Mechanics and Mechanical Engineering
Vol. 7, No. 1 (2004) 153–160

Capillary Instability of Fluid Cylinder Under the Effect of Transverse Varying Electric Field

Ahmed E. RADWAN

Helmy A. RADWAN
Mathematics Department, Faculty of Science, Ain Shams University
Cairo, Egypt

Received (9 December 2003)
Revised (15 January 2004)
Accepted (31 January 2004)

The electrodynamic instability of a capillary dielectric fluid cylinder (radius a and density ρ) penetrated by a uniform axial electric field surrounded by a transverse varying electric field is investigated. A general dispersion relation to all possible axisymmetric modes of perturbation for all short wavelengths and long wavelengths is derived and discussed in detail. The model is capillary stable to axisymmetric modes if the longitudinal wave number normalized with respect to the jet radius is equal to or greater than 1.05757 and vice versa. The axial electric fields pervading the interior and the exterior of the cylinder are stabilizing or destabilizing for all disturbance modes according to some restrictions. The transverse varying electric field is purely stabilizing in the axisymmetric disturbances for all wavelengths. The electrodynamic force has a strong stabilizing influence in the axisymmetric mode and can suppress the capillary instability above a certain value of the basic electric field.

Keywords: dielectric fluid, wavelength, electrodynamic instability

1. Introduction

The research on the interaction of electromagnetic field with fluid media has heavily emphasized the Lorentz force which forms the basis of magnetohydrodynamics [1].

Electrical components of the electromagnetic stress tensor [2] were given attention under the heading of electrohydrodynamics. Rayleigh [3] has shown that the critical length of a cylinder jet is $2\pi a$, where a is the radius of the cylinder.

Chandrasekhar and Fermi ([4] & [5]) elaborated the related problem of a fluid jet acting on its own attraction to small axisymmetric disturbances. Such a study has a correlation with the understanding of the dynamical behaviour of the spiral arms of galaxies [5]. Mohamed [6] has studied the electrodynamic stability of a rotating jet under the influence of an axial electric field. Radwan ([7] & [8]) has recently modified the stability criterion in the refs. [4] and [5] by including the effect of the electrodynamic force on the capillary force pervaded by a homogeneous and uniform electric field, see also [12]. Recently Elazab ([9] & [10]) has studied the stability of a liquid cylinder under the influence of surface tension and axial electric currents.

The main aim of this work is to investigate capillary instability of fluid cylinder under the effect of transverse varying electric field.

2. Formulation of the problem

Consider a dielectric fluid cylinder of radius a and dielectric constant $\epsilon^{(1)}$ with dielectric capillary vacuum of dielectric constant $\epsilon^{(2)}$. The fluid is assumed to be homogeneous, incompressible, non-viscous and with uniform density ρ. The superscripts (1) and (2) characterize the variables interior to the fluid cylinder and exterior to the fluid cylinder.

In the basic state it is proposed that there are no surface charges at the fluid boundary surface and consequently the surface charge density will be considered to be zero during the perturbation. The fluid is acting upon the capillary, inertia and electrodynamic forces.

In the region surrounding the fluid, the only existing force is the electrodynamic force. We shall use the cylindrical coordinates system (r, ϕ, z) with the $z-$axis coinciding with the axis of the fluid cylinder. The dielectric fluid cylinder is pervaded by the longitudinal uniform electric field

$$\underline{E}_o^{(1)} = (0, 0, E_o), \tag{1}$$

and the surrounding medium of the liquid cylinder (in the vacuum) is assumed to be pervaded by

$$\underline{E}_o^{(2)} = (0, \frac{\beta E_o a}{r}, \alpha E_o) \tag{2}$$

where β and α are some parameters satisfying certain conditions, see equation (21).

The basic equations for such a problem under consideration are a combination of the pure hydrodynamic, capillary and Maxwell equations. Under the present circumstances they are the following.

In the liquid cylinder of radius a:

$$\rho\Big(\frac{\partial}{\partial t} + (\underline{u} \cdot \nabla)\Big)\underline{u} = -\nabla p + \frac{1}{2}\epsilon^{(1)}\nabla(\underline{E} \cdot \underline{E})^{(1)}, \tag{3}$$

$$\begin{aligned} \nabla \cdot \underline{u} &= 0, & (4)\\ \nabla \cdot (\epsilon\underline{E})^{(1)} &= 0, & (5)\\ \nabla \wedge \underline{E}^{(1)} &= 0. & (6) \end{aligned}$$

In the region surrounding the liquid cylinder, we have

$$\begin{aligned} \nabla \cdot (\epsilon\underline{E})^{(2)} &= 0, & (7)\\ \nabla \wedge \underline{E}^{(2)} &= 0, & (8) \end{aligned}$$

and along the liquid interface:

$$p_s = T(\nabla \cdot \underline{N}_s), \tag{9}$$

where ρ is the mass density, $\underline{u}$ velocity vector, p kinetic pressure, $\underline{E}^{(1)}$ and $\underline{E}^{(2)}$ are the electric field intensities acting inside and outside the liquid cylinder, respectively, and p_s the pressure due to the capillary force, T the surface tension coefficient. $\underline{N}_s$ is the unit outward normal vector to the gas liquid interface,

$$\underline{N}_s = \frac{\nabla F(r,\phi,z,t)}{|\nabla F(r,\phi,z,t)|}, \tag{10}$$

such that

$$F(r,\phi,z,t) = 0, \tag{11}$$

where surface tension coefficient T is assumed to be constant.

3. Equilibrium State

In the equilibrium state, which is the unperturbed state, the basic electrodynamical equations take the form:

In the liquid cylinder of radius a:

$$\begin{aligned} -\nabla p_o + \frac{1}{2}\epsilon^{(1)}\nabla(\underline{E}_o \cdot \underline{E}_o)^{(1)} &= 0, && (12)\\ \nabla \cdot \underline{u}_o &= 0, && (13)\\ \nabla \cdot (\epsilon \underline{E}_o)^{(1)} &= 0, && (14)\\ \nabla \wedge \underline{E}_o^{(1)} &= 0. && (15) \end{aligned}$$

In the region outside the liquid cylinder:

$$\begin{aligned} \nabla \cdot (\epsilon \underline{E}_o)^{(2)} &= 0, && (16)\\ \nabla \wedge \underline{E}_o^{(2)} &= 0, && (17) \end{aligned}$$

subscript (0) here and henceforth indicates equilibrium quantities. The fundamental equations (12)-(17) of the unperturbed state are simplified and solved with azimuthal and longitudinal symmetries

$$\frac{\partial}{\partial z} = 0, \qquad \frac{\partial}{\partial \phi} = 0. \tag{18}$$

The fluid hydrostatic pressure is identified and given by

$$p_o = \frac{T}{a} + \frac{1}{2}E_o^2\big(\epsilon^{(1)} - \epsilon^{(2)}(\beta^2 + \alpha^2)\big). \tag{19}$$

It is clear as $p_o \geq 0$ at the fluid boundary surface $r = a$, the restriction

$$\epsilon^{(1)} \geq \epsilon^{(2)}(\beta^2 + \alpha^2) \tag{20}$$

must be satisfied, where the equality holds a limiting case with zero fluid pressure.

If the medium surrounding the cylinder has the same permitivity constant as that of the fluid, the conditions are given explicity by

$$\beta^2 + \alpha^2 \leq 1. \tag{21}$$

4. Perturbation State

Let the basic state be perturbed, then for small departure from the equilibrium state every perturbed quantity $Q(r, \phi, z, t)$ can be expanded as

$$Q(r, \phi, z, t) = Q_o(r) + Q_1(r, \phi, z), \tag{22}$$

with

$$Q_1(r, \phi, z, t) = Q_1(r)\epsilon(t)e^{i(kz+m\phi)}, \tag{23}$$

where $Q(r, \phi, z, t)$ stands for $\underline{E}^{(1)}$, $\underline{E}^{(2)}$, p or $\underline{u}$ and subscript (1) indicates the perturbed quantities. k (a real number) is the longitudinal wavenumber, m (an integer) is the azimuthal wavenumber. $\epsilon(t)$ is the amplitude of the surface wave at time t take the form

$$\epsilon(t) = \epsilon_o e^{\sigma t}, \tag{24}$$

and hence

$$Q_1(r, \phi, z, t) = Q_1(r)\epsilon_o e^{\sigma t + i(kz+m\phi)}, \tag{25}$$

where ϵ_o is the initial amplitude at $t = 0$ and σ is the growth rate. If $\sigma = i\omega$, $i = \sqrt{-1}$ is imaginary then $\frac{\omega}{2\pi}$ is the wave oscillation frequency. The perturbed radial distance of the interface of the compound fluid cylinder

$$r = a + \epsilon_o e^{\sigma t + i(kz+m\phi)}. \tag{26}$$

The linearization of equations (3)–(8) by means of equations (22)–(26) leads to the following relevant perturbation equations.

In the liquid cylinder of radius a:

$$\rho\sigma\underline{u}_1 = -\nabla\Pi_1, \tag{27}$$

$$\nabla \cdot \underline{u}_1 = 0, \tag{28}$$

$$\nabla \cdot \underline{E}_1^{(1)} = 0, \tag{29}$$

$$\nabla \wedge \underline{E}_1^{(1)} = 0. \tag{30}$$

where

$$\Pi_1 = p_1 - \frac{1}{2}\epsilon^{(1)}(2\underline{E}_o \cdot \underline{E}_1)^{(1)}. \tag{31}$$

In the region outside the liquid cylinder:

$$\nabla \cdot \underline{E}_1^{(2)} = 0, \tag{32}$$

$$\nabla \wedge \underline{E}_1^{(2)} = 0, \tag{33}$$

Taking the divergence of the vector equation of motion (27) and utilizing equation (28) we get

$$\nabla^2\Pi_1 = 0. \tag{34}$$

The circulation equations (30) and (33) concerning the perturbed electric field intensities interior and exterior to the liquid cylinder means that $\underline{E}_1^{(1)}$, $\underline{E}_1^{(2)}$ and can be derived by means of the scalar functions $\Psi_1^{(1)}$ and $\Psi_1^{(2)}$

$$\underline{E}_1^{(1)} = \nabla\Psi_1^{(1)}, \tag{35}$$

$$\underline{E}_1^{(2)} = \nabla \Psi_1^{(2)}. \tag{36}$$

Inserting equations (35) and (36) into equations (29) and (32), we get

$$\nabla^2 \Psi_1^{(1)} = 0, \tag{37}$$

$$\nabla^2 \Psi_1^{(2)} = 0. \tag{38}$$

By using the equations (34), (37) and (38) we note that Π_1, $\Psi_1^{(1)}$ and $\Psi_1^{(2)}$ satisfied Lapace's equation. The non-singular solutions are given by

$$\begin{aligned}
\Pi_1 &= A I_m(kr)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (39)\\
\underline{u}_1 &= \frac{-1}{\rho\sigma}\nabla \Pi_1, && (40)\\
\Psi_1^{(1)} &= B I_m(kr)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (41)\\
\Psi_1^{(2)} &= C K_m(kr)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (42)\\
\underline{E}_1^{(1)} &= B\nabla I_m(kr)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (43)\\
\underline{E}_1^{(2)} &= C\nabla K_m(kr)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (44)\\
p_{1s} &= \frac{-T}{a}(1 - m^2 - k^2a^2)\epsilon_o e^{\sigma t + i(kz+m\phi)}, && (45)
\end{aligned}$$

where $I_m(kr)$ and $K_m(kr)$ are modified Bessel functions of the first and second kind of the order m and A, B and C are constants of integration to be determined by using the boundary conditions.

The solutions (39)-(45) of the relevant perturbation equations (27)-(33) must satisfy certain boundary conditions. For the problem at hand these approximate boundary conditions at $r = a$ are the following.

4.1. *Kinematic Conditions*

The normal component of the velocity vector must be compatible with the velocity of the boundary surface particles across the interface (26) at $r = a$, i.e.

$$u_{1r} = \frac{\partial R}{\partial t}, \tag{46}$$

where

$$R = \epsilon_o e^{\sigma t + i(kz+m\phi)}, \tag{47}$$

then

$$A = \frac{-\rho\sigma^2}{k I_m'(x)}. \tag{48}$$

where $x(= ka)$ is the dimensionless longitudinal wavenumber.

4.2. *Electrodynamic Conditions*

1. The electric potential Ψ must be continuous across the fluid interface (26) at $r = a$

$$\Psi_1^{(1)} = \Psi_1^{(2)}. \tag{49}$$

2. The normal component of the electric displacement current must also be continuous across the perturbed interface (26) at $r = a$. That is,

$$\epsilon^{(1)}(\underline{N} \cdot \underline{E}^{(1)}) = \epsilon^{(2)}(\underline{N} \cdot \underline{E}^{(2)}), \tag{50}$$

where $\underline{E}^{(1)}$ and $\underline{E}^{(2)}$ are the total electric fields and $\underline{N}$ is the unit normal vector to the interface pointing outwards.

By using the conditions (1) and (2), we get

$$B = \frac{iE_o K_m(x)}{x\big(\epsilon^{(1)} I'_m(x) K_m(x) - \epsilon^{(1)} I_m(x) K'_m(x)\big)} \Big(x\epsilon^{(1)} - \epsilon^{(2)}(m\beta + \alpha x)\Big), \tag{51}$$

$$C = \frac{iE_o I_m(x)}{x\big(\epsilon^{(1)} I'_m(x) K_m(x) - \epsilon^{(1)} I_m(x) K'_m(x)\big)} \Big(x\epsilon^{(1)} - \epsilon^{(2)}(m\beta + \alpha x)\Big), \tag{52}$$

where

$$x = ka \tag{53}$$

4.3. *The Dynamical Condition*

The normal component of the total stress tensor must also be continuous across the disturbed boundary interface (26) at $r = a$. This implies that

$$\begin{aligned} p_1 \quad + \quad & R\frac{\partial p_o}{\partial r} + \frac{1}{2}\epsilon^{(1)}\Big(2\underline{E}_o^{(1)} \cdot \underline{E}_1^{(1)} + R\frac{\partial}{\partial r}\big(\underline{E}_o^{(1)} \cdot \underline{E}_1^{(1)}\big)\Big) \\ = \quad & \frac{1}{2}\epsilon^{(2)}\Big(2\underline{E}_o^{(2)} \cdot \underline{E}_1^{(2)} + R\frac{\partial}{\partial r}\big(\underline{E}_o^{(2)} \cdot \underline{E}_1^{(2)}\big)\Big) + p_{1s}. \end{aligned} \tag{54}$$

5. Dispersion Relation

Afterwards, we will obtain the analytical expressions of the dispersion relation, by the use of the equations (1), (2), (39)-(45), (48), (51), (52) for the dynamical condition (54).

The following dispersion relation is obtained:

$$\begin{aligned} \sigma^2 \quad = \quad & \frac{T}{\rho a^3}\left(1 - m^2 - x^2\right)\frac{xI'_m(x)}{I_m(x)} - \\ & \frac{E_o^2}{\rho a^2}\left[\frac{\big(x\epsilon^{(1)} - \epsilon^{(2)}(m\beta + \alpha x)\big) I'_m(x) K_m(x)}{\epsilon^{(1)} I'_m(x) K_m(x) - \epsilon^{(2)} I_m(x) K'(x)} - \frac{\epsilon^{(2)}\beta^2 x I'_m(x)}{I_m(x)}\right] \end{aligned} \tag{55}$$

then we can write this equation as follows

$$\begin{aligned} \frac{\sigma^2}{T/\rho a^3} \quad = \quad & \left(1 - m^2 - x^2\right)\frac{xI'_m(x)}{I_m(x)} - \\ & \left(\frac{E_o}{E_s}\right)^2\left[\frac{\big(x\epsilon^{(1)} - \epsilon^{(2)}(m\beta + \alpha x)\big) I'_m(x) K_m(x)}{\epsilon^{(1)} I'_m(x) K_m(x) - \epsilon^{(2)} I_m(x) K'(x)} \frac{\epsilon^{(2)}\beta^2 x I'_m(x)}{I_m(x)}\right] \end{aligned} \tag{56}$$

where

$$E_s = \sqrt{\frac{a}{T}} \tag{57}$$

For the rotationally axisymmetric mode $m = 0$, the dispersion relation (56) reduces to

$$\begin{aligned}\frac{\sigma^2}{T/\rho a^3} &= \frac{xI_1(x)}{I_o(x)}(1-x^2) - \\ &\Big(\frac{E_o}{E_s}\Big)^2\left[\frac{x^2\big(\epsilon^{(1)}-\alpha\epsilon^{(2)}\big)^2 I_1(x)K_o(x)}{\epsilon^{(1)}I_1(x)K_o(x)+\epsilon^{(2)}I_o(x)K_1(x)} - \epsilon^{(2)}\beta^2\frac{xI_1(x)}{I_o(x)}\right],\end{aligned}$$

and if non-axisymmetric $m = 1$, the dispersion relation (56) reduces to

$$\begin{aligned}\frac{\sigma^2}{T/\rho a^3} &= \frac{-x^3(I_o(x)+I_2(x))}{2I_1(x)} - \\ &\Big(\frac{E_o}{E_s}\Big)^2\left[\frac{\big(x\epsilon^{(1)}-\epsilon^{(2)}(\beta+\alpha x)\big)^2 K_1(x)\big(I_o(x)+I_2(x)\big)}{\epsilon^{(1)}K_1(x)\big(I_o(x)+I_2(x)\big)+\epsilon^{(2)}I_1(x)\big(K_o(x)+K_2(x)\big)} - \right. \\ &\left.\epsilon^{(2)}\beta^2\frac{x\big(I_o(x)+I_2(x)\big)}{2I_1(x)}\right]. \qquad (58)\end{aligned}$$

6. Stability Analysis

Equation (56) is the desired dispersion relation of a capillary stationary fluid cylinder of radius a under the influence of an axial electric field. The electric field surrounding the cylinder has the component $(\frac{\beta E_o a}{r})$ in the $\phi-$direction and (αE_o) in the $z-$direction where β and α are constants. The dispersion relation (56) relates the growth rate σ or rather the oscillation frequency (if σ is imaginary, $\sigma = i\omega$) with $\big(\sqrt{\frac{T}{\rho a^3}}\big)$ as a unit of time, the wave number x, the density ρ, the constants β and α, the electric field intensity E_o, the modified Bessel functions $I_m(x)$, $K_m(x)$, $I'_m(x)$ and $K'_m(x)$, the dielectric constant $\epsilon^{(1)}$ and $\epsilon^{(2)}$.

The influence of the uniform electric field only represented upon neglecting the capillary force influence by putting $T = 0$. The dispersion relation (56) degenerates to

$$\sigma^2 = -\Big(\frac{E_o^2}{\rho a^2}\Big)\left[\frac{\big(x\epsilon^{(1)}-\epsilon^{(2)}(m\beta+\alpha x)\big)I'_m(x)K_m(x)}{\epsilon^{(1)}I'_m(x)K_m(x)-\epsilon^{(2)}I_m(x)K'(x)} - \frac{\epsilon^{(2)}\beta^2 I'_m(x)}{xI_m(x)}\right]. \qquad (59)$$

From the equation (60) with $\beta = 0$ it is clear that σ^2 is negative for all values of α and x and $\epsilon^{(1)} \geq \epsilon^{(2)}$, since the following inequality

$$\frac{x^2\big(\epsilon^{(1)}-\alpha\epsilon^{(2)}\big)^2 I'_m(x)K_m(x)}{\epsilon^{(1)}I'_m(x)K_m(x)+\epsilon^{(2)}I_m(x)K'_m(x)} > 0 \qquad (60)$$

is satisfied for all values of α and x, see refs. [10] and [11] thus, the stability or instability depends on the values of β, as can be shown numerically.

If we take $E_o = 0$ and the effect of a capillary force only, the dispersion relation (56) becomes

$$\sigma^2 = \frac{T}{\rho a^3}(1-m^2-x^2)\frac{xI'_m(x)}{I_m(x)}. \qquad (61)$$

This dispersion relation is deduced by Chandrasekhar [4].

Moreover, the general analytical results are verified numerically by using the dispersion relation (56) for all short wavelengths and long wavelengths in which the nondimensional wave number x is taken to be $0 \leq x \leq 3.0$ and the corresponding values of σ and/or ω in the normal unit $\left(\sqrt{\frac{T}{\rho a^3}}\right)$ are obtained where σ is the growth rate of instability and ω is the oscillation frequency of the stability states. This has already been done for different values of $\beta = 0.2, 0.4, 0.6, 0.8, 1.0$ and $\alpha = 1.0, 3.0, 6.0, 8.0$ and for each value of β and α the basic electric field E_o relative to E_s is considered for the different values $\left(\frac{E_o}{E_s}\right) = 0.0, 0.5, 1.0, 1.5, 2.0$ and $(\epsilon^{(1)}, \epsilon^{(2)})$ has a definite value $(9.0, 3.0)$.

References

[1] **Cowling TG**: *Magnetohydrodynamics*, Wiley Interscience, New York, (1977).

[2] **Melcher JR**: *Field coupled surface waves*, Technology Press, Cambridge, Massachusets, (1963).

[3] **Rayleigh, JW**: *The theory of sound*, Dover Publ., New York, (1945).

[4] **Chandrasekhar, S**: *Hydrodynamic and Hydromagnetic Stability*, Dover Publ., New York, (1981).

[5] **Chandrasekhar, S** and **Fermi, E**: Problems of gravitational stability in the presence of a magnetic field, *Astrophysical J.*, **118**, (1953), 116-141.

[6] **Mohamed AA** *et al.*, Electrohydrodynamic stability of $m = 0$ mode of a rotating jet under a periodic field, *Physica Scripta*, **31**, (1985), 193-197.

[7] **Radwan, AE**: Electrification capillary instability of a charging annular fluid jet, *Kyungpook Math. J.*, **38**, (1998), 1-12.

[8] **Radwan, AE**: Electrogravitational instability of a fluid cylinder under modulating varying electric field by utilizing the energy prinicple, *Jnanabha*, **24**, (1994), 157-175.

[9] **Elzab, SS**: Effect of axial currents on the capillary instability of a gas-core liquid jet, *J. Phys. Soc. Japan*, **60**, (1991), 1233-1236.

[10] **Elzab, SS**: Electrodynamic stability of a self-gravitating fluid cylinder, *Physica Scripta*, **52**, (1995), 56-59.

[11] **Abramowitz, M** and **Stgun, I**: *Handbook of mathematical functions*, Dover Publisher, New Yorl, (1970).

[12] **Radwan, AE** and **Elazab, SS**: Magnetogravitational stability of variable streams, *J. Phys. Soc. Japan*, **57**, (1988), 461-463.