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# Magnetohydrodynamic Axisymmetric Oscillation of Bounded Hollow Jet

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The magnetohydrodynamic axisymmetric stability of a gas jet surrounded by finite liquid pervaded by a magnetic field and endowed with surface tension is studied. The fundamental equations are solved and the eigenvalue relation is derived and discussed. The radii ratio of the liquid gas cylinder $q$ has a stability effect. The capillary force is destabilizing for long wavelengths while it is stabilizing for all the rest as the wavelength is short. The axial magnetic field in the liquid region is stabilizing while the transverse magnetic field is destabilizing. Under certain restriction the destabilizing effect could dispressed and stability sets in. Here the capillary stability results are in good agreement with Kendall's results (1986).

*Keywords*: gas jet, magnetic field, destabilization.

## 1. Introduction

The stability of a full liquid jet in vacuum has been documented capillary and subject to other effects by Rayleigh (1945) and Chandrasekhar (1981).

The stability of the mirror case of a hollow jet i.e. cylinder of negligible inertia surrounded by a liquid is indicated by Chandrasekhar (1981) for axisymmetric perturbation. In (1986) Kendall performed very interesting experiments for examining the stability of this model under the capillary force. Moreover he, (1986), attracted the attention for investigating the stability of such model due to its cruitial application in several domains of science. Radwan and Elazab (1987) tried to find out the effect of viscosity on the capillary force. Soon afterwards a lot of researchers in the international congresses discussed the oscillation of such model.

Here we study the magnetohydrodynamic stability of axisymmetric hollow jet endowed with surface tension and pervaded by magnetic fields.

## 2. Formulation of the problem

We consider a gas cylinder of radius $R_o$ surrounded by finite liquid radically of radius $qR_o$ with $1 < q < \infty$. The inertia of the liquid is predominant over that of the gas. The gas is acting upon the electromagnetic force due to the pervading magnetic field

$$\underline{H}_o^g = \left(0, \frac{\beta H_o r}{R_o}, 0\right) . \tag{1}$$

The liquid is acted by the pressure gradient, capillary and electromagnetic force. The liquid is pervaded by

$$\underline{H}_o = (0, 0, H_o). \tag{2}$$

Here $H_o$ is the intensity of the magnetic field in the liquid, $\beta$ is parameter and $(r, \phi, z)$ are the cylindrical coordinates will be used for discussing this problem.

The liquid is considered to be incompressible, non viscous and perfectly conducting.

The magnetohydrodynamic basic equations are the following

$$\rho(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}) = -\nabla P + \mu(\nabla \wedge \underline{H}) \wedge \underline{H} \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \tag{4}$$

$$\frac{\partial \underline{H}}{\partial t} = curl(\underline{u} \wedge \underline{H}) \tag{5}$$

$$\nabla \cdot \underline{H} = 0 \tag{6}$$

$$P_s = T(\nabla \cdot \underline{N}) \tag{7}$$

$$\underline{N} = (\frac{1}{r_1} + \frac{1}{r_2}) = \frac{\nabla f}{|\nabla f|}, \qquad f(r, \phi, z) = 0 \tag{8}$$

$$\nabla \cdot \underline{H}^{gas} = 0 \tag{9}$$

$$\nabla \wedge \underline{H}^{gas} = 0 \tag{10}$$

Here $\rho$, $\underline{u}$, and $P$ are the liquid mass density, velocity vector and kinetic pressure; $\underline{H}$ the magnetic field intensity, $\mu$ the magnetic field permeability coefficient, $\underline{N}$ the unit outward normal vector to the gas liquid interface indicated as $r$ does, $P_s$ the curvature pressure due to the capillary force, $T$ the surface tension coefficient. $\underline{H}$ and $\underline{H}^g$ are the magnetic field intensities in the liquid and gas regions.

## 3. Perturbation Analysis

Let the fluids interface be perturbed for small departure from the initial state, the variables concerning the motion of the fluids could be expanded as

$$Q(r,\phi,z,t) = Q_o(r) + \varepsilon(t)Q_1(r,\phi,z), \ \mid Q_1 \mid << Q_o \tag{11}$$

Here $Q$ stands for $\underline{u}$, $\rho$, $\underline{H}$, $\underline{H}^g$, $\underline{N}$, $P_s$ and the radial distance of the gas cylinder, $Q_o$ represents the variables in unperturbed state while $Q_1$ represents small increments of the variable $Q$. The radial distance of the gas cylinder based on the linear perturbation technique, is given by

$$r = R_o + \varepsilon(t)R_1 + \cdots \tag{12}$$

with

$$R_1 = \exp[ikz + \sigma t] \tag{13}$$

is the elevation of the surface wave measured from the unperturbed state, where $\sigma$ is growth rate.

By inserting the expansion (11) into the basic equations (3)-(10) and equating the coefficients we obtain two different systems of partial differential equations. They are unperturbed and perturbed systems of equations.

The unperturbed system of equations is solved with taking into account equations (1) and (2) and that $\underline{u}_o = (0,0,0)$. The liquid kinetic pressure $P_o$ is given by

$$P_o = \frac{-T}{R_o} + P_o^g + \frac{\mu H_o^2}{2}(\beta^2 - 1) \tag{14}$$

where $P_o^g$ is the gas constant pressure in the initial state. In equation (14) if $\beta = 1$, we must have

$$P_o^g >> \frac{T}{R_o} \tag{15}$$

otherwise the model collapses. Clearly $(\frac{T}{R_o})$ is the contribution of the capillary force while $\frac{\mu H_o^2}{2}(\beta^2 - 1)$ is the term due to the effect of the electromagnetic forces in the gas and liquid regions.

The perturbed system of partial differential equations is given as

$$\rho\frac{\partial \underline{u}_1}{\partial t} - \mu(\underline{H}\cdot\nabla)\underline{H} = -\nabla P_1 - \mu\nabla(\underline{H}_o\cdot\underline{H}_1) \tag{16}$$

$$\nabla\cdot\underline{H}_1 = 0 \tag{17}$$

$$\nabla\cdot\underline{u}_1 = 0 \tag{18}$$

$$\frac{\partial \underline{H}_1}{\partial t} = \nabla\wedge(\underline{u}_1\wedge\underline{H}_o) \tag{19}$$

$$\nabla\cdot\underline{H}_1^{gas} = 0 \tag{20}$$

$$\nabla\wedge\underline{H}_1^{gas} = 0 \tag{21}$$

$$P_{1s} = \frac{T}{R_o^2}\left(R_1 + \frac{\partial^2 R_1}{\partial \phi^2} + R_o^2 \frac{\partial^2 R_1}{\partial z^2}\right) \tag{22}$$

For a single Fourier term and based on the linear perturbation technique, every perturbed quantity $Q_1(r, \phi, z; t)$ could be expressed as

$$Q_1(r, \phi, z; t) = Q_1(r) \exp(ikz + \sigma t) \tag{23}$$

The linear system of equations (16)-(22) taking into account the time-space dependence (23) has been solved.

Apart from the singular solution, we obtained

$$\underline{H}_1^{gas} = A\nabla(I_o(kr)R_1) \tag{24}$$

$$\Pi_1 = (BI_o(kr) + CK_o(kr))R_1 \tag{25}$$

$$\underline{H}_1 = \frac{-H_o}{\rho(\sigma^2 + \Omega_A^2)}\nabla\left(\frac{\partial \Pi_1}{\partial z}\right) \tag{26}$$

$$\underline{u}_1 = \frac{-\sigma}{\rho(\sigma^2 + \Omega_A^2)}\nabla\Pi_1 \tag{27}$$

$$P_{1s} = \frac{T}{R_o^2}(1 - x^2)R_1 \tag{28}$$

with

$$\Omega_A = \left(\frac{\mu H_o^2 k^2}{\rho}\right)^{\frac{1}{2}} \tag{29}$$

$$\rho\Pi_1 = P_1 + \mu(\underline{H}_o \cdot \underline{H}_1) \tag{30}$$

Here $I_0(kr)$ and $K_0(kr)$ are the modified Bessel functions of the first and second kind of the order zero, $A$, $B$ and $C$ are constants of integration to be determined and $\Omega_A$ is the Alfven wave frequency defined in terms of $H_o$.

## 4. Dispersion relation

Appropriate boundary conditions across the fluids interfaces at $r = R_o$ and $r = qR_o$ are applied. The constants $A$, $B$ and $C$ are identified and finally the dispersion relation is derived in the form

$$\begin{aligned} \sigma^2 \quad = \quad & \frac{T}{\rho R_o^3}(1 - x^2)\frac{x(-I_1(y)K_1(x) + I_1(x)K_1(y))}{(I_0(x)K(y) + I_1(y)K_0(x))} + \\ & \frac{\mu H_o^2}{\rho R_o^2}\left[-x^2 - \beta^2 \frac{x(-I_1(y)K_1(x) + I_1(x)K_1(y))}{(I_0(x)K_(y) + I_1(y)K_0(x))}\right] \end{aligned} \tag{31}$$

where

$$x = kR_o \tag{32}$$

$$y = qx \tag{33}$$

are the dimensionless longitudinal wavenumbers.

## 5. Stability Discussions

Equation (31) is the magnetohydrodynamics stability criterion of the present model of a gas cylinder surrounded by bounded liquid under axisymmetric perturbation. It relates the growth rate $\sigma$ with the longitudinal wavenumbers $x$, $y$, the magnetic field parameters $\beta$, the surface tension coefficient and the other parameters $\rho$, $R_o$, $\mu$ and $H_o$ of the problem.

In the limitig case as $H_o = 0$ and $q \to \infty$, the relation (31) reduces to

$$\sigma^2 = \frac{T}{\rho R_o^3}(1 - x^2)\frac{xK_1(x)}{K_0(x)} \tag{34}$$

that coincides with the dispersion relation indicated by Chandereskher (1981).

If we suppose that $T = 0$, the relation (31) yields some of our results (Radwan, Elazab and Hydia (2002) as we put $\alpha = 0$ and $m = 0$).

In order to discuss the stability analysis of the present study, we have to write down about the behaviour and characteristics of the modified Bessel functions.

Consider the recurrence relation for $m \geq 0$ (cf. Abramowitz and Stegun (1970))

$$2I'_m(x) = I_{m-1}(x) + I_{m+1}(x) \tag{35}$$

$$2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x) \tag{36}$$

Taking into account $I_m(x) > 0$, $K_m(x) > 0$, one may show that $I'_m(x) > 0$ and $K'_m(x) < 0$, also $I'_0(x) = I_1(x)$ and $K'_0(x) = K_1(x)$ since $y = qx$, $1 < q < \infty$ we have $y > x$ so

$$I_0(y) > I_0(x), \qquad I_1(y) > I_1(x) \tag{37}$$

$$K_0(x) > K_0(y), \qquad K_1(y) > K_1(x) \tag{38}$$

Therefore, we get

$$I_0(x)K_1(y) + I_1(y)K_0(x) > 0 \tag{39}$$

$$I_1(x)K_1(y) - I_1(y)K_1(x) < 0 \tag{40}$$

Consequently, for $x \neq 0$, we have

$$\frac{x(I_1(x)K_1(y) - I_1(y)K_1(x))}{(I_0(x)K_1(y) + I_1(y)K_0(x))} < 0 \tag{41}$$

Now, let us returning to our task of investigating the stability of the present model.

In the absence of the magnetic field $H_o = 0$, the capillary dispersion relation is obtained from (31) in the form

$$\sigma^2 = \frac{T}{\rho R_o^3}(1 - x^2)\frac{x(I_1(x)K_1(y) - I_1(y)K_1(x))}{(I_0(x)K_1(y) + I_1(y)K_0(x))} \tag{42}$$

In view of the inequalities (37)-(41) we see that

$$\frac{\sigma^2}{\frac{T}{\rho R_o^3}} > 0, \qquad \text{as} \qquad 0 < x < 1 \tag{43}$$

$$\frac{\sigma^2}{\frac{T}{\rho R_o^3}} < 0, \qquad \text{as} \qquad 1 < x < \infty \tag{44}$$

$$\frac{\sigma^2}{\frac{T}{\rho R_o^3}} = 0, \qquad \text{as} \qquad x = 1 \tag{45}$$

This means that the present model is unstable for long wave lengths $0 < x < 1$ and ordinary stable for short wave length $1 < x < \infty$ and marginally stable as the perturbed wave length $\lambda(= 2\pi R_o)$ equal to the circumference of the gas cylinder. Note that $\lambda = \frac{2\pi}{k}$ or $\lambda = \frac{2\pi R_o}{x}$, as $x = 1$ we have $\lambda = 2\pi R_o$. The magnetodynamic dispersion relation of the present model, may be obtained from the equation (31), as $T = 0$, in the form

$$\sigma^2 = \frac{\mu H_o^2}{\rho R_o^2}\left[-x^2 - \beta^2 \frac{x(I_1(x)K_1(y) - I_1(y)K_1(x))}{(I_o(x)K_1(y) + I_1(y)K_o(x))}\right] \tag{46}$$

The axial magnetic field in the liquid region is represented by the term $(-x^2)$ following $\frac{\mu H_o^2}{\rho R_o^2}$. It is negative for all values of $x \neq 0$. So it is stabilizing and is independent of the kind of perturbation. The effect of the magnetic field in the gas region is represented by the terms including $\beta$. This term, by means of the inequalities (37)-(41) is always positive definite. This means that the transverse magnetic field pervaded into gas region is strongly destabilizing for all short and long wavelengths. Therefore as $\beta$ is infinitesmly small, the model under consideration is magnetodynamic stable.

Combining the foregoing discussions concerning the capillary instability and magnetodynamic one, the destabilizing character of this model could be supposed and then stability sets in.

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