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Stability of Functionally Graded Plate under In-Plane Time-Dependent Compression

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Functionally graded materials have gained considerable attention in the high temperature applications. Linear dynamics equations taking into account a coupling of in-plane and transverse motions are used. Material properties are graded in the thickness direction of the plate according to volume fraction power law distribution. An oscillating temperature causes generation of in-plane time-dependent forces destabilizing the plane state of the plate equilibrium. The asymptotic stability and almost-sure asymptotic stability criteria involving a damping coefficient and loading parameters are derived using Liapunov's direct method. Effects of power law exponent on the stability domains are studied.

Keywords: functionally graded material, plates, stability, Liapunov method.

1. Introduction

Functionally graded materials have gained considerable attention in the high temperature applications.Functionally graded materials are composite materials, which are microscopically inhomogeneous, and the mechanical properties vary smoothly or continuously from one surface tothe other. It is this continuous change that results in gradient properties in functionally graded materials (FGM). Commonly, these materials are made from a mixture of ceramic and metal or a combination of different metals. The ceramic material provides high temperature resistance due to its low thermal conductivity while the ductile metal component prevents fracture due to thermas stresses and secures a suitable strength and stiffness. Many studies have examined FGM as thermal bariers. With the increased usage of these materials it is also important to understand the dynamics of FGM structures. A few studies have addressed this. Transient thermal stresses in a plate made of functionally gradient material were examined by Obata and Noda (1993). Vibration analysis of functionally graded cylindrical shells was performed by Loy, Lam and

Reddy (1999). Recently, Lam, Liew, and Reddy (2001) presented dynamic stability analysis of functionally graded cylindrical shells under periodic axial loading. In this paper, the parametric vibrations or dynamic stability of functionally graded rectangular plate described by linear partial differential equations is studied using the direct Liapunov method. Small deflection equations taking into account a coupling of in-plane and transverse motions are used. Due to a small thickness coupling, and rotary inertia terms are neglected. Material properties are graded in the thickness direction of the plate according to volume friction power law distribution. The viscous model of external damping with a constant coefficient is assumed. An oscillating temperature causes generation of in-plane time-dependent forces destabilizing plane state of the plate equilibrium. The asymptotic stability and almost-sure asymptotic stability criteria involving a damping coefficient and loading parameters are derived. Effects of power law exponent on the stability domains are studied.

2. Problem formulation

Consider the thin functionally graded rectangular plate with in-plane dimensions a and b. In-plane and transverse displacements are denoted by u, v, and w, respectively. Taking into account the Kirchhoff hypothesis on nondeformable normal element the governing partial differential equations are given as follows

$$N_{x,x} + N_{xy,y} = 0 \tag{1}$$

$$N_{xy,x} + N_{y,y} = 0 \tag{2}$$

$$w_{,tt}+2\beta\rho h w_{,t}+(\bar{N}_{x0}+\bar{N}_x)w_{,xx}+(\bar{N}_{y0}+\bar{N}_x)w_{,yy}-M_{x,xx}-2M_{xy,xy}-M_{y,yy} = 0 \tag{3}$$

$$(x, y) \in \Omega \equiv (0, a) \times (0, b)$$

where β is a damping coefficient, $\bar{N}_{x0}$ and $\bar{N}_{y0}$ are constant components of membrane forces, $\bar{N}_x$ and $\bar{N}_y$ are time-dependent components of membrane forces, and ρ is the equivalent density of the plate, h is the total thickness. The membrane forces are stochastic with means equal to zero and known probability distributions. The processes are physically realizable and sufficently smooth in order the solution of dynamics equations exist. We use the extensional, coupling and bending stiffnesses A_{ij}, B_{ij}, and D_{ij} $(i, j) = 1, 2, 6)$, defined as follows

$$\{A_{ij},\ B_{ij},\ D_{ij}\} = \int_{-h/2}^{h/2} Q_{ij}\left\{1,\ \ z,\ \ z^2\right\} dz \tag{4}$$

The stiffnesses Q_{ij} for isotropic materials are given by

$$\mathbf{Q} = \begin{vmatrix} \frac{E_{eff}}{1-\nu_{eff}^2} & \frac{\nu_{eff}E_{eff}}{1-\nu_{eff}^2} & 0 \\ \frac{\nu_{eff}E_{eff}}{1-\nu_{eff}^2} & \frac{E_{eff}}{1-\nu_{eff}^2} & 0 \\ 0 & 0 & \frac{E_{eff}}{2(1+\nu_{eff})} \end{vmatrix} \tag{5}$$

In-plane and moments are expressed by displacements as follows

$$\begin{vmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{11} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{vmatrix} \begin{vmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{vmatrix} \tag{6}$$

The effective elastic modulus and effective Poisson's ratio of the functionall graded plate are denoted by E_{eff} and ν_{eff}, respectively. In order to precisely model the material properties of functionally graded materials, the properties must be both temperature and position dependent. This is achieved by using a rule of mixtures for the mechanical parameters (E_{eff}, ν_{eff}, ρ). The volume fraction is a spatial function and the properties of the constituents are functions of the temperature. The combination of these functions gives the effective material properties of functionally graded materials and can be expressed as follows

$$F_{eff}(T, z) = F_c(T)V(z) + F_m(T)(1 - V(z)) \tag{7}$$

where F_{eff} is the effective material property of the functionally graded material, F_c and F_m are the properties of the ceramic and the metal, respectively, and V is the volume fraction of the ceramic constituent of the functionally graded material. A simple power law exponent of the volume fractions is used to describe the amount of ceramic and metal in the functionally graded material as follows

$$V(z) = \left(\frac{z + h/2}{h} \right)^q \tag{8}$$

where q is the power law exponent ($0 \leq q < \infty$).

Using equation (5) the constitutive equation (6) can be rewritten in the form

$$\begin{vmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{11} & 0 & B_{12} & B_{11} & 0 \\ 0 & 0 & (A_{11} - A_{12})/2 & 0 & 0 & (B_{11} - B_{12})/2 \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{11} & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & (B_{11} - B_{12})/2 & 0 & 0 & (D_{11} - D_{12})/2 \end{vmatrix} \begin{vmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{vmatrix} \tag{9}$$

The plate is assumed to be simply supported along each edge. The conditions imposed on displacements and internal forces and moments, called according to Almroth's [1] classifications S2, can be written down as

$$w = 0 \quad M_x = 0 \quad N_x = 0 \quad v = 0 \quad \text{at} \quad x = 0, a \tag{10}$$

$$w = 0 \quad M_y = 0 \quad N_y = 0 \quad u = 0 \quad \text{at} \quad y = 0, b \tag{11}$$

It is assumed that the plate is subjected to the time-varying in-plane axial forces $\bar{N}_x$ and $\bar{N}_y$ leading to parametric vibrations. Dividing equations (1)-(3) by ρh and denoting

$$\bar{n}_x = \bar{N}_x/\rho h \qquad \bar{n}_y = \bar{N}_y/\rho h$$

$$\bar{n}_{x0} = \bar{N}_{x0}/\rho h \qquad \bar{n}_{y0} = \bar{N}_{y0}/\rho h$$
$$n_x = N_x/\rho h \qquad n_y = N_y/\rho h$$
$$n_{xy} = N_{xy}/\rho h \qquad m_x = M_x/\rho h$$
$$m_y = M_y/\rho h \qquad m_{xy} = M_{xy}/\rho h$$

leads to the basic equations of motion

$$n_{x,x} + n_{xy,y} = 0 \tag{12}$$

$$n_{xy,x} + n_{y,y} = 0 \tag{13}$$

$$w_{,tt} + 2\beta w_{,t} + (\bar{n}_{x0} + \bar{n}_x)w_{,xx} + (\bar{n}_{y0} + \bar{n}_x)w_{,yy} - m_{x,xx} - 2m_{xy,xy} - m_{y,yy} = 0 \tag{14}$$

$$(x, y) \in \Omega$$

The plate motion is described by the uniform equations (12)–(14) with the trivial solution $w = 0$, $w_{,t} = 0$ corresponding to the plane (undisturbed) state. The trivial solution is called almost sure asymptotically stable if

$$P\{\lim_{t\to\infty} \|w(.,t)\| = 0\} = 1 \tag{15}$$

where $\|w(.,t)\|$ is a measure of disturbed solution w, $w_{,t}$ from the equilibrium state, and P is a probability measure. The crucial point of the method is a construction of a suitable Liapunov functional, which is positive for any motion of the analyzed system. It follows that the measure of distance can be chosen as the square root of Liapunov functional $\|w(.,t)\| = V^{1/2}$.

3. Stability analysis

The energy-like Liapunov functional has the form of a sum of modified kinetic energy $\mathcal{T}$ and potential energy of the plate and can be chosen in the form similar to the functional involved in stability analysis of laminated plates (Tylikowski, 1989)

$$\mathcal{V} = \frac{1}{2}\int_\Omega \big[v^2 + 2\beta vw + 2\beta^2 w^2 - m_x w_{,xx} - m_y w_{,yy} - 2m_{xy} w_{,xy} - \bar{n}_{x0} w_{,x}^2 - \bar{n}_{y0} w_{,y}^2\big]\, d\Omega \tag{16}$$

It is assumed that the in-plane forces are periodic or stochastic non-white stationary and sufficiently smooth ergodic process. Therefore, it is legitimate to use the classical differentation rule. Upon differentiation with respect to time, substituting dynamic equations (12)-(14) and using the boundary conditions we obtain the time derivative of functional in the form

$$\frac{d\mathcal{V}}{dt} = -2\lambda\mathcal{V} + 2\mathcal{U} \tag{17}$$

where the auxiliary functional $\mathcal{U}$ is defined as follows

$$\mathcal{U} = \frac{1}{2}\int_\Omega \big[2\beta^2 w w_{,t} + 2\beta^3 w^2 + (w_{,t} + \beta w)(\bar{n}_x w_{,xx} + \bar{n}_y w_{,yy})\big]\, d\Omega \tag{18}$$

Therefore, the stability analysis depends on the construction of the bound

$$\mathcal{U} \leq \lambda \mathcal{V} \tag{19}$$

or we look for a function λ defined as a maximum over all admissible functions u, v, w, w_t and satisfying the boundary conditions of the ratio $\mathcal{U}/\mathcal{V}$. As a maximum is a particular of stationary point we put to zero a variation of $\mathcal{U}/\mathcal{V}$. The associated Euler equations are linear in the case of the second-order functionals. Solving the associated Euler problem we find the function λ as follows

$$\lambda = \max_{m,n=1,2,\ldots} \left\{ \frac{\left| \hat{\beta}^2 + \frac{1}{2}\big(\hat{f}_x m^2 r^2 + \hat{f}_y n^2\big) \right|}{\sqrt{\hat{\beta}^2 + \Omega_{mn}^2}} \right\} \tag{20}$$

where the dimensionless damping coefficient, the dimensionless forces are, respectively

$$\hat{\beta} = \beta\Big(\frac{b}{\pi}\Big)^2 \sqrt{\frac{\rho h}{D_{11}}} \tag{21}$$

$$\hat{f}_x = \frac{\bar{N}_x b^2}{D_{11}\pi^2} \qquad \hat{f}_y = \frac{\bar{N}_y b^2}{D_{11}\pi^2} \tag{22}$$

$$\hat{f}_{x0} = \frac{\bar{N}_{x0} b^2}{D_{11}\pi^2} \qquad \hat{f}_{y0} = \frac{\bar{N}_{y0} b^2}{D_{11}\pi^2} \tag{23}$$

Ω_{mn} is the dimensionless natural frequency given as

$$\Omega_{mn}^2 = \big(m^2 r^2 + n^2\big)^2 \Big[1 + \Big(\frac{B_{11}}{D_{11}}\Big)^2 \frac{m^2 r^2 \Delta_{22} + n^2 \Delta_{11} - 2mrn\Delta_{12}}{\Delta_{11}\Delta_{22} - \Delta_{12}^2} + \\ - m^2 r^2 \hat{f}_{x0} - n^2 \hat{f}_{y0}\Big] \tag{24}$$

where $r = b/a$ is the plate aspect ratio and

$$\Delta_{11} = m^2 r^2 A_{11}/D_{11} + n^2 A_{66}/D_{11} \tag{25}$$

$$\Delta_{22} = m^2 r^2 A_{66}/D_{11} + n^2 A_{11}/D_{11} \tag{26}$$

$$\Delta_{12} = mrn(A_{12} + A_{66})/D_{11} \tag{27}$$

Using the property of function λ in equality (17) leads to the first-order differential inequality, the solution of which has the form

$$\mathcal{V}(t) \leq \mathcal{V}(0) \exp\big[-\big(\beta - \frac{1}{t}\int_0^t \lambda(\tau) d\tau\big)\big] \tag{28}$$

Therefore,the sufficient criterion of the asymptotic stability has the form

$$\hat{\beta} \geq \lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda(\tau) d\tau \tag{29}$$

If the processes $\bar{N}_x$ and $\bar{N}_y$ satisfy an ergodic property, the sufficient condition of the almost sure asymptotic stability can be written down as follows

$$\beta \geq \mathbf{E}\lambda \tag{30}$$

where $\mathbf{E}$ denotes the mathematical expectation.

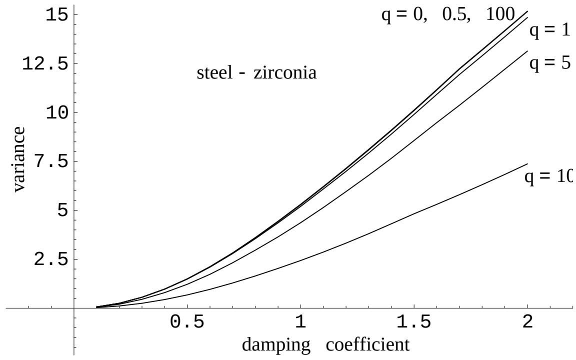

Figure 1 Stability domains of square plate for the zero mean Gaussian uniaxial force $\hat{f}_x$

4. Numerical results

The ceramic material used in this study is zirconia and the metal material is steel. Mechanical properties are given in Table 1.

Table 1. Mechanical properties of constituents of the FGM

Material	Steel	Nickel	Zirconia	Titanium alloy Ti-6Al-4V	Silicon Nitride
ρ kg/m^3	7850	8900	5490	4500	2370
E N/m^2	2.1×10^{11}	2.2395×10^{11}	1.51×10^{11}	1.16×10^{11}	3.4843×10^{11}
ν	0.25	0.31	0.33	0.33	0.24

The plate thickness is as follows: $h = 0.005$ m. Formulae (20) and (30) give us the possibility to calculate a maximal excitation intensity (e.g. square root of variance) of dimensionless in-plane forces ($\hat{f}_x$, $\hat{f}_y$) guaranteeing the almost sure asymptotic stability for given values of power law exponent q. The stability region is calculated for a Gaussian zero-mean process with variance σ^2.

5. Conclusions

The applicability of the direct Liapunov method has been extended to functionally graded plates subjected to time-dependent, in-plane forces. The major conclusion is that The influence of the power law exponent on the critical value of stability domains (expressed by the variance of dimensionless forces) is shown.

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