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Oscillatory Non-Newtonian Viscoelastic Fluid Flow Past a Porous Bed

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An analytical study has been performed to evaluate the flow of viscoelastic fluid through and past an infinite porous bed. A most general form of Darcy law has been taken into account when the free stream oscillates with or without a non-zero mean. In modeling the flow in the bed a separation of variable technique was utilized to represent the governing equations with appropriate boundary layer assumption. The effect of flow inertia was taken into consideration. Results obtained include velocity distributions of the fluid in the bed. The dependence of velocity profile on elasticity parameter of the fluid and permeability parameter of the medium with the variation of the time and position was also obtained and discussed graphically.

Keywords: viscoelastic fluid, bed flow, non-Newtonian.

1. Introduction

The non-Newtonian fluids received a considerable importance, because of their use in industry. Also, the study of the physics of flow through porous media has become the base of many scientific and engineering applications. The type of flow is of importance to the petroleum engineer concerned with the movement of oil, gas or water through the reservoir of an oil or gas field and to the chemical engineer in connection with filtration [1–4]. The Oldroyed model has been modified by Walters [5]. This modified model is generally accepted as valid for viscoelastic fluids such as oils md polymers, etc.

There are many authors who interested with this subject. The problem of pulsatile magnetohydrodynamic viscoelastic flow through a channel bounded by two permeable parallel plates is studied by Eldabe and Elmohandis [6]. Soundalgekar and Puri [7] have studied the flow of non-Newtonian elastico-viscous fluid past an infinite fiat plate with variable suction under the condition of very small elastic parameter.

The purpose of this work is to study oscillatory viscoelastic fluid flow analytically past a porous bed. We make use of the generalized Darcy law to consider the fluid motion within the porous medium. Following [8] most general boundary conditions are derived and made use of, to study the flow in which the free stream is oscillating parallel or inclined to the plane bounding the porous bed. While the flow remain linear, we obtain results which are similar to those encountered in non-linear problems. Because of the interaction of oscillatory tangential component with time dependent suction, a mean flow is induced both inside and outside the porous bed.

2. Basic equations

The constitutive equation for incompressible visco-elastic fluids suggested by Walters [5] is,

$$\tau_{ij} = 2\mu d_{ij} - 2\lambda E_{ij} + 4\phi d_i^{\alpha} d_{\alpha j}\,, \tag{1}$$

where

$$d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)\,,$$

$$E_{ij} = \frac{1}{2}\left(a_{i,j} + a_{j,i} + 2v_i^m v_{m,j}\right)\,,$$

acceleration vector

$$a_i = \frac{\partial v_i}{\partial t} + v^m v_{i,m}\,,$$

v_i is the velocity vector. A comma followed by an index implies covariant differentiation, μ, λ, and ϕ are the material constants of viscosity, elastico-viscosity, and cross-viscosity coefficients of the fluid in order.

The equations governing the motion of an incompressible fluid in the porous medium are given by the generalized Darcy law:

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{v}{k}V_i + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\,, \tag{2}$$

alongwith the equation of continuity

$$\frac{\partial V_i}{\partial x_i} = 0\,, \tag{3}$$

where P is the pressure, ρ is the density, τ_{ij} is defined in equation (1), v is the kinematic viscosity of the fluid and k is the permeability of the porous bed.

The fluid flow in the pure fluid region is governed by the Navier-Stokes equations

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\,, \tag{4}$$

$$\frac{\partial v_i}{\partial x_i} = 0\,, \tag{5}$$

where v_i, and p are the velocity and pressure of the fluid in the pure fluid region.

The appropriate boundary conditions at the bounding surface of the porous bed are:

(i) The mass flow across the surface must be continuous, i. e.,

$$V_n = v_n \,, \tag{6}$$

where V_n and v_n are the normal components of the velocity at regions outside and inside the surface respectively.

(ii) The tangential components of the velocity in the pores at the surface must be continuous, i. e.,

$$V_t = v_t \,, \tag{7}$$

(iii) The conservation of momentum across the surface leads to

$$(\tau_{ij})_t - (T_{ij})_t = \rho v_n (v_t - V_t) = 0 \,, \tag{8}$$

$$(\tau_{ij})_n - (T_{ij})_n = \rho (p - P) = 0 \,, \tag{9}$$

where $(\tau_{ij})_t$, and $(\tau_{ij})_n$ are the tangential and normal components of the general form of the stress tensor on the surface of the porous medium in the ee uid region. $(T_{ij})_t$, and $(T_{ij})_n$ are the corresponding stress components in the porous bed.

3. Analysis

Now, we shall deal with the case of flowing viscoelastic fluid through an infinite porous bed. The Cartesian coordinate axes are considered so that the bounding surface is a plane surface $y = 0$. The free stream is oscillating parallel to the bounding surface with velocity $U_1 = U_0 \cos \omega t$, where ω is the frequency of oscillations.

Consider the velocity components in two regions as follows $(u(y,t), 0, 0)$ and $(U(y,t), 0, 0)$ so that equation (9) immediately yields $p = P$ at $y = 0$.

Since $u \to U_1 = U_0 \cos \omega t$ as $y \to \infty$, this gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U_1}{\partial t} \,. \tag{10}$$

From previous assumptions, the governing equations for the fluid motion in the two regions become

$$\frac{\partial u}{\partial t} = \frac{\partial U_1}{\partial t} - v\frac{\partial^2 u}{\partial y^2} - \frac{\lambda}{\rho}\frac{\partial^3 u}{\partial t \partial y^2} \,, \quad 0 \le y < \infty \,, \tag{11}$$

$$\frac{\partial U}{\partial t} = \frac{\partial U_1}{\partial t} - \frac{v}{k}U + v\frac{\partial^2 U}{\partial y^2} - \frac{\lambda}{\rho}\frac{\partial^3 U}{\partial t \partial y^2} \,, \quad -\infty < y \le 0 \,, \tag{12}$$

with the boundary conditions:

$$u \to U_1 = U_0 \cos \omega t \text{ as } y \to \infty \,, \tag{13}$$

$$u = U \,, \quad \mu\frac{\partial u}{\partial y} - \lambda\frac{\partial^2 u}{\partial t \partial y} = \mu\frac{\partial U}{\partial y} - \lambda\frac{\partial^2 U}{\partial t \partial y} \text{ at } y = 0 \,. \tag{14}$$

We now define the non-dimensionless variables as

$$\left.\begin{array}{lll} u^* = \frac{u}{U_0}\,, & y^* = \frac{y}{\frac{\mu}{\rho U_0^2}}\,, & t^* = \frac{t}{\frac{\mu}{\rho U_0^2}}\,, \\ \omega^* = \frac{\omega}{\frac{\rho U_0^2}{\mu}}\,, & \lambda^* = \frac{\lambda}{\frac{\mu^2}{\rho U_0^2}}\,, & k^* = \frac{k}{\frac{\mu^2}{\rho^2 U_0^2}}\,. \end{array}\right\} \tag{15}$$

After dropping the star mark equations (11) and (12) with the boundary conditions (13) and (14) become:

$$\frac{\partial u}{\partial t} = \frac{\partial U_1}{\partial t} + \frac{\partial^2 u}{\partial y^2} - \lambda \frac{\partial^3 u}{\partial t \partial y^2}\,, \quad 0 \le y < \infty\,, \tag{16}$$

$$\frac{\partial U}{\partial t} = \frac{\partial U_1}{\partial t} - \frac{U}{k} + \frac{\partial^2 U}{\partial y^2} - \lambda \frac{\partial^3 U}{\partial t \partial y^2}\,, \quad -\infty < y \le 0\,, \tag{17}$$

with the boundary conditions:

$$u \to \cos \omega t \text{ as } y \to \infty\,, \tag{18}$$

$$u = U\,, \quad \frac{\partial u}{\partial y} - \lambda \frac{\partial^2 u}{\partial t \partial y} = \frac{\partial U}{\partial y} - \lambda \frac{\partial^2 U}{\partial t \partial y} \text{ at } y = 0\,. \tag{19}$$

Now, to solve the system of equation (16) and (17) subjected to the boundary conditions (18) and (19), we shall assume that

$$u(y,t) = u_1(y) + \mathrm{e}^{i\omega t} u_2(y)\,, \tag{20}$$

$$U(y,t) = U_1(y) + \mathrm{e}^{i\omega t} U_2(y)\,. \tag{21}$$

Substituting (20) and (21) into (16)–(19), we have

$$\frac{d^2 u_1}{dy^2} = 0\,, \tag{22}$$

$$\frac{d^2 u_2}{dy^2} + \frac{\omega}{\omega\lambda + i} u_2 = \frac{\omega}{\omega\lambda + i}\,, \tag{23}$$

$$\frac{d^2 U_1}{dy^2} - \frac{U_1}{k} = 0\,, \tag{24}$$

$$\frac{d^2 U_2}{dy^2} + \frac{\omega - \frac{i}{k}}{\omega\lambda + i} u_2 = \frac{\omega}{\omega\lambda + i}\,, \tag{25}$$

where

$$u_1 = 0\,, \quad u_2 = 0 \text{ as } y \to \infty\,, \tag{26}$$

$$u_1 = U_1\,, \quad u_2 = U_2 \quad \frac{du_1}{dy} = \frac{dU_1}{dy}\,, \quad \frac{du_2}{dy} = \frac{dU_2}{dy}\,, \text{ at } y = 0\,. \tag{27}$$

Solving Eqs. (22)–(25) with the aid of Eqs. (26) and (27), the form of the velocity distributions of the viscoelastic fluid which is flowing through and past an infinite porous bed can be written as:

$$u = (h_2(y) \cos \omega t + h_3(y) \sin \omega t) + i\,(h_2(y) \sin \omega t - h_3(y) \cos \omega t)\,, \quad 0 \le y < \infty\,, \tag{28}$$

and

$$U=(h_6(y)\cos\omega t-h_7(y)\sin\omega t)+i\,(h_6(y)\sin\omega t+h_7(y)\cos\omega t)\;,\quad -\infty<y\leq 0\,, \tag{29}$$

where $h_0(y)\to h_7(y)$ are defined in the Appendix A.

4. Numerical discussion

In this section we studied the oscillatory non-Newtonian viscoelastic fluid flow past a porous medium. The expressions in Eqs. (28) and (29) are evaluated numerically and some of qualitatively interesting results are presented in illustrations. Since the velocity components are periodic functions of t, the effect of the elasticity parameter λ of the fluid is to increase or decrease these velocities as seen in Figures 1 and 2. Also the velocity component U decreases or increases as the permeability parameter K of the medium increases. This illustrated in Figure 3.

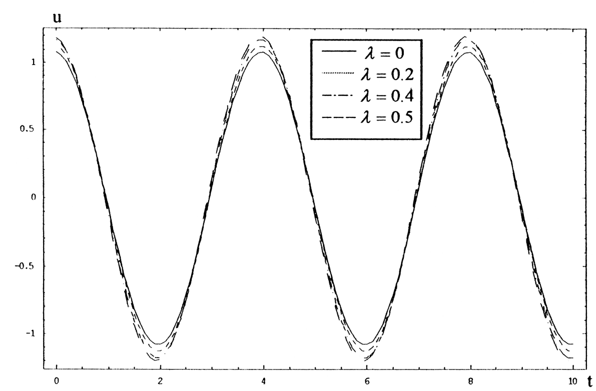

Figure 1 The velocity component u plotted versus time with the effect of elasticity parameter λ

5. Fluctuating viscoelastic flow through a porous bed

In this section we shall deal with the case of an infinite viscoelastic porous bed with a plane bounding surface $y=0$, where the free stream is oscillating about a non zero mean and makes an angle β with the line of greatest slope of the bounding surface.

Consider the velocity components in two regions which are $(u,v,0)$ and $(U,V,0)$. Since the plane surface is an infinite in extent, hence the solution of the velocities will be a function of y (distance normal to the plane) and t only, where t is a time. It is clear that the continuity equation in both regions reduces to $\frac{\partial v}{\partial y}=0$ and $\frac{\partial V}{\partial y}=0$, this means v and V at the most is functions of time and should therefore be equal

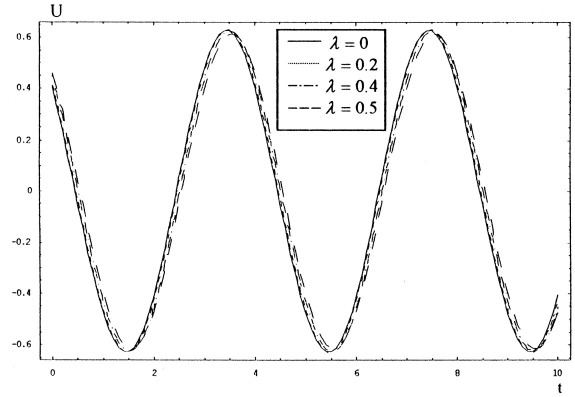

Figure 2 The velocity component U plotted versus time with the effect of elasticity parameter λ

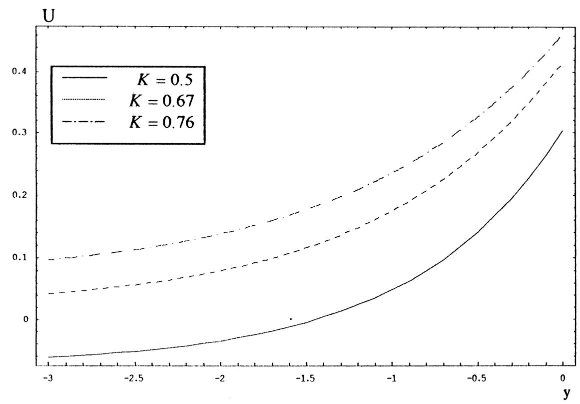

Figure 3 The velocity component U plotted versus position with the effect of permeability parameter K

in both the regions. Now, the equations governing the viscoelastic fluid flow in the two regions can be written as:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\lambda}{\rho}\left(\frac{\partial^3 u}{\partial t \partial y^2} + \nu\frac{\partial^3 u}{\partial y^3}\right) \\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad 0 \le y < \infty\,, \tag{30}$$

$$\left.\begin{aligned} \frac{\partial U}{\partial t} + V\frac{\partial U}{\partial y} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\nu}{k}U + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\lambda}{\rho}\left(\frac{\partial^3 U}{\partial t \partial y^2} + V\frac{\partial^3 U}{\partial y^3}\right) \\ \frac{\partial V}{\partial t} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{\nu V}{k} \end{aligned}\right\} \quad -\infty < y \le 0\,. \tag{31}$$

From the conservation of momentum across the surface and the dependence of the velocities on the y and t only, we can obtain $p = P$ at the plane surface $y = 0$.

Considering the free stream velocity to be $U_1 = U_0\left(1 + \epsilon e^{i\omega t}\right)$, we have in the limit when $y \to \infty$, that $u = U_1 \cos\alpha$, $v = -U_1 \sin\alpha$ and $-\frac{1}{\rho}\frac{\partial p}{\partial x} = \cos\alpha\frac{dU_1}{dt}$.

Therefore our system of equations yield

$$\begin{aligned}\frac{\partial u}{\partial t} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial u}{\partial y} &= \cos\alpha \frac{\partial U_1}{\partial t} + v \frac{\partial^2 u}{\partial y^2} - \\ \frac{\lambda}{\rho}\left[\frac{\partial^3 u}{\partial t \partial y^2} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial^3 u}{\partial y^3}\right], &\qquad 0 \le y < \infty\,, \end{aligned} \tag{32}$$

$$\begin{aligned}\frac{\partial U}{\partial t} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial U}{\partial y} &= \cos\alpha \frac{\partial U_1}{\partial t} - \frac{\nu}{k} U + v \frac{\partial^2 u}{\partial y^2} - \\ \frac{\lambda}{\rho}\left[\frac{\partial^3 U}{\partial t \partial y^2} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial^3 U}{\partial y^3}\right], &\qquad -\infty < y \le 0\,, \end{aligned} \tag{33}$$

with the boundary conditions

$$\left.\begin{array}{ll} u = U_0 \cos\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) & \text{as } y \to \infty \\ u = U\,, \quad \mu \frac{\partial u}{\partial y} - \lambda \left[\frac{\partial^2 u}{\partial t \partial y} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial^2 u}{\partial y^2}\right] = & \\ \mu \frac{\partial U}{\partial y} - \lambda \left[\frac{\partial^2 U}{\partial t \partial y} - U_0 \sin\alpha \left(1+\epsilon \mathrm{e}^{i\omega t}\right) \frac{\partial^2 U}{\partial y^2}\right] & \text{at } y = 0 \end{array}\right\} \tag{34}$$

Introducing the following dimensionless quantities

$$\left.\begin{array}{ll} u^* = \frac{u}{U_0 \cos\alpha}\,, & U^* = \frac{U}{U_0 \cos\alpha}\,, \\ y^* = \frac{y}{\frac{\nu}{U_0 \sin\alpha}}\,, & \lambda^* = \frac{\lambda}{\frac{\mu^2}{\rho U_0^2 \sin^2\alpha}}\,, \\ t^* = \omega t\,, \quad A^* = \frac{\nu\omega}{U_0^2 \sin^2\alpha} & k^* = \frac{k}{\frac{\nu^2}{U_0^2 \sin^2\alpha}}\,, \end{array}\right\} \tag{35}$$

and substituting (35) into (32)–(34) we obtain after drop star mark the following system of equations with the boundary conditions:

$$A\frac{\partial u}{\partial t} - \left(1+\epsilon \mathrm{e}^{it}\right)\frac{\partial u}{\partial y} = A\epsilon i \mathrm{e}^{it} + \frac{\partial^2 u}{\partial y^2} - \lambda\left[A\frac{\partial^3 u}{\partial t \partial y^2} - \left(1+\epsilon \mathrm{e}^{it}\right)\frac{\partial^3 u}{\partial y^3}\right], \tag{36}$$

for $0 \le y < \infty$,

$$A\frac{\partial U}{\partial t} - \left(1+\epsilon \mathrm{e}^{it}\right)\frac{\partial U}{\partial y} = A\epsilon i \mathrm{e}^{it} - \frac{U}{k} + \frac{\partial^2 U}{\partial y^2} - \lambda\left[A\frac{\partial^3 U}{\partial t \partial y^2} - \left(1+\epsilon \mathrm{e}^{it}\right)\frac{\partial^3 U}{\partial y^3}\right], \tag{37}$$

for $0 \le y < \infty$,

$$\left.\begin{array}{ll} u = \left(1+\epsilon \mathrm{e}^{it}\right)\,, & \text{as } y \to \infty \\ u = U\,, \quad \frac{\partial u}{\partial y} - \lambda \left[A\frac{\partial^2 u}{\partial t \partial y} - \left(1-\epsilon \mathrm{e}^{it}\right) \frac{\partial^2 u}{\partial y^2}\right] = & \\ \frac{\partial U}{\partial y} - \lambda \left[A\frac{\partial^2 U}{\partial t \partial y} - \left(1+\epsilon \mathrm{e}^{it}\right) \frac{\partial^2 U}{\partial y^2}\right] & \text{at } y = 0 \end{array}\right\} \tag{38}$$

The system of equations (36)–(38) is reduced to governing the flow of a Newtonian fluid through a porous bed if $\lambda = 0$. In order to solve these equations, we assume the expression for the velocity fields as:

$$\left.\begin{array}{ll} u(y,t) = u_1(y) + \epsilon \mathrm{e}^{it} u_2(y) & \\ \text{and} & \epsilon \ll 1 \\ U(y,t) = U_1(y) + \epsilon \mathrm{e}^{it} U_2(y) & \end{array}\right\} \tag{39}$$

Equations (36)–(38) reduce to the following system of equations after equating terms of ϵe^{it}

$$\lambda\frac{d^3u_1}{dy^3}+\frac{d^2u_1}{dy^2}+\frac{du_1}{dy}=0\,, \tag{40}$$

$$\lambda\frac{d^3u_2}{dy^3}+\frac{d^2u_2}{dy^2}(1-\lambda Ai)+\frac{du_2}{dy}-Aiu_2=-\lambda\frac{d^3u_1}{dy^3}-\frac{du_1}{dy}-Ai\,, \tag{41}$$

$$\lambda\frac{d^3U_1}{dy^3}+\frac{d^2U_1}{dy^2}+\frac{dU_1}{dy}-\frac{U_1}{k}=0\,, \tag{42}$$

$$\lambda\frac{d^3U_2}{dy^3}+\frac{d^2U_2}{dy^2}(1-\lambda Ai)+\frac{dU_2}{dy}-(Ai+\frac{1}{k})U_2=-\lambda\frac{d^3U_1}{dy^3}-\frac{dU_1}{dy}-Ai\,, \tag{43}$$

With the boundary conditions

$$\left.\begin{array}{ll} u_1\to 1\,,\quad u_2\to 1\,, & \text{as } y\to\infty \\ u_1=U_1\,,\quad u_2=U_2\,,\ \frac{du_1}{dy}+\lambda\frac{d^2u_1}{dy^2}=\frac{dU_1}{dy}+\lambda\frac{d^2U_1}{dy^2}\,, & \\ \frac{\partial u_2}{\partial y}-\lambda\left(Ai\frac{du_2}{dy}-\frac{d^2u_2}{dy^2}-\frac{d^2u_1}{dy^2}\right)=\frac{dU_2}{dy}-\lambda\left(Ai\frac{dU_2}{dy}-\frac{d^2U_2}{dy^2}-\frac{d^2U_1}{dy^2}\right) & \text{at } y=0 \end{array}\right\} \tag{44}$$

Now, to solve the system of equations (40)–(43) subjected to the boundary conditions (44), the following perturbation technique is made

$$\left.\begin{array}{l} u_1=u_{10}+\lambda u_{11}\,,\quad u_2=u_{20}+\lambda u_{21}\,, \\ U_1=U_{10}+\lambda U_{11}\,,\quad U_2=U_{20}+\lambda U_{21}\,,\quad \lambda\ll 1 \end{array}\right\} \tag{45}$$

Substituting from (45) into (40)–(44) and equating terms of λ, we obtain:

$$u''_{10}+u'_{10}=0\,, \tag{46}$$

$$u''_{11}+u'_{11}=-u'''_{10}\,, \tag{47}$$

$$u''_{20}+u'_{20}-Aiu_{20}=-u'_{10}-Ai\,, \tag{48}$$

$$u''_{21}+u'_{21}-Aiu_{21}=-u'''_{10}-u'_{11}-u'''_{20}+Aiu''_{20}\,, \tag{49}$$

$$U''_{10}+U'_{10}-\frac{U_{10}}{k}=0\,, \tag{50}$$

$$U''_{11}+U'_{11}-\frac{U_{11}}{k}=-U'''_{10}\,, \tag{51}$$

$$U''_{20}+U'_{20}-\left(Ai+\frac{1}{k}\right)U_{20}=-U'_{10}-Ai\,, \tag{52}$$

$$U''_{21}+U'_{21}-\left(Ai+\frac{1}{k}\right)U_{21}=-U'''_{10}-U'_{11}-U'''_{20}+AiU''_{20}\,, \tag{53}$$

With boundary conditions:

$$\left.\begin{array}{ll} u_{10}\to 1\,,\quad u_{11}\to 0\,,\quad u_{20}\to 1\,,\quad u_{21}\to 0\,, & \text{as } y\to\infty \\ u_{10}=U_{10}\,,\quad u_{11}=U_{11}\,,\quad u_{20}=U_{20}\,,\quad u_{21}=U_{21}\,, & \\ u'_{10}=U'_{10}\,,\quad u'_{11}+u''_{10}=U'_{11}+U''_{10}\,,\quad u'_{20}=U'_{20}\,, & \\ u'_{21}-Aiu'_{20}+u''_{20}+u''_{10}=U'_{21}-AiU'_{20}+U''_{20}+U''_{10}\,, & \text{at } y=0 \end{array}\right\} \tag{54}$$

The expressions for the flow velocity $u(y,t)$ and $U(y,t)$ can be written as:

$$u(y,t) = f_0(y) + \epsilon\left[(f_{11}(y)\cos t - f_{12}(y)\sin t) + i\,(f_{11}(y)\sin t + f_{12}(y)\cos t)\right],$$

for $0 \le y < \infty$, and

$$U(y,t) = f_{13}(y) + \epsilon\left[(f_{25}(y)\cos t - f_{26}(y)\sin t) + i\,(f_{25}(y)\sin t + f_{26}(y)\cos t)\right],$$

for $-\infty < y \le 0$, where $f_0(y) \to f_{26}(y)$ are defined in the Appendix B.

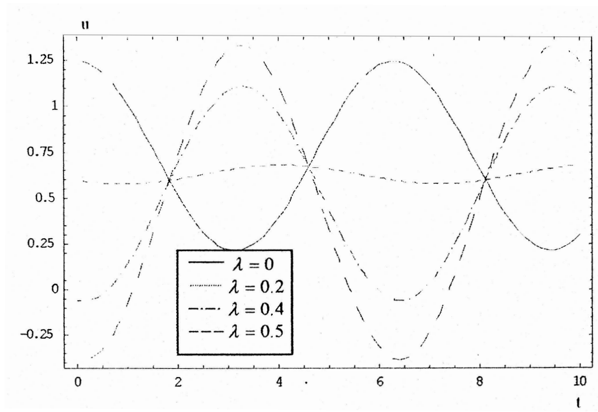

Figure 4 The velocity component u plotted versus time with the effect of elasticity parameter λ

6. Numerical Discussion

As mentioned before, since the velocity components of the viscoelastic fluid flowing through porous bed are periodic functions of the time, therefore the effect of the parameters of the problem cause decrease or increase of these velocities. In Figures 4 and 5, the velocity components u and U increase (or decrease) with the increasing of the elasticity parameter λ of the fluid. While, Figure 6 illustrate the effect of the permeability of the medium on the velocity U. It is clear that these velocity decreases with increasing the permeability parameter K.

References

[1] **Oldroyd, JG**: *Proc. R. Soc. London A*, (1950), **200**, 923.
[2] **Colman, BD** and **Noll, W**: *Arch. Rat. Mech. Anal.*, (1960), **6**, 355.
[3] **Walters, K**: *J. Mec.*, (1962), **1**, 479.
[4] **Walters, K**: *Non-Newtonian Fluid Mech.*, (1979), **5**, 113.
[5] **Walters, K**: *Second-order effect in elasticity, plasticity and fluid dynamics*, (1964), Pergamon Press Ltd, Oxford, 507.

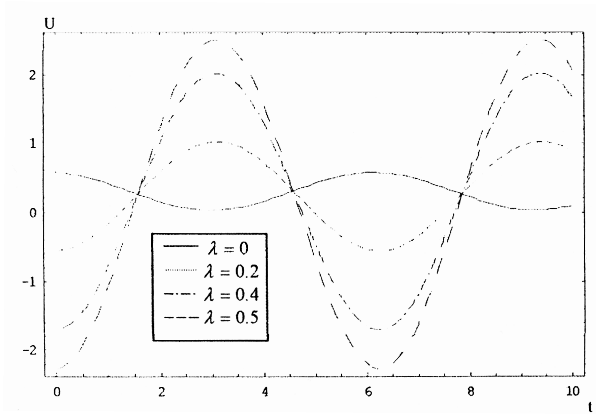

Figure 5 The velocity component U plotted versus time with the effect of elasticity parameter λ

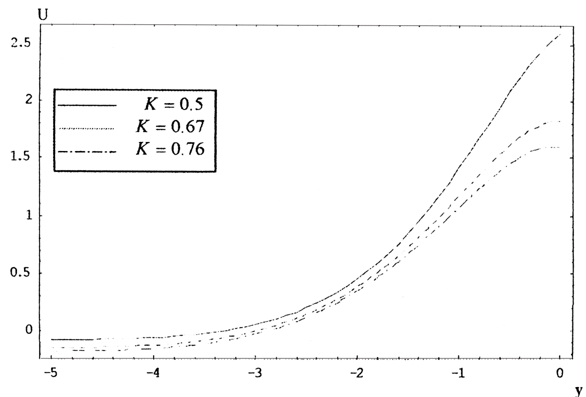

Figure 6 The velocity component U plotted versus position with the effect of permeability parameter K

[6] **Eldabe, NT** and **Elmohandis, SMG**: *J. Phys. Soc. Jap.*, (1995), **64**, 11, 4165.

[7] **Soundalgekar, NM** and **Puri, P**: *Fluid Mech.*, (1969), **35**, 561.

[8] **Chawla, SS** and **Singh, SK**: *Acta Mechanica*, (1979), **24**, 205-218.

Appendix A

$s_0 = \omega\,\lambda$, $s_1 = \left(s_0^2 + 1\right)^{1/2}$, $s_2 = \omega\,(s_0 + s_1)$,

$s_3 = \left(s_2/2s_1^2\right)^{1/2}$, $s_4 = \omega\,(s_1 - s_0)$,

$s_5 = \left(s_4/2s_1^2\right)^{1/2}$, $s_6 = \omega\,s_0 - 1/k$,

$s_7 = \lambda/k + 1$, $s_8 = \left(s_6^2 + \omega^2\,s_7^2\right)^{1/2}$,

$s_9 = \left\{(s_8 + s_6)/2s_1^2\right\}^{1/2}$, $s_{10} = \left\{(s_8 - s_6)/2s_1^2\right\}^{1/2}$,

$s_{11} = s_3 + s_9$, $s_{12} = s_5 + s_{10}$,

$s_{13} = s_{12} - \omega\,k s_{11}$, $s_{14} = s_{11} + \omega\,k s_{12}$,

$s_{15} = s_{10}s_{13} + s_9 s_{14}$, $s_{16} = s_9 s_{13} - s_{10}s_{14}$,

$s_{17} = s_{13}^2 + s_{14}^2$, $s_{18} = s_{15}\,/s_{17}$,

$s_{19} = s_{16}\,/s_{17}$, $s_{20} = \omega^2 + 1/k^2$,

$s_{21} = s_5 s_{13} + s_3 s_{14}$, $s_{22} = -s_5 s_{14} + s_3 s_{13}$,

$s_{23} = \omega^2\,/s_{20}$, $s_{24} = \omega\,/k\,s_{20}$,

$s_{25} = s_{21}\,/s_{17}$, $s_{26} = s_{22}\,/s_{17}$,

$h_0(y) = \left(s_{18}\,Cos\,(s_3 y) + s_{19}\,Sin\,(s_3 y)\right)$,

$h_1(y) = \left(s_{19}\,Cos\,(s_3 y) - s_{18}\,Sin\,(s_3 y)\right)$,

$h_2(y) = 1 - e^{-s_5 y}\,h_0(y)$, $h_3(y) = e^{-s_5 y}\,h_1(y)$,

$h_4(y) = \left(s_{25}\,Cos\,(s_9 y) - s_{26}\,Sin\,(s_9 y)\right)$,

$h_5(y) = \left(s_{25}\,Sin\,(s_9 y) + s_{26}\,Cos\,(s_9 y)\right)$,

$h_6(y) = s_{23} + e^{s_{10} y}\,h_4(y)$, $h_7(y) = s_{24} + e^{s_{10} y}\,h_5(y)$,

Appendix B

$$l_0 = \left(\sqrt{1+4/k}-1\right)\Big/2, \qquad l_1 = \left(1+\sqrt{\frac{\sqrt{1+16A^2}+1}{2}}\right)\Bigg/2$$

$$l_2 = \left(\sqrt{\frac{\sqrt{1+16A^2}-1}{2}}\right)\Bigg/2, \qquad l_3 = \left(\sqrt{\frac{\sqrt{(1+4/k)^2+16A^2}+(1+4/k)}{2}}-1\right)\Bigg/2$$

$$l_4 = \left(\sqrt{\frac{\sqrt{(1+4/k)^2+16A^2}-(1+4/k)}{2}}\right)\Bigg/2$$

$$l_5 = l_1 + l_3, \qquad l_6 = l_2 + l_4,$$

$$l_7 = \left(\sqrt{\frac{\sqrt{1+16A^2}+1}{2}}\right), \qquad l_8 = \left(\sqrt{\frac{\sqrt{(1+4/k)^2+16A^2}+(1+4/k)}{2}}\right),$$

$$n_0 = 1/(l_0+1), \qquad n_1 = -l_0/(l_0+1),$$

$$n_2 = l_0\Big/\left(\sqrt{1+4/k}\right), \qquad n_3 = -2l_0n_0^2 + n_1^2(n_2-1)$$

$$n_4 = \left(1+k^2A^2\right), \qquad n_5 = l_3 + kAl_4$$

$$n_6 = -l_4 + kAl_3, \qquad n_7 = l_0/A + n_6/n_4$$

$$n_8 = n_5/n_4, \qquad n_9 = l_5^2 + l_6^2$$

$$n_{10} = l_6n_7 - l_5n_8, \qquad n_{11} = l_5n_7 + l_6n_8$$

$$n_{12} = n_{10}/n_9, \qquad n_{13} = n_{11}/n_9$$

$$n_{14} = l_0^3 - n_2l_0^2(2l_0+1) + 2, \qquad n_{15} = n_1n_{14}/A^2$$

$$n_{16} = l_0^2n_2 - l_0^2 - 2l_0, \qquad n_{17} = l_3n_{15} - l_0l_4/A + 2n_1/A^2$$

$$n_{18} = l_4n_{15} + l_0l_3/A - n_0^2n_{16}/A + 3n_1/A, \qquad n_{19} = l_1^3 - 3l_1l_2^2 - 2l_1l_2A$$

$$n_{20} = -l_2^3 + 3l_2l_1^2 + A\left(l_1^2 - l_2^2\right), \qquad n_{21} = l_7^2 + 4l_2^2$$

$$n_{22} = l_0^4\left(n_2(2l_0+1)+l_0\right)/A^2, \qquad n_{23} = l_0^3(2n_2-1)/A$$

$n_{24} = l_1 + Akl_2$, $n_{25} = l_2 - Akl_1$

$n_{26} = -l_3^3 + 3l_3l_4^2 - 2l_3l_4A$, $n_{27} = l_4^3 - 3l_4l_3^2 + A(l_3^2 - l_4^2)$

$n_{28} = l_8^2 + 4l_4^2$, $n_{29} = l_3^2 - l_4^2 + l_4A$

$n_{30} = l_3(2l_4 - A)$, $n_{31} = l_2^2 - l_1^2 + Al_2$

$n_{32} = l_1(2l_2 + A)$, $n_{33} = l_6n_7/n_9 - l_5n_5/n_4n_9$

$n_{34} = l_5n_7/n_9 + l_6n_5/n_4n_9$, $n_{35} = (l_7n_{19} + 2l_2n_{20})/n_{21} + n_{30}$

$n_{36} = (l_7n_{20} - 2l_2n_{19})/n_{21} - n_{32}$, $n_{37} = -l_0/A + n_{25}/n_4$

$n_{38} = l_6n_{37}/n_9 + l_5n_{24}/n_4n_9$, $n_{39} = l_5n_{37}/n_9 - l_6n_{24}/n_4n_9$

$n_{40} = (l_8n_{26} + 2l_4n_{27})/n_{28} + n_{29}$, $n_{41} = (l_8n_{27} - 2l_4n_{26})/n_{28} + n_{30}$

$n_{42} = n_{17} - n_0n_{22} - n_{33}n_{35}$, $n_{43} = n_{34}n_{36} - n_{38}n_{40} + n_{39}n_{41}$

$n_{44} = n_{42} + n_{43}$, $n_{45} = n_{18} - n_0n_{23} - n_{33}n_{36}$

$n_{46} = n_{34}n_{35} + n_{38}n_{41} + n_{39}n_{40}$, $n_{47} = n_{45} - n_{46}$

$n_{48} = (l_5n_{44} + l_6n_{47})/n_9$, $n_{49} = (l_5n_{47} - l_6n_{44})/n_9$

$n_{50} = -l_1n_{15} + l_0l_2/A + 2n_1/A^2$, $n_{51} = l_2n_{15} + l_0l_1/A + n_0^2n_{16}/A - 3n_1/A$

$n_{52} = n_{50} - n_0n_{22} - n_{33}n_{35}$, $n_{53} = n_{43} + n_{52}$

$n_{54} = n_{51} + n_0n_{23} + n_{33}n_{36}$, $n_{55} = n_{46} + n_{54}$

$n_{56} = (l_5n_{53} - l_6n_{55})/n_9$, $n_{57} = (l_6n_{53} + l_5n_{55})/n_9$

$n_{58} = 4l_1l_2^2 + l_7(l_1^2 - l_2^2)$, $n_{59} = l_1l_2l_7 - l_2(l_1^2 - l_2^2)$

$n_{60} = (l_1n_{58} - 2(l_2 + A)n_{59})/n_{21}$, $n_{61} = (2l_1n_{59} + (l_2 + A)n_{58})/n_{21}$

$n_{62} = n_{12}n_{60} - n_{13}n_{61}$, $n_{63} = n_{13}n_{60} + n_{12}n_{61}$

$n_{64} = kA/n_4$, $n_{65} = kAn_{64}$

$n_{66} = 4l_3l_4^2 + l_8(l_3^2 - l_4^2)$, $n_{67} = l_3l_4l_8 - l_4(l_3^2 - l_4^2)$

$n_{68} = (l_3n_{66} + 2(A - l_4)n_{67})/n_{28}$, $n_{69} = (-2l_3n_{67} + (A - l_4)n_{66})/n_{28}$

$n_{70} = n_{38}n_{68} + n_{39}n_{69}$, $n_{71} = n_{38}n_{69} - n_{39}n_{68}$

$$n_{72} = l_0^3 n_0 n_2 / A, \qquad n_{73} = (2l_0 + 1) n_{72} / A$$

$$n_{74} = n_{72} / l_0 - l_0^4 n_0 / A^2 - n_3 l_0 / A, \qquad n_{75} = n_{74} - n_{73}$$

$$f_0(y) = 1 + e^{-y}\{n_1 + \lambda(n_3 + n_1 y)\}, \qquad f_1(y) = 1 - e^{-y} \lambda \frac{2n_1}{A^2}$$

$$f_2(y) = n_{12} \, Cos \; l_2 y + n_{13} \, Sin \; l_2 y, \qquad f_3(y) = n_{48} \, Cos \; l_2 y + n_{49} \, Sin \; l_2 y$$

$$f_4(y) = n_{62} \, Cos \; l_2 y + n_{63} \, Sin \; l_2 y, \qquad f_5(y) = e^{-y} \frac{n_1}{A} + \lambda \left(\frac{n_3 - n_1}{A} + \frac{n_1 y}{A} \right)$$

$$f_6(y) = n_{13} \, Cos \; l_2 y - n_{12} \, Sin \; l_2 y, \qquad f_7(y) = n_{49} \, Cos \; l_2 y - n_{48} \, Sin \; l_2 y$$

$$f_8(y) = n_{63} \, Cos \; l_2 y - n_{62} \, Sin \; l_2 y, \qquad f_9(y) = e^{-l_1 y} \{ f_2(y) + \lambda (f_3(y) - y f_4(y)) \}$$

$$f_{10}(y) = e^{-l_1 y} \{ f_6(y) + \lambda (f_7(y) - y f_8(y)) \}, \qquad f_{11}(y) = f_1(y) + f_9(y)$$

$$f_{12}(y) = f_5(y) + f_{10}(y), \qquad f_{13}(y) = e^{l_0 y} \{ n_0 + \lambda (n_3 - n_0 l_0^2 n_2 y) \}$$

$$f_{14}(y) = n_{38} \, Cos \; l_4 y - n_{39} \, Sin \; l_4 y, \qquad f_{15}(y) = n_{64} - \frac{n_0 l_0}{A} e^{l_0 y}$$

$$f_{16}(y) = n_{38} \, Cos \; l_4 y + n_{39} \, Sin \; l_4 y, \qquad f_{17}(y) = n_{56} \, Cos \; l_4 y + n_{57} \, Sin \; l_4 y$$

$$f_{18}(y) = n_{70} \, Cos \; l_4 y + n_{71} \, Sin \; l_4 y, \qquad f_{19}(y) = -n_{57} \, Cos \; l_4 y + n_{56} \, Sin \; l_4 y$$

$$f_{20}(y) = n_{71} \, Cos \; l_4 y - n_{70} \, Sin \; l_4 y, \qquad f_{21}(y) = n_{65} + e^{l_3 y} f_{14}(y)$$

$$f_{22}(y) = \lambda \left(n_{72} y e^{l_0 y} + e^{l_3 y} (f_{17}(y) - y f_{18}(y)) \right), \qquad f_{23}(y) = f_{15}(y) + e^{l_3 y} f_{16}(y)$$

$$f_{24}(y) = \lambda \left(n_{75} e^{l_0 y} + e^{l_3 y} (f_{19}(y) + y f_{20}(y)) \right), \qquad f_{25}(y) = f_{21}(y) + f_{22}(y)$$

$$f_{26}(y) = f_{23}(y) + f_{24}(y)$$