac{at}{V} \,.$$

If $V = 1$ m^3 and $a = 1$ mm$^2 = 10^{-6}$ m^2 therefore the physical time

$$t = \frac{11.7 \times 1}{748 \times 10^{-6}} \text{ sec} = 4.5 \text{ hour} \,,$$

and for $P^* = 50$ the time is 4 hr. As for the temperature it reaches $0.02 \times 300 = 6$ K for $P^* = 100$ after 4.5 hr and reaches $0.04 \times 300 = 12$ K for $P^* = 50$ after 4 hr. The figures show the graphs for $P^* = 100$ and 50.

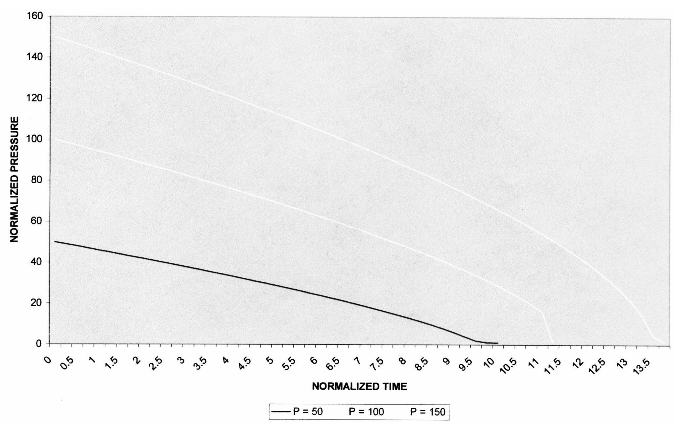

Figure 1 Normalize pressure shown in function of normalized time

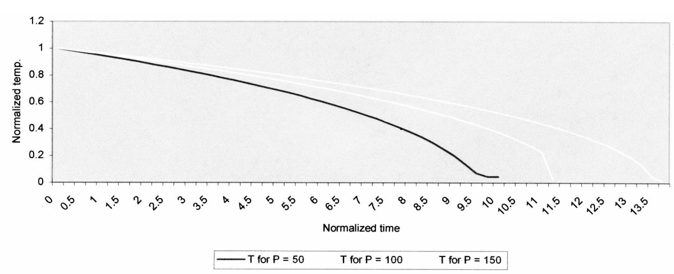

Figure 2 Normalize temperature shown in function of normalized time

These numerical results are obtained by finite difference integration using Euler's method [3] with $\Delta\tau = 0.05$ which is equivalent to 67.5 sec., the temperature T_a is considered 300 K i.e., 27°C. Fig. 1 shows the normalized pressure P/P_a versus the normalized time τ),

$$\tau = \frac{a}{V}\sqrt{\frac{2\gamma R T_a}{\gamma - 1}}\,t\,,$$

but Fig. 2 shows the normalized temperature T/T_a versus the normalized time τ where a is the area of the orifice and V is the volume of vessel.

References

[1] **Lee, JF** and **Sears, FW**: *Thermodynamics*, Adison-Wesly, Califomia, (1969), p.101.

[2] **Lee, JF** and **Sears, FW**: *Thermodynamics*, Adison-Wesly, Califomia, (1969), p.111.

[3] **Gerald, CF** and **Wheatley, PO**: *Applied Numerical Analysis*, Adison-Wesly, NY, (1994), pp.400-401.