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The Dynamics of a Coupled Three Degree of Freedom Mechanical System

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In this paper, a nonlinear coupled three degree-of-freedom autoparametric vibration system with elastic pendulum attached to the main mass is investigated numerically. Solutions for the system response are presented for specific values of the uncoupled normal frequency ratios and the energy transfer between modes of vibrations is observed. Curves of internal resonances for free vibrations and external resonances for vertical exciting force are shown. In this type system one mode of vibration may excite or damp another one, and except different kinds of periodic vibration there may also appear chaotic vibration. Various techniques, including chaos techniques such as bifurcation diagrams and: time histories, phase plane portraits, power spectral densities, Poincarè maps and exponents of Lyapunov, are used in the identification of the responses. These bifurcation diagrams show many sudden qualitative changes, that is, many bifurcations in the chaotic attractor as well as in the periodic orbits. The results show that the system can exhibit various types of motion, from periodic to quasi-periodic and to chaotic, and is sensitive to small changes of the system parameters.

Keywords: pendulum, parametric vibrations, chaos.

1. Introduction

In this work the nonlinear dynamics of a three-degree-of-freedom system with elastic pendulum is studied. Dynamically systems with elements of the mathematical or physical pendulum type have important applications. If the pendulum is suspended to the flexible element, in this system may occur the autoparametric excitation as a result of inertial coupling. In the systems of this type may occur the internal resonance of a parametric type [1,2,3,4]. It is well known, that internal resonances can give rise to coupling between the responses of various modes. Similarly is in one mass system with spring pendulum (pendulum type elastic oscillator), where was observe autoparametric nonlinear coupling between the angle of the pendulum and the elongation of the spring [5,6,7,8]. Fundamental is the influence of values different parameters of the system on conditions of internal or external resonances, because the equations of the autoparametric system have coupled nonlinearities and in this type systems can occurs except steady- state also chaotic vibrations. It depends on various amplitudes of excitation, frequencies ratio and different parameters of the systems, for example various coefficients of the damping.

In the present paper the nonlinear response of three degree of freedom system, in which a elastic pendulum suspended to the flexible element is investigated

A number of research have been discussed the complicated motion that may occur chaos in nonlinear mechanical systems with external excitation. A typical example of the governing equation is Duffing's equation with damping and harmonic or parametrically excited systems were presented by Moon [9]; Bajaj and Johnson [10]; Bajaj and Tousi [11]; Szempliska-Stupnicka, [12,13]. Chaos in a nonlinear single-degree-of-freedom, parametrically excited system was observed by Szempliska-Stupnicka at all [14]. There the excitation appears as a coefficient in the equation of motion (similarly as in the autoparametric systems).

Chaos for two degree-of-freedom autoparametric system was investigated by Hatwall at all [15]. Those authors used the harmonic balance method showed that for higher excitation levels, the response was found to be chaotic. This system was next investigated, using the averaging analysis, by Bajaj at all [16] and by Benerjee at all [17]. Those authors showed bifurcation analysis and Poincaré maps of the period and chaotic solution for different various detuning of frequency. There was assumed that system is weakly nonlinear. The analysis of transitions from periodic regular motion to chaotic motion for two degree-of-freedom systems were presented by Gonzales at all [18] or by Pust and Szöllös [19], and for autoparametric system by Mustafa and Ertas [20], by Tondl [21] or by Verhulst [22] and author [23,24,25].

Many researchers studied the effect of parametric or autoparametric excitation on three-mass systems. Three mass system chain self-excited system was investigated by Tondl and Nabergoj [26]. Chaos for three degree-of-freedom autoparametric system with double pendulum was investigated by author at all [27,28]. It was shown that except different kinds periodic vibrations there may also appear chaotic vibration. There the bifurcation diagrams were used to assess the effect of changes in system parameters on the qualitative motion of the systems.

This paper describes the numerical simulation of a nonlinear two-mass autoparametric system with elastic pendulum hangs down from the flexible suspended body. It was shown that in this three degree-of freedom system one mode of vibrations may excite or damp another mode, and that in the neighbourhood internal and external resonances except different kinds periodic vibrations may appear also chaotic vibration. In this work the bifurcation diagrams for different damping parameters are constructed. When a bifurcation diagram is plotted, several phenomena can be observed: existence of a simple attractor with low period, or existence of a chaotic attractor, and various bifurcations. All these phenomena have to be verified in the phase space [29,30,31]. So in the present paper the time histories, phase plane portraits, power spectral densities, the Poincaré maps and exponents of Lyapunov also are constructed.

2. System description and equation of motion

The investigated system is shown in Fig. 1. It consists of the flexible pendulum of length l_0, rigidity k_2 and mass m hangs down from the body of mass M suspended by an element characterized by linear elasticity of rigidity $k_1$and viscous damping. It is admitted that a linear viscous damping force acts upon the body M and a damping force proportional to angular velocity applied in the hinge opposes the motion of the pendulum. The body of mass M is subjected to harmonic vertical excitation $F(t) = F_0 \cos \eta t$. This system has three degrees of freedom. As generalised coordinates are assumed the vertical displacement y of the body of mass M measured from the equilibrium position, the angle φ of deflection of the body of mass m measured from the vertical line and the elongation x of the spring.

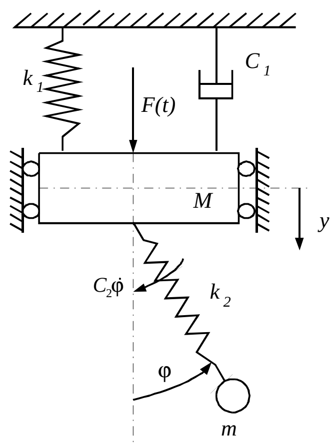

Figure 1 Schematic diagram of system

The kinetic energy E and the potential energy V of the system are

$$\begin{aligned} E &= \frac{(M+m)\dot{y}}{2} + \frac{m\dot{x}^2}{2} + \frac{m}{2}(l_0 + y_{pst} + x)^2\dot{\varphi}^2 \\ &\quad + m\dot{y}\dot{x}\cos\varphi - m\dot{y}\dot{\varphi}(l_0 + y_{pst} + x)\sin\varphi \\ V &= -(M+m)gy - mg(l_0 + y_{pst} + x)\cos\varphi + mg(l_0 + y_{pst}) + \\ &\quad \frac{k_1(y+y_{st})^2}{2} + \frac{k_2(y_{pst}+x)^2}{2} \end{aligned} \tag{1}$$

where

$$y_{st} = \frac{(M+m)g}{k_1}, \quad y_{pst} = \frac{mg}{k_2}$$

and g – acceleration of gravity.

Applying the Lagranges equations the equations of motion of the system take the following

$$\begin{aligned} &(M+m)\ddot{y} + m\ddot{x}\cos\varphi - 2m\dot{x}\dot{\phi}\sin\phi - m\ddot{\phi}\,(l_0 + y_{2st} + x)\sin\varphi \\ &-m\dot{\varphi}^2\,(l_0 + y_{2st} + x)\cos\varphi - (M+m)g + k_1(y+y_{st}) + C_1\dot{y} \quad = \quad F_0\cos\nu t \end{aligned}$$

$$\begin{aligned} m(l_0+y_{2st}+x)^2\ddot{\varphi}-m\ddot{y}\,(l_0+y_{2st}+x)\sin\varphi & \\ +2m\,(l_0+y_{2st}+x)\,\dot{\varphi}\dot{x}+mg\,(l_0+y_{2st}+x)\sin\varphi+C_2\dot{\varphi} &= 0 \\ -m\ddot{x}+m\ddot{y}\cos\varphi-m\,(l_0+y_{2st}+x)\,\varphi^2-mg\cos\varphi+k_2(y_{2st}+x) &= 0\,, \end{aligned} \tag{2}$$

By introducing the dimensionless time $\tau=\omega_1 t$ and the following definitions

$$\begin{array}{ccccc} y_1=\frac{y}{l_0} & y_{1st}=\frac{y_{st}}{l_0} & y_{2st}=\frac{y_{pst}}{l_0} & x_1=\frac{x}{l_0} & d=\frac{m}{M} \\ a=\frac{d}{1+d} & \omega_1=\frac{k_1}{M+m} & \omega_2=\frac{g}{l} & \omega_3=\frac{k_2}{m} & \beta_1=\frac{\omega_2}{\omega_1} \\ \beta_2=\frac{\omega_3}{\omega_2} & \mu=\frac{\eta}{\omega_1} & \gamma_1=\frac{C_1}{(M+m)\omega_1} & \gamma_2=\frac{C_2}{ml_0^2\omega_1} & q=\frac{F_0}{m_2 l\omega_1^2} \end{array} \tag{3}$$

We can transform (2) into

$$\begin{aligned} \ddot{y}_1 \;=\;& \frac{1}{(a-1)(1+y_{2st}+x_1)^2}\left\{a(1+y_{2st}+x_1)^2\left[(1+y_{2st}+x_1)\dot{\varphi}^2+\right.\right. \\ & \left.\beta_1^2(1+y_{2st})\cos\varphi-\beta_1^2\beta_2^2(y_{2st}+x_1)\right]\cos\varphi+\left[2(1+y_{2st}+x_1)\dot{\varphi}\dot{x}_1+\right. \\ & \left.\beta_1^2(1+y_{2st}+x_1)(1+y_{2st})\sin\varphi+\gamma_2\dot{\varphi}\right]a(1+y_{2st}+x_1)\sin\varphi- \\ & \left[2a\,\dot{x}_1\dot{\varphi}\sin\varphi+a\,\dot{\varphi}^2(1+y_{2st}+x_1)\;\cos\varphi-y_1-\gamma_1\dot{y}_1+q\;\cos\mu\tau\right] \\ & \left.(1+y_{2st}+x_1)^2\;\right\} \end{aligned}$$

$$\begin{aligned} \ddot{\varphi} \;=\;& \frac{1}{(a-1)(1+y_{2st}+x_1)^2}+\{[2(1+y_{2st}+x_1)\varphi\dot{x}_1 \\ & +\beta_1^2(1+y_{2st}+x_1)(1+y_{2st})\sin\varphi+\gamma_2\dot{\varphi}\right](1-a^2\cos\varphi)-\left[2a\dot{x}_1\dot{\varphi}\sin\varphi+\right. \\ & \left.a\dot{\varphi}^2(1+y_{2st}+x_1)\cos\varphi-y_1-\gamma_1\dot{y}_1+q\cos\mu\tau\right](1+y_{2st}+x_1)\sin\varphi+ \\ & \left[(1+y_{2st}+x_1)\dot{\varphi}^2+\beta_1^2(1+y_{2st})\cos\varphi-\beta_1^2\beta_2^2(y_{2st}+x_1)\right] \\ & a(1+y_{2st}+x_1)\cos\varphi\sin\varphi\} \end{aligned} \tag{4}$$

$$\begin{aligned} \ddot{x} \;=\;& \frac{1}{(a-1)(1+y_{2st}+x_1)^2}\left\{\left[2a\dot{x}_1\dot{\varphi}\sin\phi+a\dot{\varphi}^2(1+y_{2st}+x_1)\cos\varphi-\right.\right. \\ & \left.+y_1-\gamma_1\dot{y}_1+q\cos\mu\tau\right](1+y_{2st}+x_1)^2\cos\varphi+ \\ & \left[(1+y_{2st}+x_1)\dot{\varphi}^2+\beta_1^2(1+y_{2st})\cos\varphi-\beta_1^2\beta_2^2(y_{2st}+x_1)\right] \\ & (1+y_{2st}+x_1)^2(a\sin^2\varphi-1)-[2(1+y_{2st}+x_1)\phi\dot{x}_1+ \\ & \left.\left.\beta_1^2(1+y_{2st}+x_1)(1+y_{2st})\sin\varphi-\gamma_2\dot{\varphi}\right]a(1+y_{2st}+x_1)\sin\varphi\cos\varphi\right\} \end{aligned}$$

3. Numerical results

Equations (4) were solved numerically by using the Runge-Kutta procedure with a variable step length. Calculations have been done for different values of the system parameters. Exemplary results the energy transfer are presented in Fig. 2 for the initial values $y_1(0)=0.05$ and for the following parameters of the system: $d=0.02$, $\beta_1=0.5$, $\beta_2=2$. As it can be seen from the presented diagrams, the amplitudes grow and diminish periodically. In Fig. 3 are demonstrated the internal resonant curves for the initial values $y_1(0) = 0.05$ and $\varphi(0) = 0.005^o$. The external resonant curves of the mass M and of the pendulum for no damping effect for the conditions of an autoparametric internal resonances are shown in Fig. 4.

When the excitation acts vertically on the body of mass M (Fig. 4), one can observe three resonant amplitudes y_1, x and φ for frequencies: $\mu = 0.93$, $\mu = 1$ and $\mu = 1.08$.

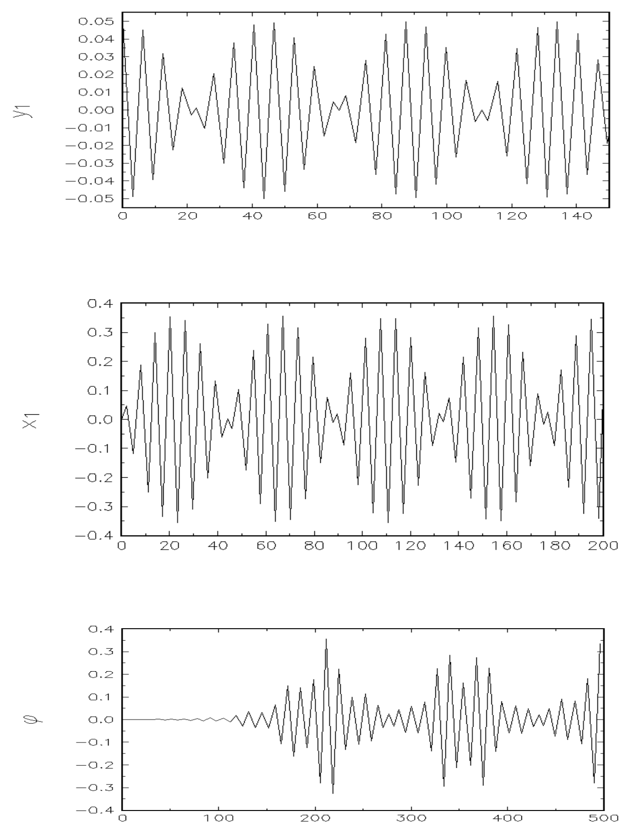

Figure 2 Time history corresponding to coordinates y_1, x_1 and φ for: $d = 0.02$, $\beta_1 = 0.5$, $\beta_2 = 2$, $q = 0$, $\gamma_1 = \gamma_2 = 0$, $y_1(0) = 0.05$

Near the internal and external resonances depending on a selection of physical system parameters the amplitudes of vibrations of coupled bodies may be related differently, motions: y_1 and pendulum are periodic or quasiperiodic vibrations, but sometimes the motions are chaotic. For characterizing a irregular chaotic response forms a transition zone between one and another type of regular steady resonant motion, in the present paper the bifurcation diagrams for different damping pa-

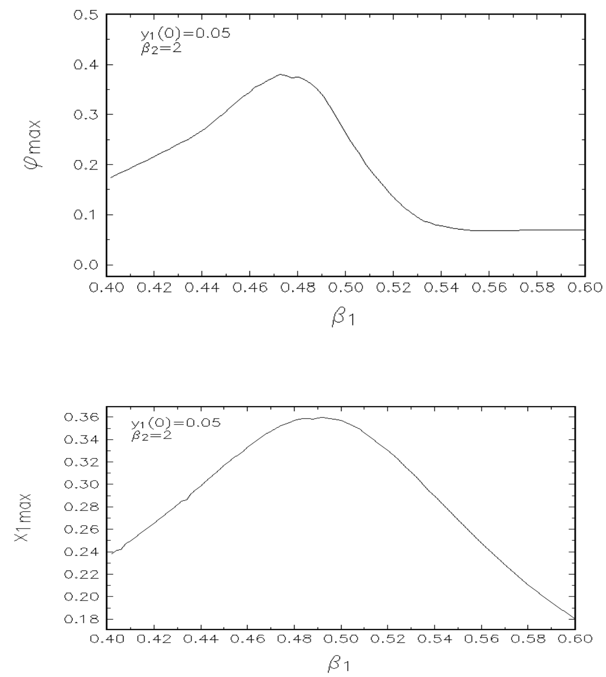

Figure 3 Amplitudes of the pendulum versus frequency ratio β_1 (internal resonant curves) for $y_1(0) = 0.05$ and $d = 0.02$, $\beta_2 = 2$, $\gamma_1 = \gamma_2 = 0$, $q = 0$

rameters are constructed. Exemplary results, for small damping put on pendulum, near internal and external resonances (near principal autoparametric resonances for $\beta_1 = 0.5$, $\beta_2 = 2$ and near eternal resonance for $\mu = 1$), are presented in Fig. 5, where are showed displacements and velocities of the beam end of the pendulum versus amplitude of excitation.

As we can seen from diagrams presented in Fig. 2 in this case ($\beta = 0.45$) for small value of the excitation amplitude q ($q < 0.00034$) motions: body of mass M (displacement y_1) and pendulum(displacements x and φ) are periodic, but for greater value of parameter q character these motions may be irregular.

Even small change the parameters of the system gives different character of the body of mass M and of the pendulum motion. As can be seen from these bifurcation diagrams, several phenomena can be observed: existence of a simple or a chaotic attractor, and various bifurcations. All these phenomena have to be verified in the phase space. Next than the time histories, phase plane portraits (Poincaré maps), power spectral densities (their fast Fourier transform – FFT), and the exponents of Lyapunov are constructed. This descriptors are available to observe chaos, and to

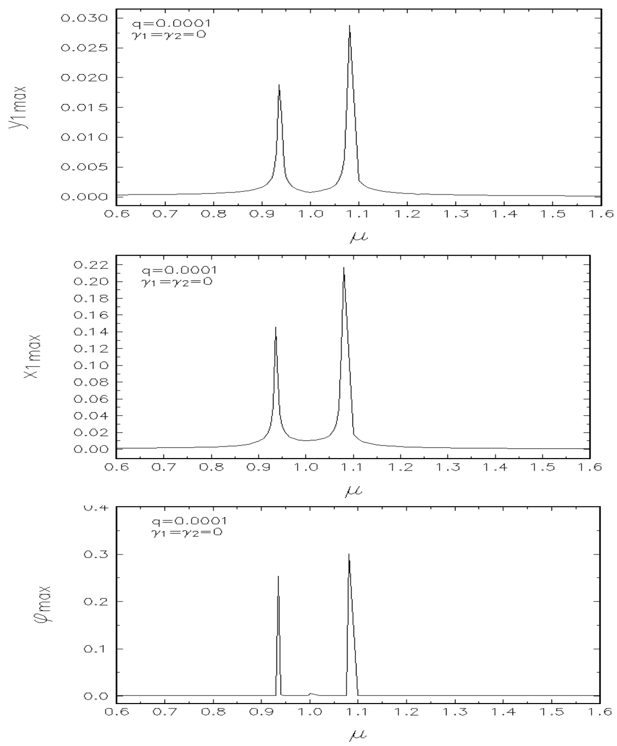

Figure 4 Resonant curves of the mass M and of the pendulum ($d = 0.02$, $\beta_1 = 0.5$, $\beta_2 = 2$, $q = 0.0001$, $\gamma_1 = \gamma_2 = 0$)

better understand it.

Exemplary results chaotic motions are presented in Fig. 6 (time histories, power spectral densities (FFT), Poincaré maps and the maximum exponents of Lyapunov corresponding to coordinate y_1 as well as to coordinates x and φ).

As can be seen from Fig. 6 this response is chaotic.We see that in this case the motion looks like irregular, the Poincarè maps trace the 'strange attractors', the frequency spectrums are continuous and the largest exponents of Lyapunov are positive.

4. Conclusions

Influence of parameters on the behaviour of the autoparametric flexible element –elastic pendulum system near the internal and external resonances frequencies is very interesting and important. In autoparametric system the energy is transferred from one degree of freedom to the others. Depending on a selection of physical system parameters the amplitudes of vibrations of both coupled bodies (and of three modes) may be related differently and were observed two types internal resonances.

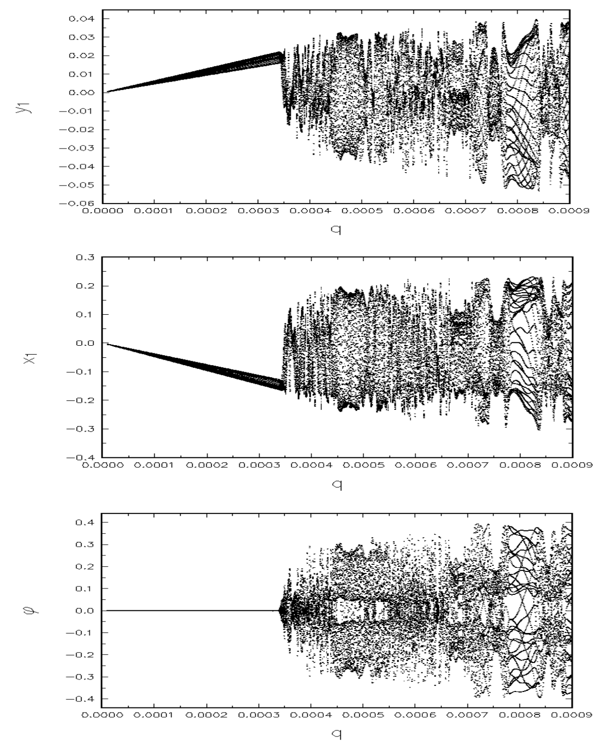

Figure 5 Bifurcation diagram for y_1, x and φ for $\beta_1 = 0.5$, $\beta_2 = 2$, $d = 0.02$, $\gamma_2 = 0.002$, $\mu_1 = 1.07$

It was also shown that near the internal and external resonances depending on a selection of physical system parameters, the amplitudes of vibrations of coupled modes may be related differently. It was observed that except different kinds periodic vibrations may appear also different kinds irregular vibrations. Nonperiodic attractors are traced by solving an initial value problem. The maximum Lyapunov exponents have been calculated in order to characterize the chaotic orbits. Sensitivity to initial conditions occurs where this exponent is positive.

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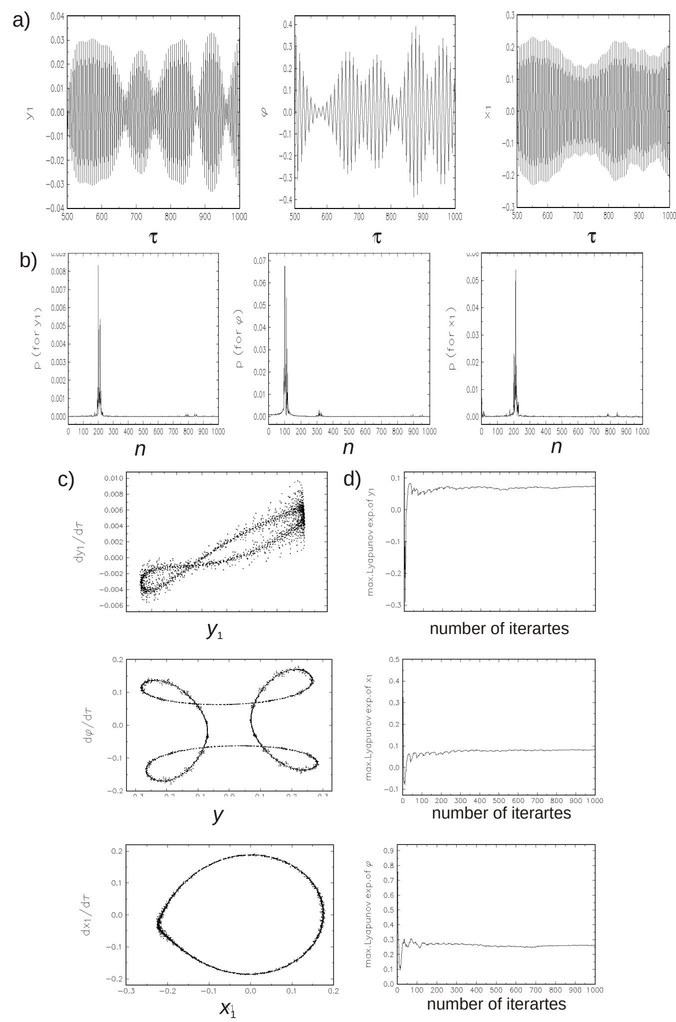

Figure 6 Time history (a) , power spectral density (FFT) (b), Poincaré map (c) and maximum Lyapunov exponent(d) corresponding to coordinates y_1 ,x_1 and φ for: $\beta_1 = 0.5$, $\beta_2 = 2$, $d = 0.02$, $\gamma_1 = 0$, $\gamma_2 = 0.0002$, $q = 0.0005$, $\mu = 1.07$

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