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Stability of self-resonance mechanisms in nonlinear interaction between two primary harmonic waves

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In the present work the stability criterion for two coupled nonlinear Schrödinger equations having parametric terms is derived. In this investigation, two different types of coupled nonlinear Schrödinger equations are discussed. Two coupled parametric nonlinear Schrödinger equations govern the wave behavior at the self-secondary resonance interaction and other two coupled parametric equations describe the wave-wave interaction at self-cubic resonance case. Stability criterion governing resonance mechanisms is performed in view of temporal periodic perturbations. Moreover, stability criterion at the perfect resonance case is achieved. Further, some numerical calculations are made to screen the stability pictures at the self-second resonance case.

Keywords: nonlinear systems, nonlinear coupled parametric Schrödinger equations, stability criteria, nonlinear resonance interaction

1. Introduction

There are wide applications and deep researches for the nonlinear Schrödinger equation. This equation is now so widely used in many branches of physics and dynamics that it forms a separate class of equations investigated thoroughly by many researches. But the correspondence between the initial full system and the nonlinear Schrödinger equation is not so clearly stated. It is well known that the nonlinear Schrödinger equation is a generic equation describing unidirectional wave modulation (see e.g., [1, 2]). It has been shown to describe the spatial and temporal evolution of the envelope of a sinusoidal wave with phase $kx - \omega(k)\,t$, which draws potential energy from some background field [3].

Lee [4] has used the method of multiple scales to analyze the second harmonic resonance of nonlinear progressive waves on the surface of a fluid column in the presence of a magnetic field. The dynamic equations governing the second harmonic resonance have been obtained. The interaction equations, truncated at second order, have solutions which develop a singularity after a finite time. Also, he points out that there exists a special case when a wave and its own second harmonic wave interact and cause a resonant triad of the form $k_1 = 2k_2$ and $\omega_1 = 2\omega_2$. Such cases have been investigated extensively by McGoldrick [5] and Nayfeh [6] for waves on a single interface.

Nayfeh *et al.* [7] investigated the nonlinear waves on the interface of two incompressible inviscid fluids of different densities and arbitrary surface tension using the method of multiple scales. They obtained a second order expansion for wavenumbers near the second harmonic resonant wavenumber, for which the fundamental wave and its second harmonic have the same phase velocity and found that this resonance does not lead to instabilities.

Singla *et al.* [8] studied the weakly nonlinear theory of resonant wave interactions at a charge free surface separating two semi-infinite dielectric streaming fluids influenced by a tangential electric field.

However, during the last twenty years of active researches of wave-wave interaction, rich information on the nature resonance instability has been accumulated. Coupled nonlinear Schrödinger equations describe co-propagation of two independent modes in nonlinear medium [9]. Thus, the aim is to advertise a useful study in theoretical approach to cover stability criteria, for the following new-coupled nonlinear Schrödinger equations:

$$i\frac{\partial A}{\partial t} + P_1\frac{\partial^2 A}{\partial x^2} + \left[\sigma_1 R\bar{A}B + iK\bar{A}\frac{\partial B}{\partial x} + iHB\frac{\partial \bar{A}}{\partial x}\right]\mathrm{e}^{i\Gamma} = \left(Q_{11}\left|A\right|^2 + Q_{21}\left|B\right|^2\right)A\,, \tag{1}$$

$$i\frac{\partial B}{\partial t} + P_2\frac{\partial^2 B}{\partial x^2} + \left[\sigma_1 GA^2 + iEA\frac{\partial A}{\partial x}\right]\mathrm{e}^{-i\Gamma} = \left(Q_{12}\left|A\right|^2 + Q_{22}\left|B\right|^2\right)B\,, \tag{2}$$

These equations govern the stability behavior at a self-secondary resonance case, where i is the imaginary number, t and x are time and space variables respectively and $\Gamma = \sigma_1 x + \sigma_2\, t$, ($\sigma_1$and σ_2 are two detuning parameters) [10,11]. The unknowns $A(x,t)$and $B(x,t)$ are the envelopes of wave packets in two different degrees of freedom of the underlying physical systems. The parameters R, K, H, G and E are constant coefficients that depend on the dispersion parameters. The parameters P_j, $j = 1,2$ are, respectively, two different dispersion coefficients representing the rate of the group velocities. The nonlinear coefficients Q_{ii} are the Landau constants which describe the self-modulation of the wave packets and Q_{12} and Q_{21} are the wave-wave interaction coefficients which describe the cross-modulations of the wave packets. They are all real parameters. These equations are derived from using the multiple scale perturbations [10] to a coupled nonlinear dynamic system [9]. There are many applications including these equations, for example, nonlinear optics, geophysical fluid dynamics and plasma physics.

Further, the approach is extended to cover the stability criteria for another two coupled nonlinear Schrödinger equations that govern the self-cubic-resonance case. These equations are

$$i\frac{\partial A}{\partial t} + P_1\frac{\partial^2 A}{\partial x^2} = \left(Q_{11}\left|A\right|^2 + Q_{21}\left|B\right|^2\right)A + S_1\bar{A}^2 B\exp(i\sigma t)\,, \tag{3}$$

$$i\frac{\partial B}{\partial t} + P_2\frac{\partial^2 B}{\partial x^2} = \left(Q_{12}\left|A\right|^2 + Q_{22}\left|B\right|^2\right)B + S_2 A^3\exp(-i\sigma t)\,, \tag{4}$$

where the parameters S_j, $j = 1, 2$ are constant coefficients depending on the dispersion parameters P_1, and P_2 while σ refers to the parametric terms which are assumed to be real.

The absence of the parametric coefficients from the system of equations (1) and (2) or from the system of (3) and (4) is reduced to

$$i\frac{\partial A}{\partial t} + P_1 \frac{\partial^2 A}{\partial x^2} = \left(Q_{11} |A|^2 + Q_{21} |B|^2 \right) A \,, \tag{5}$$

$$i\frac{\partial B}{\partial t} + P_2 \frac{\partial^2 B}{\partial x^2} = \left(Q_{12} |A|^2 + Q_{22} |B|^2 \right) B \,. \tag{6}$$

This system is composed of two coupled nonlinear Schrödinger equations governing the non-resonance case. However, the above system refers to those discussed before by Tan and Boyd [12]. They discuss the stabilization of these systems, which depends on a quadratic algebraic equation without cubic term. Their dispersion relation contains eleven parameters. This causes more difficulty in discussing stability behavior. Accordingly, they discussed some special cases. In the present work we explore the stability criterion for both coupled equations (1), (2) as well as for coupled equations (3) and (4).

Inoué [13] derived coupled nonlinear Schrödinger equations with the same group velocities for the interaction of two wave-packets in an isotropic dielectric material. In transverse waves along a beam on an elastic foundation, Nayfeh [10] studied third-harmonic resonance waves, which are described by two simultaneous nonlinear Schrödinger equations having two different group velocities. The stability of localized solitary wave solutions of simultaneous nonlinear Schrödinger equations describing different types of interacting waves in plasma has been investigated by Bhakta and Gupta [14]. Recently El-Dib [9] discussed stability of the periodic solutions for two coupled nonlinear Schrödinger equations. He also studied nonlinear Schrödinger equations with time-depend coefficients. In his investigation stability criteria at resonance cases are derived.

2. Stability criteria at the self-second-harmonic resonance case

Suppose we have two uniform monochromatic wavetrains solutions

$$u(x,t) = A \exp(i\theta_1) \text{ and } v(x,t) = B \exp(i\theta_2) \tag{7}$$

corresponding to any nonlinear dispersive system having two degrees of freedom in which $\theta_j = k_j x - \omega_j t$, ω_j, $j = 1, 2$, two unequal angular frequencies which are assumed to be real, k_j two different wavenumbers which are assumed to be real and positive and A B are the amplitude of the wavetrains. Nonlinear interactions force us to distinguish between two cases, the case of θ_1 and θ_2 are different (the non-resonance case). The second case is the self-second-harmonic resonance which arises when θ_2 (say) near $2\theta_1$ (say).

We shall now, discuss the stability criteria at the resonance case (self-second-harmonic resonance). It is convenient to introduce two detuning parameters [15] σ_1 and σ_2 in order to express the nearness of θ_2 to θ_1 as

$$k_2 = 2k_1 + \sigma_1 \text{ and } \omega_2 = 2\omega_1 + \sigma_2 \,,$$

and hence

$$i\theta_2 = 2i\theta_1 + i\Gamma \text{ and } -i(\theta_1 - \theta_2) = i\theta_1 + i\Gamma\,.$$

The use of the above notation in the nonlinear analysis [9] leads to deriving the two coupled nonlinear Schrödinger equations (1) and (2) in which they govern the resonance case. If we introduce the following transformation:

$$\begin{bmatrix} A(x,t) \\ B(x,y) \end{bmatrix} = \begin{bmatrix} \alpha(x,t) \\ \beta(x,t) \end{bmatrix} \exp(i\Gamma) \tag{8}$$

hence both equations (1) and (2) will reduce to

$$\begin{aligned} &i\tfrac{\partial\alpha}{\partial t} + P_1\tfrac{\partial^2\alpha}{\partial x^2} + i\sigma_1 P_1\left(2\tfrac{\partial\alpha}{\partial x} + i\sigma_1\alpha\right) - \sigma_2\alpha + \sigma_1\left(R + H - K\right)\bar{\alpha}\beta + \\ &iK\bar{\alpha}\tfrac{\partial\beta}{\partial x} + iH\beta\tfrac{\partial\bar{\alpha}}{\partial x} = (Q_{11}\left|\alpha\right|^2 + Q_{21}\left|\beta\right|^2)\alpha\,, \end{aligned} \tag{9}$$

$$\begin{aligned} &i\tfrac{\partial\beta}{\partial t} + P_2\tfrac{\partial^2\beta}{\partial x^2} + i\sigma_1 P_2\left(2\tfrac{\partial\beta}{\partial x} + i\sigma_1\beta\right) - \sigma_2\beta + \sigma_1(G - E)\alpha^2 + iE\alpha\tfrac{\partial\alpha}{\partial x} = \\ &\left(Q_{12}\left|\alpha\right|^2 + \quad Q_{22}\left|\beta\right|^2\right)\beta\,. \end{aligned} \tag{10}$$

In view of the temporal solution, suppose that the above equations (9) and (10) have the following dependence:

$$\alpha = m\exp(i\varpi t) \text{ and } \beta = n\exp(2i\varpi t)\,, \tag{11}$$

where m and n are non-zero real amplitudes, ϖ is the frequency. Inserting (11) into (9) and (10) we obtain

$$\left[\left(\varpi + \sigma_2 + P_1\sigma_1^2\right) + \left(Q_{11}m^2 + Q_{21}n^2\right)\right] = \sigma_1\left(R + H - K\right)\,n\,, \tag{12}$$

$$\sigma_1\left(G - E\right)m^2 = \left[\left(2\varpi + \sigma_2 + P_2\sigma_1^2\right) + \left(Q_{12}m^2 + Q_{22}n^2\right)\right]n\,. \tag{13}$$

Dividing Eq. (7) by Eq. (8), we obtain

$$\begin{aligned} &\left[\left(\varpi + \sigma_2 + P_1\sigma_1^2\right) + \left(Q_{11}m^2 + Q_{21}n^2\right)\right]\,\left[\left(2\varpi + \sigma_2 + P_2\sigma_1^2\right) + \left(Q_{12}m^2 + Q_{22}n^2\right)\right] \\ &= \sigma_1^2\left(R + H - K\right)\left(G - E\right)\,m^2\,. \end{aligned} \tag{14}$$

This can be rearranged in powers of the frequency ϖ as

$$2\varpi^2 + L_1\varpi + L_0 = 0\,, \tag{15}$$

where

$$\begin{aligned} L_1 &= \sigma_1^2(2P_1 + P_2) + (2Q_{11} + Q_{12})m^2 + (Q_{22} + 2Q_{21})n^2 + 3\sigma_2, \\ L_0 &= P_1P_2\sigma_1^4 + \left\{\left[P_2Q_{11} + P_1Q_{12} + (E - G)(H - K + R)\right]m^2\right. \\ &\quad \left. + (P_2Q_{21} + P_1Q_{22})n^2\right\}\sigma_1^2 + \left(Q_{11}m^2 + Q_{21}n^2\right)\left(Q_{12}m^2 + Q_{22}n^2\right) + \sigma_2^2 \\ &\quad + \left(\sigma_1^2(P_1 + P_2) + (Q_{11} + Q_{12})m^2 + (Q_{22} + Q_{21})n^2\right)\sigma_2 \end{aligned}$$

Clearly, the dispersion relation for the unperturbed solution (11) depends on the two unknowns m and n. It is easy to show that the solution (11) is bounded provided that the discriminate of the relation (10) is positive, that is

$$\begin{aligned}\left[\sigma_1^2(2P_1-P_2)+\sigma_2+(2Q_{11}-Q_{12})m^2+(2Q_{21}-Q_{22})n^2\right]^2 & \\ +8\sigma_1^2 m^2(G-E)(R+H-K) &> 0\,. \end{aligned} \tag{16}$$

This condition is trivially satisfied when

$$(G-E)(R+H-K)>0\,. \tag{17}$$

In order to achieve the stabilization for the unperturbed solutions (11), the two amplitudes m and n need to have real values while the stability criterion is satisfied. This may be achieved in practice. For example, with

$$m^2 = 2(Q_{22}-2Q_{21})\left[\sigma_1^2(2P_1-P_2)+\sigma_2\right]\,, \tag{18}$$

$$n^2 = 2(Q_{11}-2Q_{12})\left[\sigma_1^2(2P_1-P_2)+\sigma_2\right]+8\sigma_1^2(G-E)(H-K+R)\,. \tag{19}$$

Inserting these values into (11) yields

$$\left[\sigma_1^2(2P_1-P_2)+\sigma_2\right]^2+256\sigma_1^4(G-E)^2(R+H-K)^2(2Q_{21}-Q_{22})^2>0\,. \tag{20}$$

This is automatically satisfied. Accordingly, the real nature for the amplitudes of the unperturbed solutions (11) requires that

$$(Q_{22}-2Q_{21})\left[\sigma_1^2(2P_1-P_2)+\sigma_2\right] > 0\,, \tag{21}$$

$$(2Q_{11}-Q_{12})\left[\sigma_1^2(2P_1-P_2)+\sigma_2\right]+4\sigma_1^2(G-E)(H-K+R) > 0\,. \tag{22}$$

However, we wish to examine the stability of the unperturbed solutions (11). To accomplish this we perturb each solution according to

$$\alpha=(m+\zeta(x,t))\exp(i\varpi t)\,,\quad \beta=(n+\xi(x,t))\exp(2i\varpi t)\,, \tag{23}$$

where ζ and ξ are the perturbed solutions to be determined. Linearizing in ζ and ξ we obtain

$$\begin{aligned} i\frac{\partial\zeta}{\partial t}+P_1\frac{\partial^2\zeta}{\partial x^2}-Q_{11}(\zeta+\bar{\zeta})m^2-Q_{21}(\xi+\bar{\xi})mn+i\frac{\partial}{\partial x}(K\xi m+H\bar{\zeta}n) & \\ +\sigma_1\,(R+H-K)\left(\xi m+(\bar{\zeta}-\zeta)n\right) &= 0\,, \end{aligned} \tag{24}$$

$$\begin{aligned} i\frac{\partial\xi}{\partial t}+P_2\frac{\partial^2\xi}{\partial x^2}-Q_{22}(\xi+\bar{\xi})n^2-Q_{12}(\zeta+\bar{\zeta})mn+iEm\frac{\partial\zeta}{\partial x} & \\ +2\sigma_1(G-E)\zeta m-\sigma_1(G-E)\xi\frac{m^2}{n} &= 0\,. \end{aligned} \tag{25}$$

If the solutions of the above system are proportional to $\exp(-iqx-i\Omega t)$, such that

$$\xi(x,t)=\xi_0\exp(-iqx-i\Omega t) \text{ and } \zeta(x,t)=\zeta_0\exp(-iqx-i\Omega t)\,, \tag{26}$$

with constant coefficients ξ_0 and ζ_0, then the following dispersion relation relates the modulation frequency Ω and the modulation wavenumber q by

$$\Omega^2 - h_1\Omega + h_0 = 0\,, \tag{27}$$

where

$$\begin{aligned} h_1 &= (P_1+P_2)q^2 + Q_{11}m^2 + Q_{22}n^2 + \sigma_1(R+H-K)n + \sigma_1(G-E)\frac{m^2}{n}\,, \\ h_0 &= \left[P_1q^2 + Q_{11}m^2 + \sigma_1(R+H-K)n\right]\left[P_2q^2 + Q_{22}n^2 + \sigma_1(G-E)\frac{m^2}{n}\right] - \\ &\quad \left[Q_{12}mn - qEm - 2\sigma_1(G-E)m\right]\left[Q_{21}mn - qKm - \sigma_1(R+H-K)m\right]\,, \end{aligned}$$

and

$$q = \frac{1}{nHQ_{22}}\left[\sigma_1 Q_{22}(R+H-K)n + (Q_{12}Q_{21} - Q_{11}Q_{22})\,m^2\right]\,, \tag{28}$$

Because the stability is presented when Ω is real, which requires the discrimination of the above dispersion relation to be positive, hence the stability criteria are presented as

$$\begin{gathered} \left[(P_1-P_2)q^2 + Q_{11}m^2 - Q_{22}n^2 + \sigma_1(R+H-K)n - \sigma_1(G-E)\tfrac{m^2}{n}\right]^2 + \\ 4m^2\left\{\left[q^2KE - q\left[(EQ_{21}+KQ_{12})n - E\sigma_1(R+H-K)\right.\right.\right. \;> 0 \end{gathered} \tag{29}$$

provided that condition (11) is satisfied. This stability criterion depends on the two amplitudes of the unperturbed solutions (11) as well as on the wavenumber of the perturbation disturbance. The above stability condition can be trivially satisfied when

$$\left[qE + Q_{12} - 2\sigma_1(G-E)\right]\left[qK + Q_{12}n - \sigma_1(R+H-K)\right] > 0\,. \tag{30}$$

In view of the disturbance wavenumber q given by (29) the above inequality becomes

$$\begin{gathered} \left[\sigma_1 nQ_{22}(ER-EK+3EH-2GH) + Q_{12}Q_{22}Hn^2 + m^2(Q_{12}Q_{21}-Q_{11}Q_{22})\right]\times \\ \left[\sigma_1 nQ_{22}(R+H-K)(K-H) + Q_{21}Q_{22}Hn^2 + Km^2(Q_{12}Q_{21}-Q_{11}Q_{22})\right] > 0\,. \end{gathered} \tag{31}$$

For any $\frac{\sigma_1}{n}$ the above stability condition reduces to

$$\begin{aligned} Q_{22}(ER-EK+3EH-2GH) &> 0\,, \\ Q_{22}(R+H-K)(K-H) &> 0\,, \\ Em^2(Q_{12}Q_{21}-Q_{11}Q_{22}) + Q_{12}Q_{22}Hn^2 &> 0\,, \\ Km^2(Q_{12}Q_{21}-Q_{11}Q_{22}) + Q_{21}Q_{22}Hn^2 &> 0\,. \end{aligned}$$

The last two conditions can be satisfied when

$$\begin{aligned} E(Q_{12}Q_{21}-Q_{11}Q_{22}) &> 0\,, \\ Q_{12}Q_{22}H &> 0\,, \\ K(Q_{12}Q_{21}-Q_{11}Q_{22}) &> 0\,, \\ Q_{21}Q_{22}H &> 0\,. \end{aligned}$$

Finally, stability conditions can be summarised as

$$\begin{aligned} Q_{22}(ER - EK + 3EH - 2GH) &> 0\,, \\ Q_{22}(R + H - K)(K - H) &> 0\,, \\ EK &> 0\,, \\ Q_{12}Q_{21} &> 0\,, \\ HQ_{22}(Q_{12} + Q_{21}) &> 0\,, \\ (E + K)(Q_{12}Q_{21} - Q_{11}Q_{22}) &> 0\,. \end{aligned} \tag{32}$$

At the perfect resonance both the detuning parameters σ_1 and σ_2 should approach zero. At this end, the constraint (16) is trivially satisfied. Further, the temporal solutions (11) may have the frequency $\varpi = Q_{12}Q_{21} - Q_{11}Q_{22}$ associates with the amplitudes $m^2 = Q_{22} - 2Q_{21}$ and $n^2 = 2Q_{11} - Q_{12}$. At this stage the above stability conditions (30)–(32) will reduce to

$$\begin{gathered} \left[Q_{12}Q_{22}Hn^2 + Em^2\left(Q_{12}Q_{21} - Q_{11}Q_{22}\right)\right] \times \\ \left[Q_{21}Q_{22}Hn^2 + Km^2\left(Q_{12}Q_{21} - Q_{11}Q_{22}\right)\right] > 0\,. \end{gathered} \tag{33}$$

This is the stability criterion at the perfect second resonance case and hence stability condition for the coupled nonlinear Schrödinger equations (1) and (2) has been achieved.

Inspection of the two coupled nonlinear Schrödinger equations (1) and (2) reveals that there are 11 different coefficients. These coefficients are divided into three kinds. Four coefficients (Q_{11} Q_{22}, Q_{12} and Q_{21}) for the cubic nonlinear terms, five coefficients (R, K, H, G and E) refer to the parametric and quadratic nonlinear terms and finial two coefficients (P_1 and P_2) of the linear terms and represent the rate of the initial group velocity. Clearly, these coefficients play some roles in the stability configuration. In order to screen the impact of these coefficients in the stability criteria, some numerical calculations should be made. Before proceeding in numerical calculations, we may rearrange the stability condition (29) as

$$a_4q^4 + a_2q^2 + a_1q + a_0 > 0, \tag{34}$$

where the significance of the coefficients a_j is clear from the context. The equality for the above relation refers to the marginal stability curve.

3. Numerical illustrations for the secondary-resonance case

In this section, we are interested in screening a numerical picture for the stability criteria (34), taking into account the values of the two amplitudes m and $n$given by (18) and (19). The equality of the relation (34) refers to the transition curves separating stable region from unstable region at the self-secondary resonance case. The calculations are made in order to evaluate the modulation wavenumber q versus the detuning parameterσ_1. The calculation results are displayed in the plane $(q - \sigma_1)$. The bifurcation curves are emanating from a specifying point at a certain q known as a resonance point. These curves bound an unstable region which is embedded inside the resonance case. The stable area surrounds the unstable region.

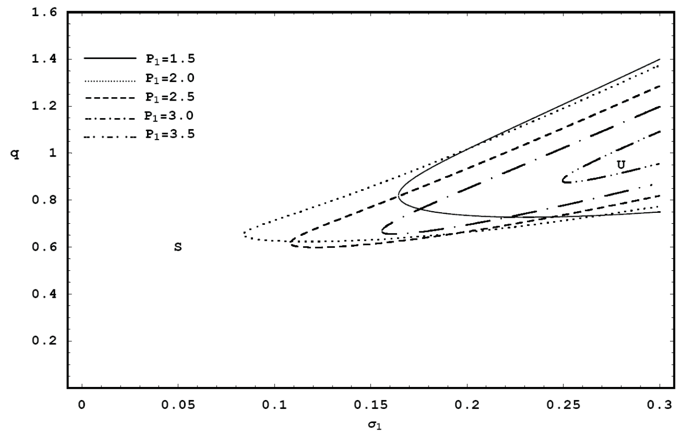

Figure 1 Represents the stability diagram at the self-second-resonance case. The diagram screen the influence of the dispersion parameter P_1 on the stability criteria. The curves refer to the equality corresponding to the condition (34). The calculations are made for $R = H = K = G = E = Q_{21} = Q_{12} = 1,\ Q_{11} = 4\,,\ Q_{22} = 3,\ P_2 = 1$ and $\sigma_2 = 0.1$

In the stability diagram, the symbol S refers to the stable region while the symbol U indicates the unstable region.

In order to catch the role of each coefficient for equations (1) and (2) in the stability criteria we proceed as follows: All the eleven coefficients for the coupled equations (1) and (2) are held fixed to the unit value except one coefficient which has slightly changed. This procedure allows us to discuss the influence of a chosen coefficient on the stability profile. The impact of the dispersion parameters P_1 or P_2 on the stability diagram has been illustrated in Figs. 1 and 2, respectively. In Fig. 1, some variation for parameter P_1 is made, while parameter P_2 has unit value. It seems as P_1 is increased, from $P_1 = 1.5$ to$P_1 = 2.0$, the unstable region increases associated with shifting the resonance point to the direction of decreasing the detuning parameter σ_1. At this stage, the dispersion parameter P_1 has a destabilizing role. A more increase in P_1, leads to some contraction in the unstable region are observed associated with shifting the resonance point into the direction of increasing the parameterσ_1. This shows that P_1 has a stabilizing role. However, we observe that there are two different roles for increasing the parameter P_1 from the value of 1.5 to the value of 3.5. At this end, increasing P_1 plays a dual role in the stability criteria. In Fig. 2, the examination is made in the stability picture for decreasing parameter P_2. It is found that the decrease in P_2 behaves in the stability configuration similar to the increase of parameter P_1.

The examination of the influence of the five coefficients of the periodic terms has been displayed in Figs. 3–6. Both the coefficients R and H behave the same role in the stability examination (see the stability criteria (28)). In Fig. 3, the stability diagram is made of variation the coefficient R. Inspection of the stability diagram leads to catch that the increase in R plays a destabilizing influence. While the influence of the parameter K has been illustrated in Fig. 4. It seems that there are two roles for increasing the coefficient K. A stabilizing influence is due to smaller

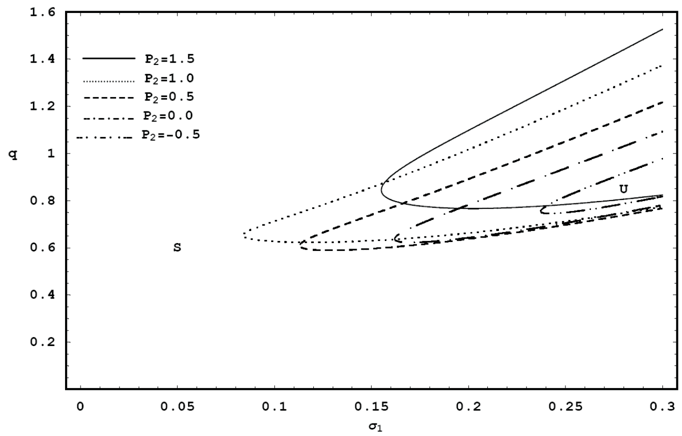

Figure 2 The variation for the parameter P_2 on the stability picture for the same system as in Fig. 1

values of the detuning parameter σ_1 and a destabilizing influence is due to larger values of σ_1. Thus, there is a dual role in the stability criteria for increasing the coefficient K. The increase in values of G leads to a contraction in the unstable region associate with shifting the resonance point into the direction of increasing both q and σ_1 as shown in Fig. 5. Therefore, the increase in the coefficient G plays a stabilizing role in the stability diagram.

Fig. 6 illustrates the influence of the variation in the coefficient E on the stability picture. It seen that for the specific value of $E = 0.7$, the resonance point occurs corresponds to the negative values for σ_1. The unstable region has a minimum width that corresponds to very small positive values of σ_1. As σ_1 is increased the width of the unstable region increases. This shows that the increase in σ_1 plays a destabilizing effect. As E is increased to the value $E = 0.9$, the unstable region has been separated into two unstable regions with two resonance points. One region corresponds to larger values of σ_1 and the other region corresponds to smaller and negative values of σ_1. For continue increasing in E, the unstable region corresponding to small values of σ_1, has an extension in its width associated with a decrease in the second unstable region. For $E = 1.5$, more extending in the unstable region is observed. In addition the resonance point occurs at $\sigma_1 \cong 0.2$. It seems that there are two different roles for the stability criteria between $E = 0.7$ and $E = 1.5$. However, the increase in E plays a dual role in the stability picture.

The examination of the influence of the cubic nonlinear coefficients Q_{ij} is illustrated in Fig. 7. The calculations showed that the increase of these coefficients ($Q_{ij} = Q\tilde{Q}_{ij}$) plays a dual role in the stability criteria.

Inspection of the variation for the detuning parameter σ_2 reveals that the increase in the parameter σ_2 plays a destabilizing influence in the stability picture as shown in Fig. 8.

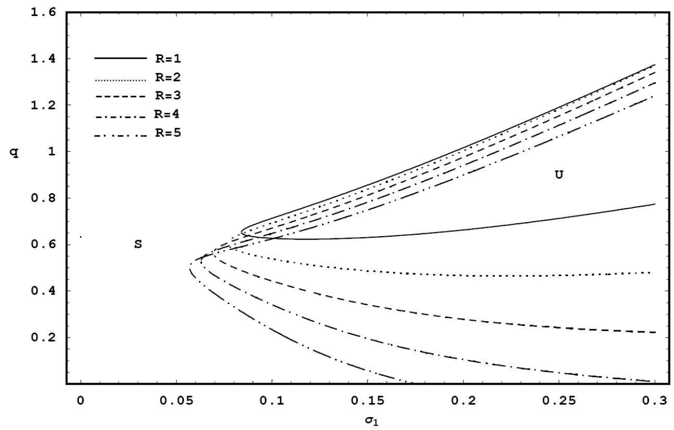

Figure 3 The illustration for the parameter R

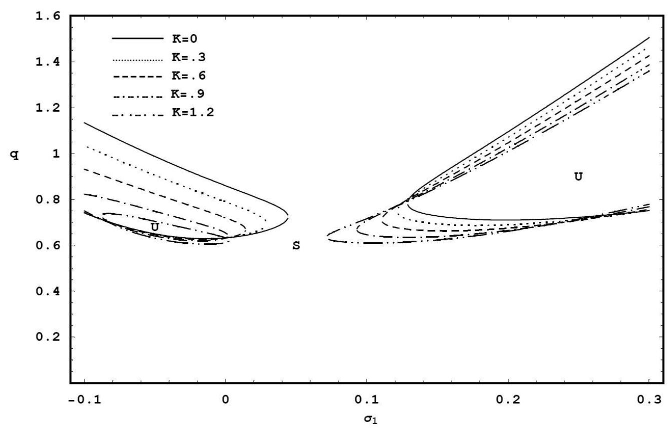

Figure 4 The influence of the parameter K

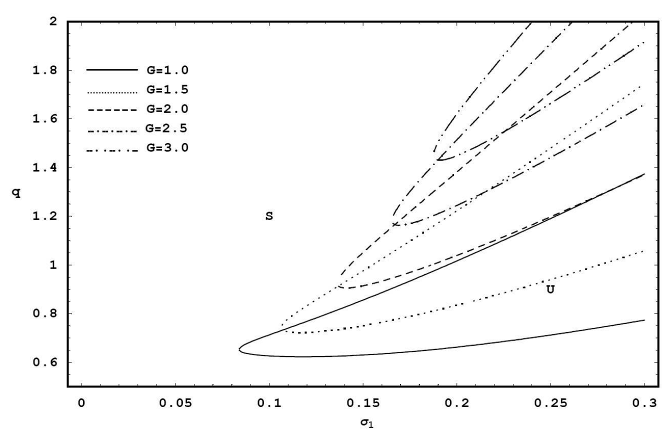

Figure 5 The illustration for the parameter G

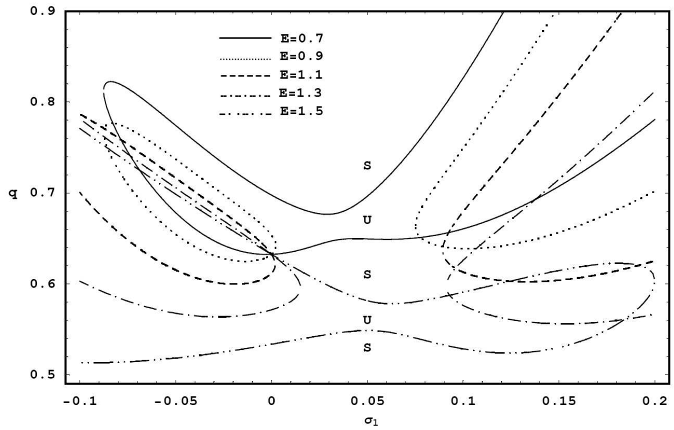

Figure 6 The impact of the parameter E on the stability configuration

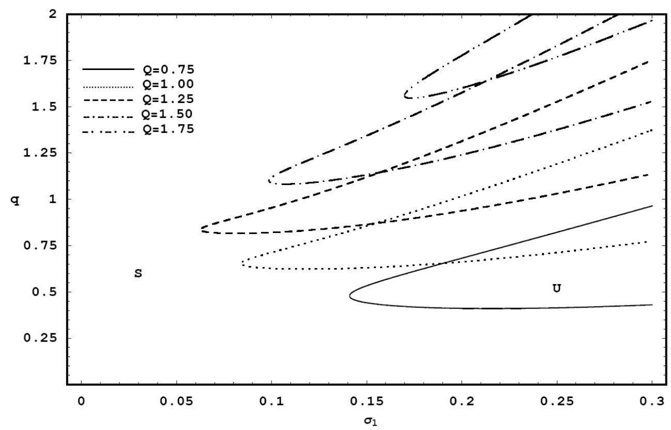

Figure 7 The impact of increasing the cubic nonlinear coefficients Q_{ij} on the stability behavior

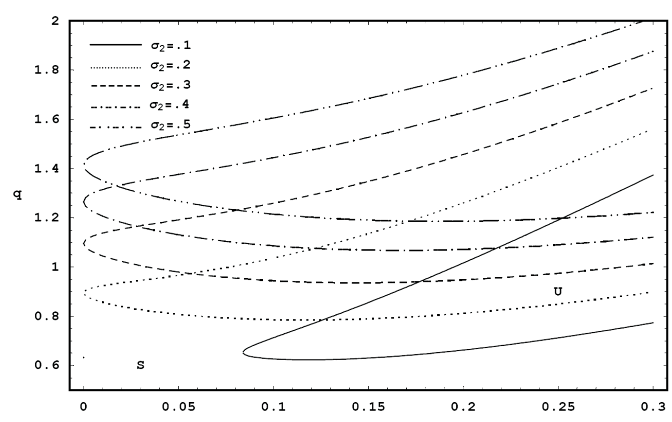

Figure 8 The influence for increasing the detuning parameter σ_2 on the stability plane

4. Cubic self-interaction of the primary waves and stability description

The derivation of the nonlinear equations (3) and (4) is due to nonlinear interaction between two primary harmonic waves having the following form:

$$u(x,t) = \big(A\exp(i\omega_1 t) + \bar{A}\exp(-i\omega_1 t)\big)\exp(ikx)\,, \tag{35}$$

$$v(x,t) = \big(B\exp(i\omega_2 t) + \bar{B}\exp(-i\omega_2 t)\big)\exp(ikx)\,, \tag{36}$$

in which $\omega_j,\ j = 1,2$ are the angular frequencies and k is the wavenumber which is assumed to be real and positive, and A & B are the amplitude of these wavetrains. The superposed bar refers to the complex conjugate variable. When the interaction between these two primary waves is centered around (k,ω_1) and (k,ω_2), where $\omega_2 \cong 3\omega_1$ (say), the self-third-harmonic resonance is presented. A nonlinear interaction forces us to distinguish between two cases. The first case deals with ω_1 and ω_2 being different (the non-resonance case that is out of our scope). The other is the self-third-harmonic resonance case which arises when ω_2 (say) near $3\omega_1$ (say). To express the nearness of these frequencies, we define

$$\omega_2 = 3\omega_1 + \sigma\,, \tag{37}$$

where σ is a small quantity which represents the detuning parameter [4]. Nonlinear interactions for equations (35) and (36) are based on the above relation (26). The analysis results are the previous two coupled nonlinear Schrödinger equations (3) and (4).

5. Stability analysis at the neighborhood of the cubic-resonance case

The stability analysis should be divided into two versions according if σ tends to zero or not. When σ tends to zero in the above system the perfect resonance case arises. Otherwise, the case of the neighborhood of the resonance is present. The absence of both S_1 and S_2 leads to the non-resonance case.

In order to accomplish the stability criteria for the coupled system of equations (3) and (4) we use the temporal modulation technique [16-19]. Suppose that the solutions of this system are stationary in x and depend, only, on the time t which are given by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \exp\left(\frac{1}{2} i\sigma t\right)\,, \tag{38}$$

where m_1 and m_2 are two real amplitudes satisfying the following equation:

$$S_2 m^3 + (Q_{12} - Q_{11})m^2 - S_1 m + (Q_{22} - Q_{21}) = 0\,. \tag{39}$$

This is a cubic equation in the stratified $m = \frac{m_1}{m_2}$. From elementary algebra, m must be real when $\alpha^2 + \beta^2 \geq 0$, where

$$\begin{aligned}
\alpha &= \frac{(Q_{12} - Q_{11})^2 - S_1 S_2}{9S_2^2}\,,\\
\beta &= -\left[\frac{S_1(Q_{12} - Q_{11}) + 3S_2(Q_{22} - Q_{21})}{6S_2^2} + \left(\frac{Q_{12} - Q_{11}}{3S_2}\right)^3\right].
\end{aligned}$$

As usual, we should perturb the time-dependent solution (38) to determine the stability conditions. Hence, the perturbation has the form:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} m_1 + \psi_1(x,t) \\ m_2 + \psi_2(x,t) \end{bmatrix} \exp\left(\frac{1}{2} i\sigma t\right) \quad , \tag{40}$$

where ψ_r are complex functions to be determined. Linearizing in ψ_r, we find that ψ_r satisfy the following system:

$$i\frac{\partial \Psi}{\partial t} + P\frac{\partial^2 \Psi}{\partial x^2} = (M\Psi + N\bar{\Psi}), \tag{41}$$

where Ψ is a vertical matrix of type 2×1 with components ψ_1 and ψ_2

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

while P, M and N are 2×2 matrices

$$\begin{aligned} P &= \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \\ M &= \begin{pmatrix} Q_{11}m_1^2 - S_1 m_1 m_2 & Q_{21}m_1 m_2 + S_1 m_1^2 \\ Q_{12}m_1 m_2 + 3S_2 m_1^2 & Q_{22}m_2^2 \end{pmatrix}, \\ N &= \begin{pmatrix} Q_{11}m_1^2 + 2S_1 m_1 m_2 & Q_{21}m_1 m_2 \\ Q_{12}m_1 m_2 & Q_{22}m_2^2 \end{pmatrix}. \end{aligned}$$

If the real and imaginary parts of Ψ are proportional to $\exp\left(iqm_1 x + i\Omega m_2^2 t\right)$, where q and Ω are the modulation wavenumber and frequency respectively, they are related by the following dispersion relation:

$$\Omega^4 - (a_1 + a_3)\,\Omega^2 + (a_1 a_3 - a_2 a_4) = 0\,, \tag{42}$$

where

$$\begin{aligned} a_1 &= P_1^2 m^2 q^4 + P_1(2Q_{11}m - 3S_1)mq^2 + 3(2Q_{21}S_2 + S_1 S_2 m - 2Q_{11}S_1)m, \\ a_2 &= ((P_1 + P_2)S_1 m + 2P_2 Q_{21})\,mq^2 + 2Q_{11}S_1 m^2, \\ a_3 &= P_2^2 m^2 q^4 + 2P_2 Q_{22} q^2 + S_1(3S_2 m + 2Q_{12})m, \\ a_4 &= (3(P_1 + P_2)S_2 m + 2P_1 Q_{12})\,mq^2 - 6(Q_{12}S_1 - Q_{22}S_2) - 9S_1 S_2 m. \end{aligned}$$

It is known that stability arises whence Ω having real roots. Since the above dispersion relation is a quadratic in Ω^2, hence the stability arises whence the two roots of equation (31) are both real and positive. This can be accomplished when

$$(a_1 + a_3) > 0\,, \quad (a_1 a_3 - a_2 a_4) > 0 \text{ and } (a_1 - a_3)^2 + 4a_2 a_4 > 0\,. \tag{43}$$

These are the stability criteria that are imposed from linear perturbation properties for the two periodic solutions (38). Clearly, these stability criteria depend on the coefficients that appear into equations (3) and (4) as well as on the ratio of the two amplitudes m_1, m_2 and finally on the disturbance wavenumber q. To seek the above stability criteria without depending on the disturbance q, we use the

Hurwitz criterion and then the stability will be available for negative real roots of the equalities for (43). These conditions are reduced as follows:

The first-condition of (43) yields

$$2P_1Q_{11}m^2 - 3P_1S_1m + 2P_2Q_{22} > 0\,, \tag{44}$$

$$6S_1S_2m^2 + 6S_2Q_{21}m + S_1(Q_{12} - 3Q_{11}) > 0\,. \tag{45}$$

The second-condition of (43) reduces to

$$S_1S_2\left[(3mS_2 + 2Q_{12})\,(mS_1 + 2Q_{21}) - 4Q_{11}Q_{22}\right] > 0\,, \tag{46}$$

$$\begin{aligned} S_1P_1P_2\,(-S_2(-3S_1 + 2mQ_{11})+ \\ 2P_1Q_{22})^3\,((3mS_2 + 2Q_{12})(mS_1 + 2Q_{21})- \\ 4Q_{11}Q_{22}) + P_2\left(6m^3S_1Q_{11}P_2^2(3S_1 - 2mQ_{11})+\right. \\ 2Q_{22}P_1^2\,(-m(3mS_2 + 2Q_{12})(mS_1 + 2Q_{21})- \\ 2\,(3S_1 - 2mQ_{11})Q_{22}) + mP_1P_2(m^2(9mS_1^2S_2- \\ 2Q_{11}(3mS_2 + 2Q_{12})(mS_1 + 2Q_{21}))+ \\ 2(3S_1 - 2mQ_{11})^2Q_{22}))(-2mS_2P_1Q_{22}+ \\ P_2((3mS_2 + 2Q_{12})(mS_1 + 2Q_{21})- \\ 2Q_{11}(m^3S_2 + 2Q_{22})))) \quad > \quad 0\,. \end{aligned} \tag{47}$$

The third-condition of (43) imposes the following inequalities:

$$(3mS_2(P_1 + P_2) + 2P_1Q_{12})\,(mS_1(P_1 + P_2) + 2P_2Q_{21}) \quad > \quad 0\,, \tag{48}$$

$$\begin{aligned} m(mS_1P_1((-3S_1 + 2mQ_{11})(3mS_2 + 2Q_{12}) + 6S_2Q_{22})+ \\ 3P_2(2m^3S_1S_2Q_{11}- \\ (mS_1 + 2Q_{21})(3mS_1S_2 + 2S_1Q_{12} - 2S_2Q_{22}))) \quad > \quad 0, \end{aligned} \tag{49}$$

$$S_1Q_{11}(3mS_1S_2 + 2S_1Q_{12} - 2S_2Q_{22}) \quad > \quad 0\,. \tag{50}$$

However the stability constraint that must be satisfied at the self-cubic-resonance for two co-propagating wavetrains has been achieved.

For a special case, when the two parameters S_1 and S_2 tend to zero into the two coupled nonlinear Schrödinger equations (3) and (4), the resulting system should be governing the non-resonance case. The system is composed of two coupled nonlinear Schrödinger equations that govern the non-resonance case as those discussed before by Tan and Boyd [12]. At this stage, equation (29) gives the value of the stratified amplitude m in the following form:

$$m^2 = \frac{Q_{22} - Q_{21}}{Q_{11} - Q_{12}}\,. \tag{51}$$

Here we explore the stability criterion for these coupled equations as a special case of (43). Accordingly, the stability conditions (43) will reduce to

$$(P_1^2 + P_2^2)m^2q^2 + 2(P_1Q_{11}m^2 + P_2Q_{22}) > 0\,, \tag{52}$$

$$P_1^2P_2^2m^2q^4 + 2P_1P_2(P_1Q_{22} + P_2Q_{11})q^2 + 4P_1P_2(Q_{11}Q_{22} - Q_{12}Q_{21}) > 0\,, \quad (53)$$

$$\left[(P_1^2 - P_2^2)m^2q^2 + 2(P_1Q_{11}m^2 - P_2Q_{22})\right]^2 + 16P_1P_2Q_{12}Q_{21}m^2 > 0\,. \quad (54)$$

These stability conditions can be satisfied, in arbitrary of the disturbance wavenumber q, whence

$$\begin{aligned} P_1Q_{11}m^2 + P_2Q_{22} &> 0\,,\\ P_1P_2(P_1Q_{22} + P_2Q_{11}) &> 0\,,\\ P_1P_2Q_{12}Q_{21} &> 0\,,\\ P_1P_2(Q_{11}Q_{22} - Q_{12}Q_{21}) &> 0\,. \end{aligned}$$

These conditions, for arbitrary of m, can be reduced to

$$P_1Q_{11} > 0, \quad P_2Q_{22} > 0 \text{ and } P_1P_2(Q_{11}Q_{22} - Q_{12}Q_{21}) > 0\,. \quad (55)$$

The above conditions represent the stability criteria at the non-resonance case.

At the perfect resonance case the parameter σ should vanish from equations (3) and (4). The formal result is given by

$$i\frac{\partial A}{\partial t} + P_1\frac{\partial^2 A}{\partial x^2} = (Q_{11}\,|A|^2 + Q_{21}\,|B|^2)A + S_1\bar{A}^2B\,, \quad (56)$$

$$i\frac{\partial B}{\partial t} + P_2\frac{\partial^2 B}{\partial x^2} = (Q_{12}\,|A|^2 + Q_{22}\,|B|^2)B + S_2A^3\,. \quad (57)$$

Since the stability conditions (43) are derived independently from the parameter σ, hence the stability criterion for the two coupled nonlinear Schrödinger equations (56) and (57) cannot be derived from the above stability conditions (43) as a special case. In what follows, we shall seek the stability conditions that cover the perfect cubic-resonance case.

6. Stability discussion at the perfect cubic-resonance case

We suppose the two-coupled nonlinear Schrödinger equations (56) and (57) to have the following time-dependent solutions:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} n_1\exp(i\varpi_1 t) \\ n_2\exp(i\varpi_2 t) \end{bmatrix}, \quad (58)$$

where n_1 and n_2 are non-zero real amplitudes, ϖ_1 and ϖ_2 are two frequency defined as

$$-\varpi_1 = Q_{11}n_1^2 + Q_{21}n_2^2 + S_1n_1n_2\exp\left(i(\varpi_2 - 3\varpi_1)t\right), \quad (59)$$

$$-\varpi_2 = Q_{12}n_1^2 + Q_{22}n_2^2 + S_2\frac{n_1^3}{n_2}\exp\left(-i(\varpi_2 - 3\varpi_1)t\right). \quad (60)$$

In the light of the relation between the primary waves at the perfect resonance ($\omega_2 = 3\omega_1$), the above solutions (43) take the form

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} n_1\exp(i\varpi t) \\ n_2\exp(3i\varpi t) \end{bmatrix}. \quad (61)$$

Consequently $n_1$and $n_2$are related by

$$n^3 S_2 + (Q_{12} - 3Q_{11})n^2 - 3S_1 n + (Q_{22} - 3Q_{21}) = 0\,; \quad n = n_1/n_2\,. \tag{62}$$

Let the linear perturbation for (46) be described as

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} n_1 & +\phi_1(x,t)\exp(i\varpi t) \\ n_2 & +\phi_2(x,t)\exp(3i\varpi t) \end{bmatrix}, \tag{63}$$

where ϕ_1 and ϕ_2 are small increments to be determined. Linearizing in ϕ_1 and ϕ_2, we find that they satisfy the following system

$$i\frac{\partial \Phi}{\partial t} + P\frac{\partial \Phi}{\partial x^2} = (U\Phi + V\bar{\Phi}), \tag{64}$$

where Φ is a vertical matrix of type 2×1 with components ϕ_1 and ϕ_2

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

and U representsa square matrix of the type 2×2

$$U = \begin{bmatrix} Q_{11}n_1^2 - S_1 n_1 n_2 & Q_{21}n_1 n_2 + S_1 n_1^2 \\ Q_{12}n_1 n_2 + 3S_2 n_1^2 & Q_{22}n_2^2 - S_2 n_1^3/n_2 \end{bmatrix}, \tag{65}$$

while the matrix V can be obtained from the matrix N by replacing the amplitudes m_j by n_j. When the real and imaginary parts of ϕ_1 and ϕ_2 are proportional to $\exp\left(iqn_1 x + i\Omega n_2^2\, t\right)$, the modulation frequency Ω will satisfy the following dispersion relation:

$$\Omega^4 - (b_1 + b_3)\,\Omega^2 + (b_1 b_3 - b_2 b_4) = 0\,. \tag{66}$$

where

$$\begin{aligned}
b_1 &= n^3 S_1(2Q_{12} + 3nS_2) + n^2(P_1 q^2 n - 3S_1)(P_1 q^2 n + 2Q_{11}n + S_1)\,,\\
b_2 &= n^2 S_1(P_2 q^2 n^2 + 2Q_{22} - n^3) + n^2(P_1 q^2 n - 3S_1)(2Q_{21} + nS_1)\,,\\
b_3 &= 3n^3 S_2(2Q_{21} + S_1 n) + n^2(P_2 q^2 n^2 + 2Q_{22} - n^3)(P_2 q^2 - n)\,,\\
b_4 &= 3n^3 S_2(P_1 q^2 n + 2Q_{11}n + S_1) + n^3(2Q_{12} + 3S_2 n)(P_2 q^2 - n)\,.
\end{aligned}$$

As discussed in the previous section the stability arises when

$$\begin{aligned}
(P_1 q^2 n + 2nQ_{11} + S_1)(P_1 q^2 n - 3S_1) & \\
+nS_1(3nS_2 + 2Q_{12}) + 3nS_2(nS_1 + 2Q_{21}) + & \\
(n - P_2 q^2)(n^3 - P_2 q^2 n^2 - 2Q_{22}) &\quad > \quad 0\,, \qquad (67)\\
(3nS_1 S_2 - 3nS_1 + q^2(3S_1 P_2 + nP_1(n - P_2 q^2)))\times & \\
(-P_2 q^2 n^2(S_1 + 2nQ_{11}) + n^3(S_1 + 3S_1 S_2 + 2nQ_{11}) + & \\
+2n^2 S_1 Q_{12} + 2n(3nS_2 + 2Q_{12})Q_{21} + & \\
nP_1 q^2(n^3 - P_2 q^2 n^2 - 2Q_{22}) - 2(S_1 + 2nQ_{11})Q_{22}) &\quad > \quad 0, \qquad (68)\\
((P_1 q^2 n - 3S_1)(S_1 + P_1 q^2 n + 2nQ_{11}) + & \\
nS_1(3nS_2 + 2Q_{12}) - 3nS_2(nS_1 + 2Q_{21}) - (n - P_2 q^2) & \\
\times(n^3 - P_2 q^2 n^2 - 2Q_{22}))^2 - 4n(3S_2(S_1 + P_1 q^2 n + 2nQ_{11}) + & \\
(P_2 q^2 - n)(3nS_2 + 2Q_{12})) & \\
\times((P_1 q^2 n - 3S_1)(nS_1 + 2Q_{21}) + S_1(P_2 q^2 n^2 - n^3 + 2Q_{22})) &\quad > \quad 0\,. \qquad (69)
\end{aligned}$$

In this discussion we have focused on the linear stability properties of two co-propagating wavetrains to determine the linear terms responsible at the prefect cubic-resonance case.

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