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Effect of Rotation in Case of 2-D Problems of the Generalized Thermoelasticity with Thermal Relaxation

Mohamed I.A. Othman
Department of Mathematics, Faculty of Education,
P.O. Box 2801, Salalah- 211, Sultanate of Oman
e-mail: m_i_othman@yahoo.com

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The mode of two-dimensional equations of generalized thermo-elasticity with one relaxation time under the effect of rotation is studied using the theory of thermo-elasticity recently proposed by Lord-Shulman. The normal mode analysis is used to obtain the exact expressions for the temperature distributions, the displacement components and thermal stresses. The resulting formulation is applied to two different concrete problems. The first concerns to the case of a heated punch moving across the surface of a semi-infinite thermo-elastic half-space subjected to appropriate boundary conditions. The second deals with a thick plate subjected to a time-dependent heat source on each face. Numerical results are given and illustrated graphically for each problem. Comparisons are made with the results predicted by the coupled theory and with the theory of generalized thermo-elasticity with one relaxation time in the absence of rotation.

Keywords: Rotation, Thermal relaxation, Thermo-elasticity, Normal mode.

1. Introduction

During the second half of the twentieth century, non-isothermal problems in the theory of elasticity have become increasingly important due to their many applications in widely diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses, reducing the strength of the aircraft structure. In the technology of modern propulsive systems, such as jet and rocket engines, the high temperatures, associated with combustion processes are the origins of severe thermal stresses. Similar phenomena are encountered in the technologies of space vehicles and missiles and in the mechanics of large steam turbines and even in shipbuilding, where, strangely enough, ship factories are often attributed to thermal stresses of moderate intensities [1].

The classical uncoupled theory of thermo-elasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms, contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot [2] introduced the theory of coupled thermo-elasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

Two generalizations to the coupled theory were introduced. The first is due to Lord and Shulman [3], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It also contains a new constant that acts as a relaxation time. Since the heat equation of this theory is of the wave type , it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations, remain the same as those for the coupled and the uncoupled theories. This theory was extended by Dhaliwal and Sherief [4] to generalize an isotropic media in the presence of heat sources. Sherief and Dhaliwal [5] solved a thermal shock problem, and Sherief [6] solved a spherically symmetric problem with a point source. Both of these problems are valid for short times. Recently, Sherief and Ezzat [7] obtained the fundamental solution for this theory that is valid for all times.

The second generalization to the coupled theory of thermo-elasticity is what is known as the theory of thermo-elasticity with two relaxation times or the theory of temperature-rate-dependent thermo-elasticity. Müller [8], in review of the thermodynamics of thermo-elastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations.

A generalization of this inequality was proposed by Green and Laws [9]. Green and Lindsay [10] obtained an explicit version of the constitutive equations. These equations were also obtained independently by Şuhubi [11]. The theory contains two constants that act as relaxation times and modifies all the equations of the coupled theory, not only the heat equation. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Şuhubi [12] studied wave propagation in finite cylinders. Ignaczak [13] studied a strong discontinuity wave and obtained a decomposition theorem of this theory [14]. Dhaliwal and Rokne solved a thermal shock problem in [15]. Ezzat [16] also obtained the fundamental solution for cylindrical region. Othman [17] studied the dependence of the modulus of elasticity on the reference temperature in two-dimensional generalized thermo-elasticity with one relaxation time.

The main objective of this work is to investigate the effect of rotation on the temperature, displacement components and stresses with one relaxation time. The resulting formulation is applied to two concrete problems. The exact expressions for temperature distribution, displacement components and thermal stresses are obtained for each problem.

2. Formulation of the problem

We shall consider an infinite isotropic, homogeneous, thermally conducting elastic medium. The medium is rotating uniformly with an angular velocity $\mathbf{\Omega} = \mathbf{\Omega n}$, where $\mathbf{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms [18]:

(i) Centripetal acceleration $\Omega \wedge (\Omega \wedge u)$ due to the time-varying motion only;

(ii) The Coriolis acceleration $2\Omega \wedge \dot{u}$.

Here, $\mathbf{u}$ is the dynamic displacement vector measured from a steady state deformed position and supposed to be small. These two terms do not appear in the equations for non-rotating media.

The fundamental equations of the generalized thermo-elasticity are:

- The constitutive law for the theory of generalized thermo-elasticity

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu\varepsilon_{ij} - \gamma (T - T_0) \delta_{ij} \,, \tag{1}$$

- The heat conduction equation

$$kT_{,ii} = \rho C_E \left(\dot{T} + \tau \ddot{T} \right) + \gamma T_0 \left(\dot{e} + \tau \ddot{e} \right) \,, \tag{2}$$

- The strain-displacement constitutive relations

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \text{ and } \varepsilon_{ii} = e = u_{i,i} \,. \tag{3}$$

The equations of motion, in the absence of body forces, are

$$\sigma_{ij,j} = \rho \left[\ddot{u}_i + \{ \Omega \wedge (\Omega \wedge u) \}_i + (2\Omega \wedge \dot{u})_i \right] \,. \tag{4}$$

where all the terms have the same significance as in [3].

Combining Eqs. (1), (3) and (3), we obtain the displacement equation of motion in the rotating frame of reference as

$$\rho \left[\ddot{u} + \{ \Omega \wedge (\Omega \wedge u) \} + (2\Omega \wedge \dot{u}) \right] = (\lambda + \mu) \nabla (\nabla . u) + \mu \nabla^2 u - \gamma \nabla T \,. \tag{5}$$

From Eqs. (1) and (3) the stress components are given by

$$\sigma_{xx} = \lambda e + 2\mu u_{,x} - \gamma (T - T_0) \,, \tag{6}$$

$$\sigma_{yy} = \lambda e + 2\mu v_{,y} - \gamma (T - T_0) \,, \tag{7}$$

$$\sigma_{xy} = \mu \left(u_{,y} + v_{,x} \right) \,, \tag{8}$$

$$\sigma_{zz} = \lambda e - \gamma (T - T_0) \,. \tag{9}$$

From Eq. (4)

$$\rho \left[\frac{\partial^2 u}{\partial t^2} - \Omega^2 u - 2\Omega \dot{v} \right] = (\lambda + \mu) \frac{\partial e}{\partial x} + \mu \nabla^2 u - \gamma \frac{\partial T}{\partial x} \,, \tag{10}$$

$$\rho\left[\frac{\partial^2 v}{\partial t^2}-\Omega^2 v+2\Omega\dot{u}\right]=(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\nabla^2 v-\gamma\frac{\partial T}{\partial y} \tag{11}$$

Now we introduce the following non-dimensional variables:

$$x_i'=c_0\eta_0 x_i',\quad u_i'=c_0\eta_0 u_i,\quad t'=c_0^2\eta_0 t,\quad \tau=c_0^2\eta_0\tau',\quad \theta=\frac{\gamma\left(T-T_0\right)}{\lambda+2\mu},$$

$$\sigma_{ij}'=\frac{\sigma_{ij}}{\mu}\quad \Omega'=\frac{\Omega}{c_0^2\eta_0}. \tag{12}$$

where the dashed quantities denote the dimensional variables.

In order to examine the effect of rotation and relaxation time on coupled elastic dilatational, shear and thermal waves, we get

$$\mathbf{\Omega}=(0,0,\Omega),$$

$$\mathbf{u}=(u(x,y,t),\quad v(x,y,t),\,0)\ ,$$

where Ω is a constant and

$$e=u_{,x}+v_{,y}\qquad \varepsilon_{xy}=\frac{1}{2}\left(u_{,y}+v_{,x}\right)\qquad \varepsilon_{xz}=\varepsilon_{y_z}=\varepsilon_{zz}=0. \tag{13}$$

In terms of the non-dimensional quantities defined in Eq. (11), the above governing equations reduce to (dropping the dashes for convenience)

$$\beta^2\left[\frac{\partial^2 u}{\partial t^2}-\Omega^2 u-2\Omega\dot{v}\right]=(\beta^2-1)\frac{\partial e}{\partial x}+\nabla^2 u-\beta^2\frac{\partial\theta}{\partial x} \tag{14}$$

$$\beta^2\left[\frac{\partial^2 v}{\partial t^2}-\Omega^2 v+2\Omega\dot{u}\right]=(\beta^2-1)\frac{\partial e}{\partial y}+\nabla^2 v-\beta^2\frac{\partial\theta}{\partial y}, \tag{15}$$

$$\nabla^2\theta=\left(\frac{\partial\theta}{\partial t}+\tau\frac{\partial^2\theta}{\partial t^2}\right)+\varepsilon(\frac{\partial e}{\partial t}+\tau\frac{\partial^2 e}{\partial t^2}). \tag{16}$$

and the components of the stress are

$$\sigma_{xx}=2u_{,x}+\left(\beta^2-2\right)e-\beta^2\theta, \tag{17}$$

$$\sigma_{yy}=(\beta^2-2)e+2v_{,y}-\beta^2\theta, \tag{18}$$

$$\sigma_{xy}=u_{,y}+v_{,x}, \tag{19}$$

$$\sigma_{zz}=(\beta^2-2)e-\beta^2\theta. \tag{20}$$

Differentiating Eq. (13) with respect to x, and Eq. (14) with respect to y, then adding, we arrive at

$$\left[\nabla^2-\frac{\partial^2}{\partial t^2}+\Omega^2\right]e=\nabla^2\theta+2\Omega\frac{\partial\zeta}{\partial t}. \tag{21}$$

Differentiating Eq. (13) with respect to y, and Eq. (14) with respect to x, then subtracting, we arrive at

$$\left[\nabla^2-\beta^2\frac{\partial^2}{\partial t^2}-\Omega^2)\right]\zeta=-2\Omega\beta^2\frac{\partial e}{\partial t}. \tag{22}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace's operator in a two-dimensional space and

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \tag{23}$$

3. Normal mode analysis

Equations (15), (20) and (21) are simplified in the usual manner by decomposing the solution of normal modes so that

$$[u,\, v,\, e,\, \zeta,\, \theta,\, \sigma_{ij}]\,(x,y,t) = \left[u^*(y),\, v^*(y),\, e^*(y),\, \zeta^*(y),\, \theta^*(y),\, \sigma^*_{ij}(y)\right] \exp(\omega t + iax)\,. \tag{24}$$

where ω is the (complex) time constant, $i = \sqrt{-1}$, a is the wave number in the x-direction and $u^*(y)$, $v^*(y)$, $e^*(y)$, $\zeta^*(y)$, $\theta^*(y)$ and $\sigma^*_{ij}(y)$ are the amplitude of the functions.

Eqs. (15), (20) and (21) after using Eq. (22) take the form

$$\left[D^2 - a^2 - \omega(1+\tau\omega)\right]\theta^*(y) = \varepsilon\omega(1+\tau\omega)e^*(y)\,, \tag{25}$$

$$\left[D^2 - a^2 - \omega^2 + \Omega^2\right]e^*(y) = (D^2 - a^2)\theta^*(y) + 2\Omega\omega\zeta^*\,, \tag{26}$$

$$\left[D^2 - a^2 - \beta^2(\omega^2 - \Omega^2)\right]\zeta^*(y) = -2\beta^2\omega\Omega e^*\,. \tag{27}$$

where $D = \frac{\partial}{\partial y}$.

Eliminating $\theta^*(y)$ and $\zeta^*(y)$ between Eqs. (23)–(25), we get the following sixth-order partial differential equation satisfied by $e^*(y)$

$$\left(D^6 - a_1 D^4 + a_2 D^2 - a_3\right)e^*(y) = 0\,. \tag{28}$$

where,

$$a_1 = 3a^2 + b_1\,, \tag{29}$$

$$a_2 = 3a^4 + 2a^2 b_1 + b_2\,, \tag{30}$$

$$a_3 = a^6 + a^4 b_1 + a^2 b_2 + b_3\,, \tag{31}$$

$$b_1 = (\beta^2+1)\omega_2 + (\varepsilon+1)\omega_1\,, \tag{32}$$

$$b_2 = \beta^2\omega_2(\omega_1 + \omega_2 + \varepsilon\omega_1) + \omega_1\omega_2 + 4\omega^2\Omega^2\beta^2\,, \tag{33}$$

$$b_3 = \beta^2\omega_1(\omega_2^2 + 4\omega^2\Omega^2)\,, \tag{34}$$

$$\omega_1 = \omega(1+\tau\omega), \quad \omega_2 = \omega^2 - \Omega^2\,. \tag{35}$$

Equation (26) can be factorized as

$$\left(D^2 - k_1^2\right)\left(D^2 - k_2^2\right)\left(D^2 - k_3^2\right)e^*(y) = 0\,. \tag{36}$$

where $k_j,\ j = 1,2,3$ are the roots of the following characteristic equation

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0\,. \tag{37}$$

The solution of Eq. (34) is given by:

$$e^*(y) = \sum_{j=1}^{3} e_j^*(y)\,. \tag{38}$$

where $e_j^*(y)$ is the solution of the equation

$$\left(D^2 - k_j^2\right) e_j^*(y) = 0 \qquad j = 1, 2, 3\,. \tag{39}$$

The solution of Eq. (37), which is bounded as $y \to \infty$, is given by

$$e_j^*(y) = R_j(a, \omega) e^{-k_j} y\,. \tag{40}$$

Substituting from Eq.(38) into the Eq. (36), we obtain:

$$e^*(y) = \sum_{j=1}^{3} R_j(a, \omega) e^{-k_j y}\,. \tag{41}$$

In a similar manner, we get

$$\theta^*(y) = \sum_{j=1}^{3} R_j'(a, \omega) e^{-k_j y}\,, \tag{42}$$

$$\zeta^*(y) = \sum_{j=1}^{3} R_j''(a, \omega) e^{-k_j y}\,. \tag{43}$$

where $R_j(a, \omega)$, $R_j'(a, \omega)$, and $R_j''(a, \omega)$ are parameters depending on a and ω.

Substituting from Eqs. (39)-(41) into Eqs. (23) and (25), we obtain

$$R_j'(a, \omega) = \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1} R_j(a, \omega), \quad j = 1, 2, 3\,, \tag{44}$$

$$R_j''(a, \omega) = \frac{-2\omega \Omega \beta^2}{k_j^2 - a^2 - \beta^2 \omega_2} R_j(a, \omega) \quad j = 1, 2, 3\,. \tag{45}$$

Substituting from Eqs. (42) and (43) into Eqs. (40) and (41), respectively, we obtain

$$\theta^*(y) = \sum_{j=1}^{3} \frac{\varepsilon \omega_1}{k_j^2 - a^2 - \omega_1} R_j(a, \omega) e^{-k_j y}\,, \tag{46}$$

$$\zeta^*(y) = \sum_{j=1}^{3} \frac{-2\omega \Omega \beta^2}{k_j^2 - a^2 - \beta^2 \omega_2} R_j(a, \omega) e^{-k_j y}\,. \tag{47}$$

Since,

$$e^* = iau^* + Dv^*\,, \tag{48}$$

$$\zeta^* = Du^* - iav^*\,. \tag{49}$$

In order to obtain the amplitude of displacement components u^* and v^*, which are bounded as $y \to \infty$, from Eqs. (39), (41), (46) and (47) we can obtain

$$u^*(y) = \sum_{j=1}^{3} \frac{1}{k_j^2 - a^2} \left[ia + \frac{2\omega\Omega\beta^2 k_j}{k_j^2 - a^2 - \beta^2\omega_2} \right] R_j(a,\omega) e^{-k_j y} \,, \tag{50}$$

$$v^*(y) = -\sum_{j=1}^{3} \frac{1}{k_j^2 - a^2} \left[k_j - \frac{2ia\omega\Omega\beta^2}{k_j^2 - a^2 - \beta^2\omega_2} \right] R_j(a,\omega) e^{-k_j y} \,. \tag{51}$$

In terms of Eq. (22), substituting from Eqs. (39), (44), (45), (48) and (49) into Eqs. (16)–(19), respectively, we obtain the stress components in the form

$$\begin{aligned} \sigma_{xx}^*(y) \;\; = \;\; & \sum_{j=1}^{3} \left\{ \left[\beta^2 - 2 - \frac{2a^2}{k_j^2 - a^2} - \frac{\varepsilon\omega_1\beta^2}{k_j^2 - a^2 - \omega_1} \right] \right. \\ & \left. + i\frac{4a\omega\Omega\beta^2 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta^2\omega_2]} \right\} R_j e^{-k_j y} \,, \end{aligned} \tag{52}$$

$$\begin{aligned} \sigma_{yy}^*(y) \;\; = \;\; & \sum_{j=1}^{3} \left\{ \frac{-4ia\omega\Omega\beta^2 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta^2\omega_2]} + \beta^2 - 2 + \frac{2k_j^2}{k_j^2 - a^2} \right. \\ - \;\; & \left. \frac{\varepsilon\omega_1\beta^2}{k_j^2 - a^2 - \omega_1} \right\} R_j e^{-k_j y} \,, \end{aligned} \tag{53}$$

$$\sigma_{xy}^*(y) = -\sum_{j=1}^{3} \left\{ \frac{2\omega\Omega\beta^2(k_j^2 + a^2)}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta^2\omega_2]} + \frac{2iak_j}{(k_j^2 - a^2)} \right\} R_j e^{-k_j y} \,, \tag{54}$$

$$\sigma_{zz}^*(y) = \sum_{j=1}^{3} \left\{ \beta^2 - 2 - \frac{\varepsilon\omega_1\beta^2}{k_j^2 - a^2 - \omega_1} \right\} R_j e^{-k_j y} \,. \tag{55}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. This assumes that all the field quantities are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Applications

4.1. *Problem I: A time-dependent heat punch across the surface of semi-infinite thermo-elastic half space [19]*

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We consider a homogeneous isotropic thermo-elastic solid occupying the region G given by

$$G = \{(x,y,z) \,|\, -\infty < x < \infty \,, \quad 0 \geq y, \quad -\infty < z < \infty\} \,.$$

The constants R_1, R_2 and R_3 have to be chosen such that the boundary conditions on the surface $y = 0$ take the form

$$\theta(x,y,t) = n(x,t) \quad \text{on} \quad y = 0 \,, \tag{56}$$

$$\sigma_{yy}(x,y,t) = P(x,t) \quad y = 0\,, \tag{57}$$

$$\sigma_{xy}(x,y,t) = 0 \quad \text{on} \quad y = 0\,, \tag{58}$$

where n, P are given functions of x and t.

Eqs. (56)–(58) in the normal mode form together with Eqs. (46), (53) and (54) respectively, give

$$L_1 R_1 + L_2 R_2 + L_3 R_3 = n^*(a,\omega)\,, \tag{59}$$

$$M_1 R_1 + M_2 R_2 + M_3 L_3 = P^*(a,\omega)\,, \tag{60}$$

$$N_1 R_1 + N_2 R_2 + N_3 R_3 = 0\,. \tag{61}$$

Eqs. (59)–(61) can be solved for the three unknowns R_1, R_2 and R_3 one obtains

$$R_1 = \frac{1}{\Delta}\left[(\lambda_1\Delta_1 + \lambda_2\Delta_2) + i(\lambda_2\Delta_1 - \lambda_1\Delta_2)\right]\,, \tag{62}$$

$$R_2 = \frac{1}{\Delta}\left[(\lambda_3\Delta_1 + \lambda_4\Delta_2) + i(\lambda_4\Delta_1 - \lambda_3\Delta_2)\right]\,, \tag{63}$$

$$R_3 = \frac{1}{\Delta}\left[(\lambda_5\Delta_1 + \lambda_6\Delta_2) + i(\lambda_6\Delta_1 - \lambda_5\Delta_2)\right]\,. \tag{64}$$

$$L_j = \frac{\varepsilon\omega_1}{\alpha_j}, \qquad j = 1,2,3\,, \tag{65}$$

$$\alpha_j = \left[k_j^2 - a^2 - \omega_1\right]\,, \quad \beta_j = \left[k_j^2 - a^2 - \beta^2\omega_2\right]\,, \quad j = 1,2,3\,, \tag{66}$$

$$M_j = (\alpha_{j1} - i\beta_{j1})\,, \quad j = 1,2,3\,, \tag{67}$$

$$N_j = (\alpha_{j2} + i\beta_{j2})\,, \quad j = 1,2,3\,, \tag{68}$$

$$\alpha_{j1} = \left[\beta^2 - 2 + \frac{2k_j^2}{k_j^2 - a^2} - \frac{\varepsilon\omega_1\beta^2}{\alpha_j}\right]\,, \quad j = 1,2,3\,, \tag{69}$$

$$\beta_{j1} = \frac{4a\omega\Omega\beta^2 k_j}{(k_j^2 - a^2)\beta_j}\,, \qquad j = 1,2,3\,, \tag{70}$$

$$\alpha_{j2} = \frac{2\omega\Omega\beta^2(k_j^2 + a^2)}{(k_j^2 - a^2)\beta_j}\,, \quad j = 1,2,3 \tag{71}$$

$$\beta_{j2} = \frac{2ak_j}{k_j^2 - a^2}\,, \quad j = 1,2,3\,, \tag{72}$$

$$\lambda_1 = n^*(\alpha_{21}\alpha_{32} + \beta_{21}\beta_{32} - \alpha_{31}\alpha_{22} - \beta_{31}\beta_{22}) - P^*(L_2\alpha_{32} - L_3\alpha_{22})\,, \tag{73}$$

$$\lambda_2 = n^*(\alpha_{21}\beta_{32} - \alpha_{32}\beta_{21} + \alpha_{22}\beta_{31} - \alpha_{31}\beta_{22}) - P^*(L_2\beta_{32} - L_3\beta_{22})\,, \tag{74}$$

$$\lambda_3 = P^*(L_1\alpha_{32} - L_3\alpha_{12}) - n^*(\alpha_{11}\alpha_{32} + \beta_{11}\beta_{32} - \alpha_{31}\alpha_{12} - \beta_{31}\beta_{12})\,, \tag{75}$$

$$\lambda_4 = P^*(L_1\beta_{32} - L_2\beta_{12}) - n^*(\alpha_{11}\beta_{32} - \alpha_{32}\beta_{11} + \alpha_{12}\beta_{31} - \alpha_{31}\beta_{12})\,, \tag{76}$$

$$\lambda_5 = n^*(\alpha_{11}\alpha_{22} + \beta_{11}\beta_{22} - \alpha_{21}\alpha_{12} - \beta_{21}\beta_{12}) - P^*(L_1\alpha_{22} - L_2\alpha_{12})\,, \tag{77}$$

$$\lambda_6 = n^*(\alpha_{11}\beta_{22} - \alpha_{22}\beta_{11} - \alpha_{21}\beta_{12} + \alpha_{12}\beta_{21}) - P^*(L_1\beta_{22} - L_2\beta_{12})\,, \tag{78}$$

$$\Delta_1 = L_1\left(\alpha_{21}\alpha_{32}+\beta_{21}\beta_{32}-\alpha_{31}\alpha_{22}-\beta_{31}\beta_{22}\right)-L_2(\alpha_{11}\alpha_{32}-\beta_{11}\beta_{32} -\alpha_{31}\alpha_{12}-\beta_{31}\beta_{12})+L_3(\alpha_{11}\alpha_{22}+\beta_{11}\beta_{22}-\alpha_{21}\alpha_{12}-\beta_{21}\beta_{12})\,, \quad (79)$$

$$\Delta_2 = L_1\left(\alpha_{21}\beta_{32}-\alpha_{32}\beta_{21}+\alpha_{22}\beta_{31}-\alpha_{31}\beta_{22}\right)-L_2(\alpha_{11}\beta_{32}-\alpha_{32}\beta_{11} +\alpha_{12}\beta_{31}-\alpha_{31}\beta_{12})+L_3(\alpha_{11}\beta_{22}+\alpha_{22}\beta_{11}-\alpha_{21}\beta_{12}+\alpha_{12}\beta_{21})\,, \quad (80)$$

$$\Delta = \Delta_1^2+\Delta_2^2\,. \quad (81)$$

4.2. *Problem II: A plate subjected to time-dependent heat sources on both sides [20]*

We shall consider a homogeneous isotropic thermo-elastic infinite conductivity thick flat plate of a finite thickness $2L$ occupying the region G^* given by:

$$G^* = \{(x,y,z)\,|-\infty<x<\infty\,,\quad -L\le y\le L\,,\quad -\infty<z<\infty\}$$

with the middle surface of the plate coinciding with the plane $y=0$.

The boundary conditions of the problem are taken as:

(i) The thermal boundary condition

$$q_n+h_0\theta_0=r(x,t)\quad \text{on}\quad y=\pm L\,, \quad (82)$$

where q_n denotes the normal component of the heat flux vector, h_0 is Biot's number and $r(x,t)$ represents the intensity of the applied heat sources. We now make use of the generalized Fourier's law of heat conduction in the non-dimensional form, namely,

$$q_n+\tau\frac{\partial q_n}{\partial t}=-\frac{\partial\theta}{\partial n} \quad (83)$$

Eq. (83) in the normal mode form

$$q_n^* = -\frac{1}{1+\tau\omega}\frac{\partial\theta^*}{\partial n}\,. \quad (84)$$

Combining Eqs. (46), (83) and (84) we arrive at

$$S_1R_1+S_2R_2+S_3R_3=(1+\tau\omega)r^*\,. \quad (85)$$

(ii) The normal and tangential stress components are zero on both surfaces of the plate; thus,

$$\sigma_{yy}=0\quad \text{on}\quad y=\pm L\,, \quad (86)$$

$$\sigma_{xy}=0\quad \text{on}\quad y=\pm L\,. \quad (87)$$

Equations (86) and (87) in the normal mode form together with Eqs. (53) and (55) respectively give:

$$M_1R_1\cosh(k_1L)+M_2R_2\cosh(k_2L)+M_3R_3\cosh(k_3L)=0\,, \quad (88)$$

$$N_1R_1\sinh(k_1L)+N_2R_2\sinh(k_2L)+N_3R_3\sinh(k_3L)=0\,. \tag{89}$$

Equations (85), (88) and (89) can be solved for the three unknowns R_1, R_2 and R_3.

$$R_1=\frac{(1+\tau\omega)r^*}{\varepsilon\omega_1\Delta^*\cosh(k_1L)}\left[(\lambda_7\Delta_3+\lambda_8\Delta_4)+i(\lambda_8\Delta_3-\lambda_7\Delta_4)\right]\,, \tag{90}$$

$$R_2=-\frac{(1+\tau\omega)r^*}{\Delta^*\cosh(k_1L)}\left[(\lambda_9\Delta_3+\lambda_{10}\Delta_4)+i(\lambda_{10}\Delta_3-\lambda_9\Delta_4)\right]\,, \tag{91}$$

$$R_3=\frac{(1+\tau\omega)r^*}{\Delta^*\cosh(k_1L)}\left[(\lambda_{11}\Delta_3+\lambda_{12}\Delta_4)+i(\lambda_{12}\Delta_3-\lambda_{11}\Delta_4)\right]\,, \tag{92}$$

where

$$S_j=\frac{\varepsilon\omega_1}{\alpha_j}\left[-k_j\sinh(k_jL)+h_0(1+\tau\omega)\cosh(k_jL)\right]\,,\quad j=1,2,3\,, \tag{93}$$

$$\lambda_7=(\alpha_{21}\alpha_{32}+\beta_{21}\beta_{32})\tanh(k_3L)-(\alpha_{31}\alpha_{22}+\beta_{31}\beta_{22})\tanh(k_2L)\,, \tag{94}$$

$$\lambda_8=(\alpha_{21}\beta_{32}-\alpha_{32}\beta_{21})\tanh(k_3L)-(\alpha_{31}\beta_{22}-\alpha_{22}\beta_{31})\tanh(k_2L)\,, \tag{95}$$

$$\begin{aligned}\lambda_9 \;=\;& \left[\frac{k_3\alpha_{22}}{\alpha_3}-\frac{k_2\alpha_{32}}{\alpha_2}\right]\tanh(k_2L)\tanh(k_3L)\\ &+h_0(1+\tau\omega)\left[\frac{\alpha_{32}}{\alpha_2}\tanh(k_3L)-\frac{\alpha_{22}}{\alpha_3}\tanh(k_2L)\right]\,,\end{aligned} \tag{96}$$

$$\begin{aligned}\lambda_{10} \;=\;& \left[\frac{k_3\beta_{22}}{\alpha_3}-\frac{k_2\beta_{32}}{\alpha_2}\right]\tanh(k_2L)\tanh(k_3L)\\ &+h_0(1+\tau\omega)\left[\frac{\beta_{32}}{\alpha_2}\tanh(k_3L)-\frac{\beta_{22}}{\alpha_3}\tanh(k_2L)\right]\,,\end{aligned} \tag{97}$$

$$\lambda_{11}=\frac{k_3\alpha_{21}}{\alpha_3}\tanh(k_3L)-\frac{k_2\alpha_{31}}{\alpha_2}\tanh(k_2L)+h_0\,(1+\tau\omega)\left(\frac{\alpha_{31}}{\alpha_2}-\frac{\alpha_{21}}{\alpha_3}\right)\,, \tag{98}$$

$$\lambda_{12}=\frac{k_2\beta_{31}}{\alpha_2}\tanh(k_2L)-\frac{k_3\beta_{21}}{\alpha_3}\tanh(k_3L)+h_0\,(1+\tau\omega)\left(\frac{\beta_{21}}{\alpha_1}-\frac{\beta_{31}}{\alpha_2}\right)\,, \tag{99}$$

$$\begin{aligned}\Delta_3 \;=\;& \frac{\lambda_1}{\alpha_1}\left[-k_1\tanh(k_1L)+h_0(1+\tau\omega)\right]-\alpha_{11}\lambda_3+\alpha_{12}\lambda_5\tanh(k_1L)\\ &-\beta_{11}\lambda_4-\beta_{12}\lambda_6\tanh(k_1L)\,,\end{aligned} \tag{100}$$

$$\begin{aligned}\Delta_4 \;=\;& \frac{\lambda_2}{\alpha_1}\left[-k_1\tanh(k_1L)+h_0(1+\tau\omega)\right]-\alpha_{11}\lambda_4+\alpha_{12}\lambda_6\tanh(k_1L)\\ &-\beta_{11}\lambda_3-\beta_{12}\lambda_5\tanh(k_1L)\,,\end{aligned} \tag{101}$$

$$\Delta^*=\Delta_3^2+\Delta_4^2\,. \tag{102}$$

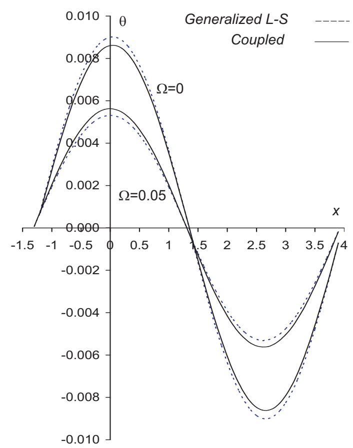

Figure 1 Temperature distribution θ for $y = 6$ of Problem I

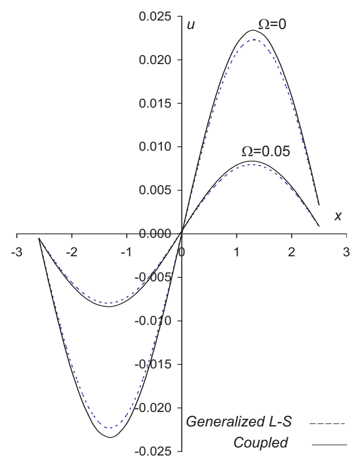

Figure 2 Displacement distribution u for $y = 6$ of Problem I

5. Numerical results

The copper material was chosen for the purpose of numerical evaluations. Since we have $\omega = \omega_0 + i\zeta$, where i is imaginary unit, $e^{\omega t} = e^{\omega_0 t}(\cos\zeta t + i\sin\zeta t)$ and for small values of time, we can take $\omega = \omega_0$ (real). The numerical constants of the problems were taken as: $\varepsilon = 0.0168$, $\beta = 3.5$, $\rho = 8954$, $\tau = 0.05$, $n^* = 100$, $P^* = 10$, $h_0 = 50$, $r^* = 10$, $a = 0.5$, $\omega = 0.5$. The computations were carried out for a value of time $t = 0.3$. The numerical technique, outlined above, was used for the real part of the thermal temperature θ distribution, the displacement distribution u and the stress distribution σ_{xx} for each problem, for problem I on the plane $y = 6$ and for problem II on plane $y = 2$, where $L = 4$ and on the middle plane $y = 0$ for two different values of $\Omega = 0$ and $\Omega = 0.01$. The results are shown in Figs. 1–9.

The graph shows the four curves predicted by different theories of thermoelasticity. In these figures, the solid lines represent the solution corresponding to

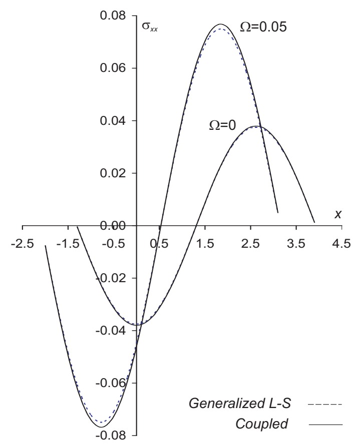

Figure 3 The distribution of stress components σ_{xx} for $y = 6$ of Problem I

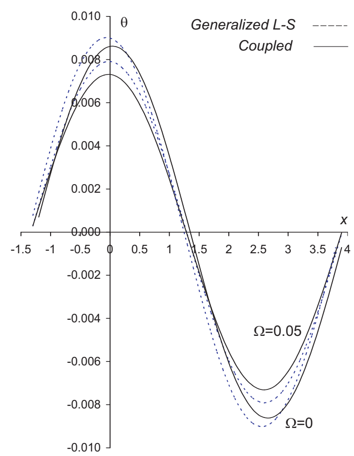

Figure 4 Temperature distribution θ on the surface for the Problem II

using the Coupled theory (CD) of heat conduction ($\tau = 0$) the dashed lines represent the solution for Lord-Shulman's theory ($\tau = 0.05$). It can be seen from these figures that the rotation acts to decrease the magnitude of the real part of the temperature, displacement and the stress component.

We notice also, that results for the temperature, the components of displacement and stress distributions when the relaxation time is appeared in the heat equation are distinctly different from those the relaxation time is not mentioned in the heat equation. This due to the fact that thermal waves in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. This demonstrates clearly the difference between the coupled and the generalized theories of thermo-elasticity.

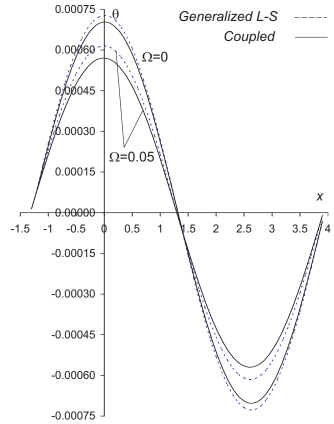

Figure 5 Temperature distribution θ on the middle plane for the Problem II

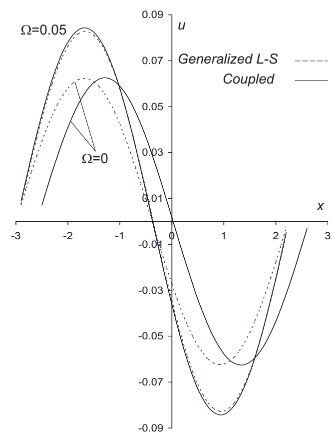

Figure 6 The horizontal displacement distribution u on the surface for the Problem I

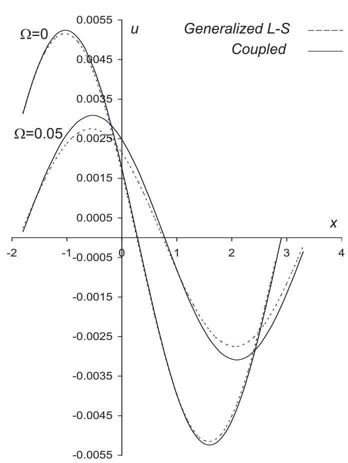

Figure 7 The horizontal displacement distribution u on the surface for the Problem II

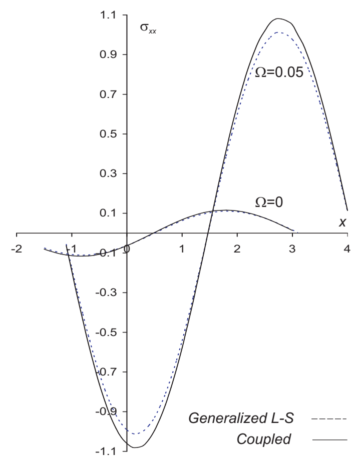

Figure 8 Stress distribution σ_{xx} on the surface for the Problem I

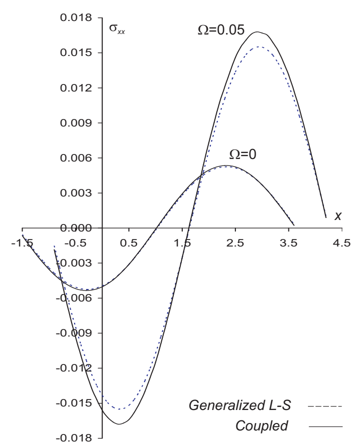

Figure 9 Stress distribution σ_{xx} on the middle plane for the Problem II

6. Concluding remarks

Due to the complicated nature of the governing equations for generalized thermo-elasticity, with thermal relaxation, few attempts have been made to solve problems in this field [21]; these attempts utilized an approximate method that is valid only for a specific range of some parameters.

In this work the method of normal mode analysis is introduced in the field of thermo-elasticity and applied to two specific problems in which the temperature, displacement and stress are coupled. This method gives exact solutions without any assumed restrictions on temperature, displacement and stress distributions.

The normal mode analysis is applied to a wide range of problems in different branches [17, 22]. It can be applied to boundary-layer problems, which are described by the linearized Navier-Stokes equations in hydrodynamic [23, 25].

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Nomenclature

λ, μ	Lame's constants
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_0	reference temperature chosen so that: $\left\lvert \frac{T-T_0}{T_0} \right\rvert \ll 1$
σ_{ij}	components of stress tensor
ε_{ij}	components of strain tensor
u_i	components of displacement vector
Ω	the rotation
k	thermal conductivity
c_0^2	$\frac{\lambda+2\mu}{\rho}$
c_2	$\sqrt{\frac{\mu}{\rho}}$ velocity of transverse waves
β^2	$\frac{c_0^2}{c_2^2}$
τ	one relaxation time
e	$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)$, the dilatation
α_t	coefficient of linear thermal expansion
γ	$(3\lambda + 2\mu)\,\alpha_t$
ε	$\gamma^2 T_0/\rho C_E\,(\lambda + 2\mu)$
η_0	$\rho C_E/k$