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Free And Forced Vibrations of a Post-Buckled Delaminated Beam-Plate Resting on a Two Parameter Elastic Foundation

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In this study, the free and forced vibrations of a delaminated beam-plate resting on a two-parameter (Pasternak type) elastic foundation are investigated. We consider a homogeneous isotropic clamped beam-plate compressed axially and subject to an external time dependent force. In order to study the vibration of the delaminated beam-plate relative to a post-buckled state we should solve the static deformations. This has been accomplished by the successive substitution method.The free and forced vibration analysis are then performed.Results are represented in both graphical and tabulated forms and shows that the natural frequencies are affected by the change in the delamination length, depth and location and also affected by the change in the elastic foundation parameters.

Keywords: Vibration, postbuckling, delaminated beam plate, elastic foundation.

1. Introduction

Composite materials have many advantages over a very wide range and especially in aerodynamics because they are high strength and have a light weight. The increasing usage of composite materials requires better understanding of their structural behavior and failure modes. Delamination (interlayer cracking) is one of the most common failure modes of laminated composite materials, and can be caused by defects or impact loading.

The problem of the growth of a general delamination has been done by Chai [1] by determining the energy release rate by differentiating the total energy with respect to the delamination length using a numerical method. Wang *et. al.* [2] studied the Free vibration of delaminated beams. They observed that the fundamental frequency was not visibly reduced due to the short delamination and was in agreement with the experimental results. Simitses *et. al.* [3] studied the effect of delamination of axially loaded homogeneous laminated plates and the results indicates that the buckling load could be considered as a measure of the load carrying capacity of the delaminated configuration of a certain geometry. Yin *et. al.* [4] studied the ultimate axial load capacity of a delaminated beam-plate. They found that the post-buckling axial load could be considerably greater than the buckling load, while the failure of the plate may or may not be governed by the delamination growth. Kardomateas and Schmueser[5] studied the deformation of delaminated composites under axial compression in a one-dimensional beam plate model and the critical instability load and postbuckling deflections were obtained by using the perturbation technique.

Le-Chung Shiau [6] studied the flutter of composite laminated beam-plates with delamination. He investigates theoretically the flutter characteristics of two-dimensional delaminated composite panels at high supersonic Mach numbers. Jane and Chen [7] studied the post-buckling deformation and vibration of a delaminated beam-plate having an arbitrary delamination location with respect to a post-buckled state using perturbation technique. Numerical results show that the delamination length and thickness affect the post-buckling deformation and vibration frequencies significantly. Chang and Liang [8] studied the free and forced vibration of postbuckled delaminated beam-plates,and they found that the natural frequencies and mode shapes were affected by the change in the size and location of delamination and by the magnitude of the axial compression that is applied axially to the one-dimensional model.

Hui-Shen Shen [9] studied the large deflection of composite laminated plates under the transverse and in-plane loads and resting on elastic foundation. He used a simply supported, composite laminated thin plate subjected to the combined uniform lateral pressure and compressive edge loading and resting on a two parameter (Pasternak type) elastic foundation. Patel and Ganapathi *et. al.* [10] studied the nonlinear free flexural vibrations/post-buckling analysis of laminated orthotropic beams/columns on a two parameter elastic foundation. Numerical results are obtained for both othotropic and cross-ply laminated beams with simply supported boundary conditions. Brandinelli *et al.* [11] studied the free vibration of through-thickness reinforced delaminated beams. A model based on the theory of bending of beams is formulated. The natural frequencies of a delaminated cantilever beam are predicted for different lengths of the delamination. The governing force equilibrium equations are solved by the separation of variables technique. The mode shapes and post-buckling deflections are obtained.

2. Formulation of the problem

Figure 1 shows a typical beam-plate model of unit width containing an arbitrary, parallel delamination at depth h_3 from the top surface of the beam-plate. The beam-plate is clamped and compressed by an axial load p at the two edges in the initial state and the plate is resting on a two parameter (Pasternak type) elastic foundation. The length of the single delamination is a and the left tip of the delamination is located at length l_1 from the left edge of the beam plate.

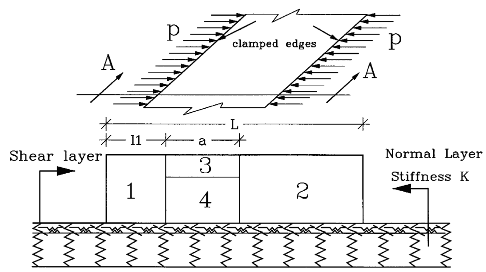

Figure 1 One-dimensional model – sec (A–A) (plate geometry)

The plate will be divided into four segments designated from 1 to 4. The interaction between the upper sub-plate and the lower one can be neglected. The governing equations can be written in the following operator form:

$$\begin{aligned} L_i(\bar{w}_i) &= \ f(x,t) \quad i=1,2,3 \quad \text{(a)} \\ L_4(\bar{w}_4) &= \ 0 \qquad\qquad\qquad\quad\ \text{(b)} \end{aligned} \tag{1}$$

where the operator L_i is given by:

$$L_i(\ ) = D_i \frac{\partial^4}{\partial \bar{x}^4} + (p_i - 2R_i)\, \frac{\partial^2}{\partial \bar{x}^2} + m_i \frac{\partial^2}{\partial t^2} + k_i\,.$$

where D_i is the bending stiffness equal to $Eh_i^3/[12(1-\nu^2)]$, t is the time, $f(x,t)$ is the external force per unit area, p_i is the axial load per unit width of the i-th segment, m_i is the mass per unit length of the i-th segment, k and R are the elastic spring constant and the stiffness of the shear layer respectively. In the previous equations $\mathbf{p_1} = \mathbf{p_2} = \mathbf{p}$ and $\mathbf{p_3} + \mathbf{p_4} = \mathbf{p}$ where $\mathbf{p}$ is greater than the buckling load of the delaminated beam plate. These equations must satisfy all boundary conditions.

The clamped boundary conditions are:

$$w_{\ 1}(0,t) \quad = \quad w_2(L,t) = 0 \tag{2}$$

$$w_1'(0,t) \quad = \quad w_2'(L,t) + 0 \tag{3}$$

The continuity of deflection at both delaminated tips can be written as:

$$w_1(l_1,t) \quad = \quad w_3(l_1,t) = w_4(l_1,t) \tag{4}$$

$$w_2(l_2,t) \quad = \quad w_3(l_2,t) = w_4(l_2,t) \tag{5}$$

The continuity of rotation at both delaminated tips can be written as:

$$w_1'(l_1,t) \quad = \quad w_3'(l_1,t) = w_4'(l_1,t) \tag{6}$$

$$w_2'(l_2,t) \quad = \quad w_3'(l_2,t) = w_4'(l_2,t) \tag{7}$$

Bending moments $\bar{M}_i$ and shear forces V_i for different segments can be balanced at both delaminated tips to give:

$$-\bar{M}_1(l_1,t)+\bar{M}_3(l_1,t)+\bar{M}_4(l_1,t)-\frac{p_4h_3}{2}+\frac{p_3h_4}{2} = 0\,, \tag{8}$$

$$-\bar{M}_2(l_2,t)+\bar{M}_3(l_2,t)+\bar{M}_4(l_2,t)-\frac{p_4h_3}{2}+\frac{p_3h_4}{2} = 0\,, \tag{9}$$

$$V_1(l_1,t)-V_3(l_1,t)-V_4(l_1,t) = 0\,, \tag{10}$$

$$V_2(l_2,t)-V_3(l_2,t)-V_4(l_2,t) = 0\,. \tag{11}$$

Equations (10) and (11) should be adopted to become homogeneous by using the following transformation:

$$\bar{w}_i(x,t)=\bar{u}_i(x,t)+a_i\bar{x}^3+b_i\bar{x}^2+c_i\bar{x}+d_i i=1,2,3,4\,. \tag{12}$$

Substituting equation (12) into equations (2) to (11) we can get:

$$d_1=0\,, \tag{13a}$$

$$c_1=0\,, \tag{13b}$$

$$a_2l_2^3+b_2l_2^2+c_2l_2+d_2=0\,, \tag{13c}$$

$$3a_2l_2^2+2b_2l_2+c_2=0\,, \tag{13d}$$

$$a_1l_1^3+b_1l_1^2+c_1l_1+d_1=a_3l_1^3+b_3l_1^2+c_3l_1+d_3\,, \tag{13e}$$

$$a_3l_1^3+b_3l_1^2+c_3l_1+d_3=a_4l_1^3+b_4l_1^2+c_4l_1+d_4\,, \tag{13f}$$

$$a_3l_2^3+b_3l_2^2+c_3l_2+d_3=a_4l_2^3+b_4l_2^2+c_4l_2+d_4\,, \tag{13g}$$

$$a_2l_2^3+b_2l_2^2+c_2l_2+d_2=a_3l_2^3+b_3l_2^2+c_3l_2+d_3\,, \tag{13h}$$

$$3a_1l_1^2+2b_1l_1+c_1=3a_3l_1^2+2b_3l_1+c_3\,, \tag{13i}$$

$$3a_3l_1^2+2b_3l_1+c_3=3a_4l_1^2+2b_4l_1+c_4\,, \tag{13j}$$

$$3a_3l_2^2+2b_3l_2+c_3=3a_4l_2^2+2b_4l_2+c_4\,, \tag{13k}$$

$$3a_2l_2^2+2b_2l_2+c_2=3a_3l_2^2+2b_3l_2+c_3\,, \tag{13l}$$

$$-D_1(6a_1l_1+2b_1)+D_3(6a_3l_1+2b_3)+D_4(6a_4l_1+2b_4)-\frac{p_4h_3-p_3h_4}{2}=0\,, \tag{13m}$$

$$-D_2(6a_2l_2+2b_2)+D_3(6a_3l_2+2b_3)+D_4(6a_4l_2+2b_4)-\frac{p_4h_3-p_3h_4}{2}=0\,, \tag{13n}$$

$$D_1(6a_1)-D_3(6a_3)-D_4(6a_4)=0\,, \tag{13o}$$

$$D_2(6a_2)-D_3(6a_3)-D_4(6a_4)=0\,. \tag{13p}$$

Equations (13a) through (13p) provides 16 linear equations in 16 variables a_i, b_i, c_i and d_i, $i=1,2,3,4$. These equations can be written in the matrix form as

$$[\ \mathrm{M}\ ]\ \{\ \mathrm{X}\ \}\ =\ \{\ \mathrm{b}\ \} \tag{14}$$

where:

$$\{\mathrm{X}\}=[a_1,b_1,c_1,d_1,a_2,b_2,c_2,d_2,a_3,b_3,c_3,d_3,a_4,b_4,c_4,d_4]^T$$

and

$$\{\mathrm{b}\} = \left[0,0,0,0,0,0,0,0,0,0,0,0, \frac{p_4h_3 - p_3h_4}{2}, \frac{p_4h_3 - p_3h_4}{2}, 0, 0\right]^T .$$

Substituting equation (12) into equations (1a) and (1b) we get,

$$L_i(\bar{u}_i) = f(x,t) - p_i(6a_i\bar{x} + 2b_i) - k_i(a_i\bar{x}^3 + b_i\bar{x}^2 + c_i\bar{x} + d_i), \quad i = 1,2,3\,, \tag{15}$$

$$L_4(\bar{u}_4) = -p_4(6a_4\bar{x} + 2b_4) - k_4(a_4\bar{x}^3 + b_4\bar{x}^2 + c_4\bar{x} + d_4)\,. \tag{16}$$

The boundary conditions given by equations (2) to (11) have been made to be homogeneous and can be written as follows:

$$\begin{aligned}
\bar{u}_1'(0,t) = \bar{u}_2'(L,t) &= 0\,, && (17)\\
\bar{u}_1(0,t) = \bar{u}_2(L,t) &= 0\,, && (18)\\
\bar{u}_1(l_1,t) = \bar{u}_3(l_1,t) &= \bar{u}_4(l_1,t)\,, && (19)\\
\bar{u}_2(l_2,t) = \bar{u}_3(l_2,t) &= \bar{u}_4(l_2,t)\,, && (20)\\
\bar{u}_1'(l_1,t) = \bar{u}_3'(l_1,t) &= \bar{u}_4'(l_1,t)\,, && (21)\\
\bar{u}_2'(l_2,t) = \bar{u}_3'(l_2,t) &= \bar{u}_4'(l_2,t)\,, && (22)\\
\bar{M}_1(l_1,t) - \bar{M}_3(l_1,t) - \bar{M}_4(l_1,t) &= 0\,, && (23)\\
\bar{M}_2(l_2,t) - \bar{M}_3(l_2,t) - \bar{M}_4(l_2,t) &= 0\,, && (24)\\
\bar{V}_1(l_1,t) - \bar{V}_3(l_1,t) - \bar{V}_4(l_1,t) &= 0\,, && (25)\\
\bar{V}_2(l_2,t) - \bar{V}_3(l_2,t) - \bar{V}_4(l_2,t) &= 0\,. && (26)
\end{aligned}$$

3. Free vibration analysis

The deflection of the beam-plate $\bar{u}_i(\bar{x},t)$ can be written as:

$$\bar{u}_i(\bar{x},t) = \bar{Y}_i(\bar{x})T_i(t)\,. \tag{27}$$

Equation (1) becomes,

$$\overline{D}_i\frac{\partial^4\bar{u}_i}{\partial\bar{x}^4} + (p_i - 2R_i)\,\frac{\partial^2\bar{u}_i}{\partial\bar{x}^2} + m_i\frac{\partial^2\bar{u}_i}{\partial t^2} + k_i\bar{u}_i = 0 \quad i = 1,2,3,4\,. \tag{28}$$

Substituting equation (27) into equation (28) yield,

$$\overline{D}_i\overline{Y}_i'''' + (p_i - 2R_i)\,\overline{Y}_i'' - \left(\overline{m}_i\lambda^2 - k_i\right)\overline{Y}_i = 0\,, \tag{29}$$

where λ represents the natural frequency of the free vibration. Define

$$Y = \frac{\overline{Y}}{h}, \quad x = \frac{\overline{x}}{L}, \quad m_i = \frac{\overline{m}_i}{m_1}, \quad D_i = \frac{\overline{D}_i}{D_1}\,. \tag{30}$$

Using equations (30) together with equation (29), yields

$$Y_i'''' + 2\sigma_iY_i'' - (M_i\omega^2 - K_i)Y_i = 0\,, \tag{31}$$

where

$$\sigma_i = \frac{p_i - R_i}{2\overline{D}_i} L^2, \quad \omega^2 = \frac{m_1 \lambda^2}{D_1} L^4, \quad M_i = \frac{m_i}{D_i}, \quad K_i = \frac{k_i L^4}{\overline{D}_i}\,. \tag{32}$$

It is clear that ω represents the non dimensional frequency of the vibration.

Assuming the mode shape as

$$Y_i = e^{s_i x} \tag{33}$$

we have

$$s_i = \pm\sqrt{-\sigma_i + \sqrt{\sigma_i^2 + (M_i\omega^2 - K_i)}} \pm \sqrt{\sigma_i + \sqrt{\sigma_i^2 + (M_i\omega^2 - K_i)}}\,j\,, \tag{34}$$

where j is the square root of -1 and we can write the values of s in the following form

$$s_i = \pm s_{1i} \pm s_{21} \tag{35}$$

and the mode shapes can be expressed in the form

$$Y_i(x) = c_{1i}\cosh(s_{1i}x) + c_{2i}\sinh(s_{1i}x) + c_{3i}\cos(s_{2i}x) + c_{4i}\sin(s_{2i}x)\,. \tag{36}$$

Substituting the mode shape into the boundary conditions, we can obtain a system of 16 linear homogeneous algebraic equations in the 16 variables c_{ij}, $i, j = 1, 2, 3, 4$, and may be expressed in the following compacted form

$$[\mathrm{N}]\{\mathrm{C}\} = 0\,, \tag{37}$$

where

$$\{\mathrm{C}\} = [c_{11}\ c_{21}\ c_{31}\ c_{41}\ c_{12}\ c_{22}\ c_{32}\ c_{42}\ c_{13}\ c_{23}\ c_{33}\ c_{43}\ c_{14}\ c_{24}\ c_{34}\ c_{44}]^T \tag{38}$$

the necessary and sufficient condition for the existence of non trivial solution is that the determinant of [N] must be vanished, i.e.

$$\det(\mathrm{N}) = 0\,. \tag{39}$$

The characteristic equation (39) gives the natural frequencies ω_j and mode shapes $Y_j(x)$ of the delaminated beam plate.

4. Forced vibration analysis

In order to perform the forced vibration analysis it is necessary to prove the orthogonality of $\bar{Y}_j(x)$.

Let us recall equation (29) which can be written as

$$\left(\bar{m}_i\lambda_j^2 - k_i\right)\bar{Y}_{ij} = \overline{D}_i\bar{Y}_{ij}'''' + (p_i - 2R_i)\,\bar{Y}_{ij}'' \tag{40}$$

Multiplying both sides by $\bar{Y}_{ik}(x)$ and integrating with respect to $\bar{x}$

$$\begin{aligned}\lambda_j^2\int_\alpha^\beta \bar{m}_i\bar{Y}_{ij}\bar{Y}_{ik}d\bar{x} \quad = \quad & \int_\alpha^\beta k_i\bar{Y}_{ij}\bar{Y}_{ik}d\bar{x} + \left[\bar{D}_i\bar{Y}_{ij}''' + (p_i - 2R_i)\,\bar{Y}_{ij}'\right]\bar{Y}_{ik}\Big|_\alpha^\beta - \\ & \bar{D}_i\bar{Y}_{ij}''\bar{Y}_{ik}'\Big|_\alpha^\beta - \int_\alpha^\beta \bar{D}_i\bar{Y}_{ij}''\bar{Y}_{ik}''d\bar{x} - \int_\alpha^\beta (p_i - 2R_i)\,\bar{Y}_{ij}'\bar{Y}_{ik}'d\bar{x}\end{aligned} \tag{41}$$

in a similar way we can obtain an expression for the k-th natural frequency

$$\lambda_k^2 \int\limits_{\alpha}^{\beta} \bar{m}_i \bar{Y}_{ij} \bar{Y}_{ik} d\bar{x} = \int\limits_{\alpha}^{\beta} k_i \bar{Y}_{ij} \bar{Y}_{ik} d\bar{x} + \left[\bar{D}_i \bar{Y}_{ij}''' + (p_i - 2R_i)\, \bar{Y}_{ij}'\right] \bar{Y}_{ik}\big|_{\alpha}^{\beta} - \bar{D}_i \bar{Y}_{ij}'' \bar{Y}_{ik}'\big|_{\alpha}^{\beta} - \int\limits_{\alpha}^{\beta} \bar{D}_i \bar{Y}_{ij}'' \bar{Y}_{ik}'' d\bar{x} - \int\limits_{\alpha}^{\beta} (p_i - 2R_i)\, \bar{Y}_{ij}' \bar{Y}_{ik}' d\bar{x}\,, \tag{42}$$

where α and β denotes the starting and end points of each segment.

Subtracting equation (42) from equation (41)

$$\left(\lambda_j^2 - \lambda_k^2\right) \int\limits_{\alpha}^{\beta} \bar{m}_i \bar{Y}_{ij} \bar{Y}_{ik} d\bar{x} = \left(\bar{Y}_{ij}''' \bar{Y}_{ik} - \bar{Y}_{ik}''' \bar{Y}_{ij}\right)|_{\alpha}^{\beta} + (p_i - 2R_i)\left(\bar{Y}_{ij}' \bar{Y}_{ik} - \bar{Y}_{ik}' \bar{Y}_{ij}\right)|_{\alpha}^{\beta} - \bar{D}_i\left(\bar{Y}_{ij}'' \bar{Y}_{ik}' - \bar{Y}_{ik}'' \bar{Y}_{ij}'\right)|_{\alpha}^{\beta} \tag{43}$$

Integrating the product of the j-th mode shape multiplied by the k-th mode shape from 0 to l, then we can obtain the following integral which contains four parts for the entire beam plate

$$\left(\lambda_j^2 - \lambda_k^2\right) \int\limits_{\alpha}^{\beta} \bar{m}_i \bar{Y}_{ij} \bar{Y}_{ik} d\bar{x} = \left(\lambda_j^2 - \lambda_k^2\right) \left[\int\limits_{0}^{l_1} \bar{m}_1 \bar{Y}_{1j} \bar{Y}_{1k} d\bar{x} + \int_{l_2}^{L} \bar{m}_2 \bar{Y}_{2j} \bar{Y}_{2k} d\bar{x} + \int\limits_{l_1}^{l_2} \bar{m}_3 \bar{Y}_{3j} \bar{Y}_{3k} d\bar{x} + \int_{l_1}^{l_2} \bar{m}_4 \bar{Y}_{4j} \bar{Y}_{4k} d\bar{x}\right] \tag{44}$$

substituting equation (44) into equation (43) and dividing the integral into three groups which are the shear forces, the axial forces and the moments.Using the boundary conditions all terms vanish, except

$$\left(\lambda_j^2 - \lambda_k^2\right) \int\limits_{0}^{l} \bar{m} \bar{Y}_j \bar{Y}_k d\bar{x} = 0\,. \tag{45}$$

If j is not equal to k, it means that the square of the j-th frequency is not equal to the square of the k-th frequency, and then we can assume that the integral of the product of j-th mode shape multiplied by the k-th mode shape is vanished, so the orthogonality condition is established.

$$\int\limits_{0}^{l} \bar{m} \bar{Y}_j \bar{Y}_k d\bar{x} = 0\,, \qquad \text{if } j \neq k\,. \tag{46}$$

For the forced vibration response,assume the displacement $\bar{u}_i(\bar{x}, t)$ as follows:

$$\bar{u}_i(\bar{x}, t) = \sum_{n=1}^{\infty} \bar{Y}_{in}(\bar{x})\, T_n(t)\,. \tag{47}$$

Substituting equation (47) into equation (15) and making use of orthogonality relation we can at last get:

$$\ddot{T}_n + \frac{\lambda_n^2 \mu_n + S_n}{\mu_n} T_n = \frac{Q_n}{\mu_n} \tag{48}$$

where the generalized mass, generalized foundation stiffness and generalized force can be respectively written as:

$$\mu_n = \int_0^{l_1} \bar{m}_1 \bar{Y}_{1n}^2 d\bar{x} + \int_{l_2}^{L} \bar{m}_2 \bar{Y}_{2n}^2 d\bar{x} + \int_{l_1}^{l_2} \bar{m}_3 \bar{Y}_{3n}^2 d\bar{x} + \int_{l_1}^{l_2} \bar{m}_4 \bar{Y}_{4n}^2 d\bar{x} \tag{49}$$

$$S_n = \int_0^{l_1} k \bar{Y}_{1n}^2 d\bar{x} + \int_{l_2}^{L} k \bar{Y}_{2n}^2 d\bar{x} + \int_{l_1}^{l_2} k \bar{Y}_{4n}^2 d\bar{x} \tag{50}$$

$$\begin{aligned} Q_n(t) \quad = \quad & \int_0^{l_1} \left[f(\bar{x},t) - (p_1 - 2R_1)(6a_1\bar{x} + 2b_1) \ -k(a_1\bar{x}^3 + b_1\bar{x}^2 + c_1\bar{x} + d_1) \right] \bar{Y}_{1n}(\bar{x}) d\bar{x} + \\ & \int_{l_2}^{L} \left[f(\bar{x},t) - (p_2 - 2R_2)(6a_2\bar{x} + 2b_2) \ -k(a_2\bar{x}^3 + b_2\bar{x}^2 + c_2\bar{x} + d_2) \right] \bar{Y}_{2n}(\bar{x}) d\bar{x} + \\ & \int_{l_1}^{l_2} \left[f(\bar{x},t) - (p_3 - 2R_3)(6a_3\bar{x} + 2b_3) \right] \bar{Y}_{3n}(\bar{x}) d\bar{x} + \\ & \int_{l_1}^{l_2} \left[-(p_3 - 2R_3)(6a_3\bar{x} + 2b_3) \ k(a_4\bar{x}^3 + b_4\bar{x}^2 + c_4\bar{x} + d_4) \right] \bar{Y}_{4n}(\bar{x}) d\bar{x} \end{aligned} \tag{51}$$

introducing the notation $\beta_n^2 = \frac{\lambda_n^2 \mu_n + S_n}{\mu_n}$, equation (48) becomes:

$$\ddot{T}_n(t) + \beta_n^2 T_n(t) = \frac{Q_n(t)}{\mu_n}\,, \tag{52}$$

The particular solution of equation (52) can be obtained as

$$T_n(t) = \frac{1}{\mu_n \beta_n} \int_0^t Q_n(t) \sin\left(\beta_n(t-\tau)\right) d\tau\,. \tag{53}$$

The homogeneous solution of equation (52) is:

$$T_n(t) = A_1 \sin(\beta_n t) + A_2 \cos(\beta_n t)\,. \tag{54}$$

The general forced vibration response is:

$$T_n(t) = A_1 \sin(\beta_n t) + A_2 \cos(\beta_n t) + \frac{1}{\mu_n \beta_n} \int\limits_0^t Q_n(\tau) \sin\left(\beta_n (t-\tau)\right) \, d\tau \,, \qquad (55)$$

Assuming only initial static deformations, yield

$$T_n(t) = T_n(0) \cos(\beta_n t) + \frac{1}{\mu_n \beta_n} \int\limits_0^t Q_n(\tau) \sin\left(\beta_n (t-\tau)\right) \, d\tau \,. \qquad (56)$$

Let us recall equation (47), multiplying by $\bar{m}_i \bar{Y}_{im}$ and integrating from 0 to l:

$$\int\limits_0^l \bar{m}_i \bar{Y}_{im}(\bar{x}) \bar{u}_i(\bar{x}, t) d\bar{x} = \int\limits_0^l \sum_{n=1}^{\infty} \bar{m}_i \bar{Y}_{im}(\bar{x}) \bar{Y}_{in}(\bar{x}) T_n(t) d\bar{x} \qquad (57)$$

due to the orthogonality property the last equation can be written as:

$$T_n(t) = \frac{1}{\mu_n} \int\limits_0^l \bar{m}_i \bar{Y}_{in}(\bar{x}) \bar{u}_i(\bar{x}, t) d\bar{x} \qquad (58)$$

and the initial generalized displacements can be expressed in the following form:

$$\begin{aligned} T_n(0) &= \frac{1}{\mu_n} \left[\int\limits_0^{l_1} \bar{m}_1 \bar{Y}_{1n}(\bar{x}) \bar{u}_1(\bar{x}, 0) d\bar{x} + \int\limits_{l_2}^{l} \bar{m}_2 \bar{Y}_{2n}(\bar{x}) \bar{u}_2(\bar{x}, 0) d\bar{x} + \right. \\ &= \int\limits_{l_2}^{l_3} \bar{m}_3 \bar{Y}_{3n}(\bar{x}) \bar{u}_3(\bar{x}, 0) d\bar{x} + \int\limits_{l_2}^{l_3} \bar{m}_4 \bar{Y}_{4n}(\bar{x}) \bar{u}_4(\bar{x}, 0) d\bar{x} \qquad (59) \end{aligned}$$

and so the general solution of the generalized displacement can be written as:

$$T_n(t) = T_n(0) \cos(\beta_n t) + \frac{1}{\mu_n \beta_n} \int\limits_0^t Q_n(\tau) \sin\left(\beta_n (t-\tau)\right) \, d\tau \qquad (60)$$

substituting equation (60) into equation (47) we can calculate the revised deformation as follows:

$$\bar{u}_i(x, t) = T_1(t) \bar{Y}_{i1} + T_2(t) \bar{Y}_{i2} + T_3(t) \bar{Y}_{i3} + \dots \,, \quad i = 1, 2, 3, 4 \,. \qquad (61)$$

The final step is to substitute (61) into equation (12), we can obtain the forced vibration solution for the post-buckled delaminated beam-plate as follows:

$$\bar{w}_i(x, t) = \bar{u}_i(x, t) + a_i \bar{x}^3 + b_i \bar{x}^2 + c_i \bar{x} + d_i \,, \quad i = 1, 2, 3, 4 \,. \qquad (62)$$

5. Results and discussion

The material Aluminum (Alloy 1100 – H14. 99.0 % Al) which has a density of 2710 kg/m^3, modulus of elasticity 70 GPa and Poisson ratio of 0.346 is chosen for the beam-plate. The beam is assumed to be 2.00 m long and 0.50 m deep and of unit width. In order to perform the parametric study certain parameters are assigned as the length ratio which is $\bar{a} = a/L$, the delamination location which is $\bar{l}_1 = l_1/L$,the depth ratio which is $\overline{h} = h_3/h$ and the load parameter σ_1.

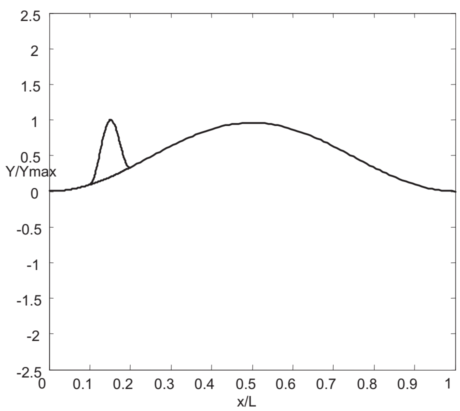

Figure 2 The first mode shape: $\bar{a} = 0.10$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

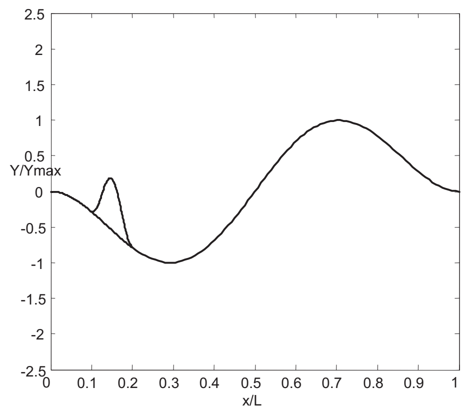

Figure 3 The second mode shape: $\bar{a} = 0.10$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

For each value of the load parameter σ_1, the lowest frequencies with respect to the post-buckled reference states are calculated numerically by searching for the successive roots of the characteristic equation (39).

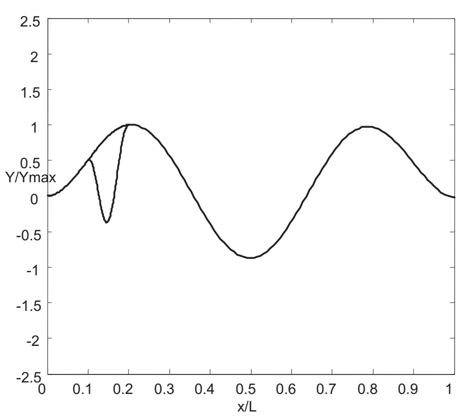

Figure 4 The third mode shape: $\bar{a} = 0.10$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

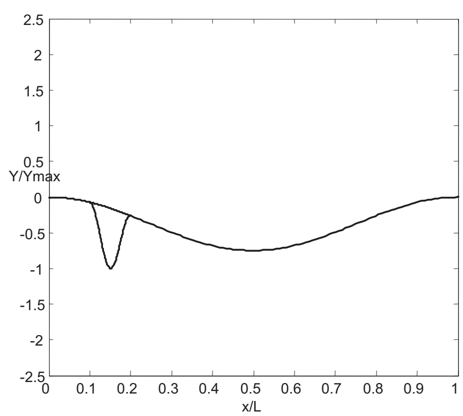

Figure 5 The first mode shape: $\bar{a} = 0.10$, $\overline{h} = 0.08$, $\overline{p} = 0.90$

For a perfect beam-plate with clamped edges and resting on a two parameter elastic foundation, the lowest frequencies can be calculated numerically by searching for the successive roots of the characteristic equation:

$$2s_{11}s_{12}\left[1-\cosh(s_{11}L)\cos(s_{21}L)\right]-\cosh(s_{11}L)\cos(s_{21}L)\left[s_{21}^{2}-s_{11}^{2}\right]=0 \qquad (63)$$

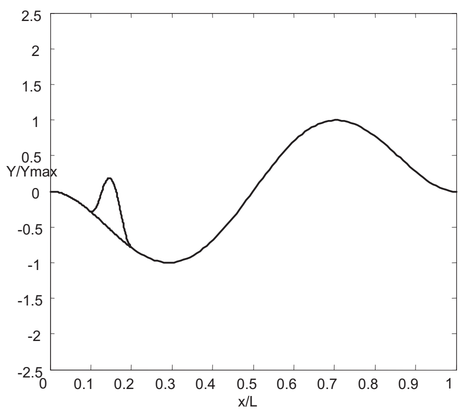

Figure 6 The second mode shape: $\overline{a}=0.10$, $\overline{h}=0.08$, $\overline{p}=0.90$

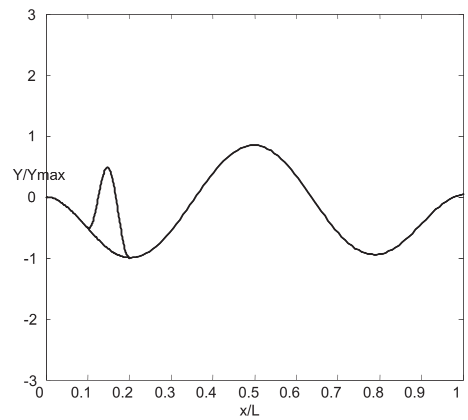

Figure 7 The third mode shape: $\overline{a}=0.10$, $\overline{h}=0.08$, $\overline{p}=0.90$

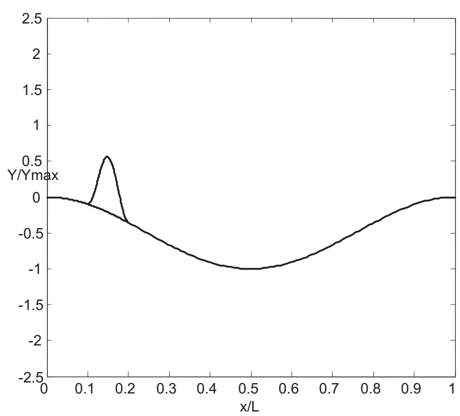

Figure 8 The thirst mode shape: $\overline{a}=0.10$, $\overline{h}=0.06$, $\overline{p}=0.60$

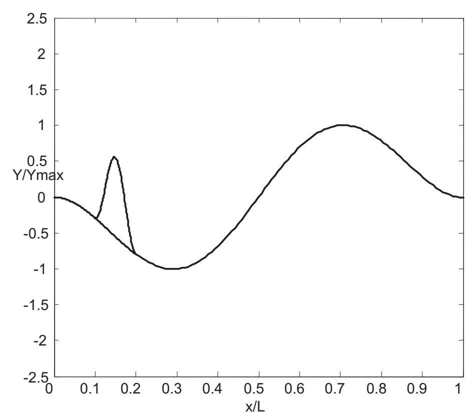

Figure 9 The second mode shape: $\overline{a}=0.10$, $\overline{h}=0.08$, $\overline{p}=0.60$

The lowest three dimensionless frequencies of the delaminated beam-plate and the perfect beam-plate for different combinations of the geometric and load parameters are listed in the Tables 1 to 4 and compared to those given by Chang & Liang [8]. By examining Table 1 we realize that as the length ratio and depth ratio are small, here $\bar{h}=0.05$ and $\bar{a}=0.06$, the natural frequency of the delaminated beam plate are much closer than other cases to the perfect beam-plate. As we increase the length ratio from 0.06 to 0.10 the frequency goes further away from that of a perfect beam plate. Also as we increase the depth ratio from 0.05 to 0.08 under the same axial load, see Table 2 and 4, the natural frequency also decreases. For two levels of the load parameter from $\sigma = 17.7629$ to $\sigma = 11.8412$, see Tables 3 and 4, we can see that as the axial load decreases the first frequency increases so drastically that we can realize that the frequency under $\bar{p} = 0.60$ is double that under $\bar{p} = 0.90$.

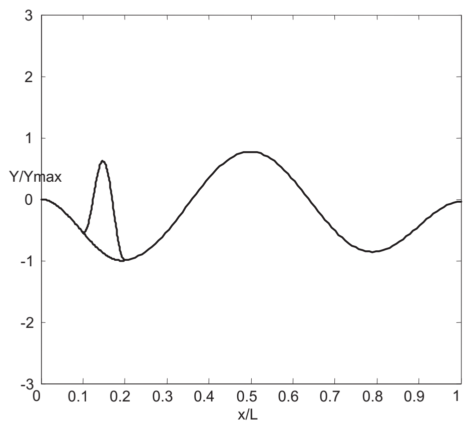

Figure 10 The third mode shape: $\bar{a} = 0.10$, $\bar{h} = 0.08$, $\bar{p} = 0.60$

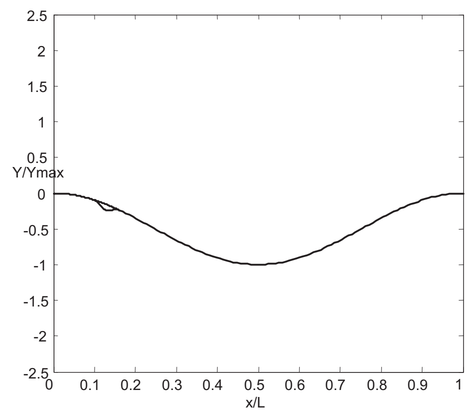

Figure 11 The first mode shape: $\bar{a} = 0.06$, $\bar{h} = 0.05$, $\bar{p} = 0.90$

Referring to Tables 1 to 4 where the results of Chang & Liang [39] are listed, the results of the present work shows good agreement with these results putting $K = 0$ and setting $R = 0$ in the load parameter σ and from these results we can conclude that the beam-plate with clamped edges an resting on a two parameter elastic foundation behaves like that with no elastic foundation but the last one frequencies are more closer to the perfect beam-plate.

To study the mode shapes of the vibrating delaminated beam-plate we should note, examining Figures 5 and 8, that the first mode shape for two levels of the axial load from $\bar{p} = 0.90$ to $\bar{p} = 0.60$ has larger amplitude for the third segment as the axial load is increased. For two levels of the thickness ratio from $\bar{h} = 0.05$ to $\bar{h} = 0.08$, see Figures 2 and 5, we can realize that the third segment has larger amplitude as we increase the thickness ratio.

Chang & Liang Ref. [39]	Perfect beam-plate	Present Work	Frequency ω
6.9444	7.2016	6.8884	ω_1
46.3862	46.4093	46.3530	ω_2
105.2696	105.3597	105.0286	ω_3

Table 1 First three non-dimensional frequencies $\bar{h} = 0.05$, $\bar{a} = 0.06$, $\sigma_1 = 17.7629$ and $K = 0.029$

For two levels of the length ratio from $\bar{a} = 0.06$ to $\bar{a} = 0.10$, see Figures 2 and 11, we can realize that the amplitude of the third segment is sensitive to the change in the length ratio.

Chang & Liang Ref. [39]	Perfect beam-plate	Present Work	Frequency ω
6.8145	7.2016	6.8452	ω_1
46.3487	46.4093	46.2740	ω_2
104.5853	105.3597	104.4165	ω_3

Table 2 First three non-dimensional frequencies $\overline{h} = 0.05$, $\bar{a} = 0.10$, $\sigma_1 = 17.7629$ and $K = 0.029$

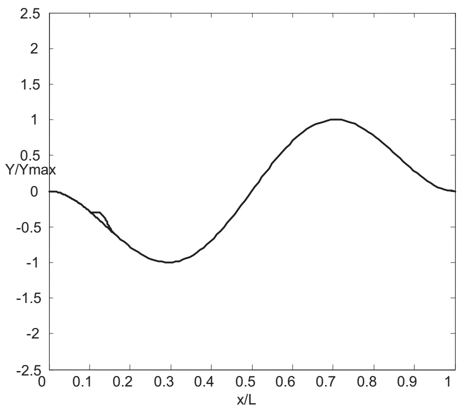

Figure 12 The second mode shape: $\overline{a} = 0.06$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

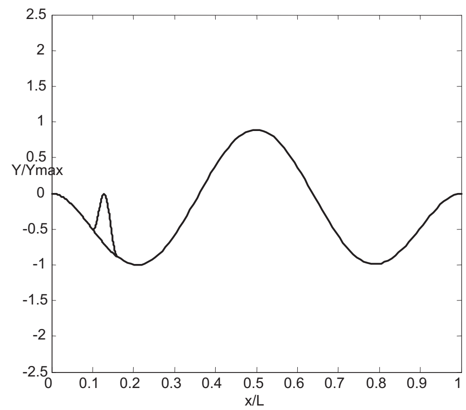

Figure 13 The third mode shape: $\overline{a} = 0.06$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

6. Effect of elastic foundation on the frequency of vibration

Table 5 contains the lowest three dimensionless frequencies setting the shear modulus $R = 0$ for three levels of the sub-grade parameter K. First we can compare these results with the results obtained by Chang & Liang [39] for the case where $R = 0$ and $K = 0$, see Table 2, and ensure the good agreement.

Upon increasing the value of K from 0 to 1.9316×10^{-6} we can notice a small increase in the frequency and if we increase the value of K to reach 0.0193 an additional increase in the frequency can be realized but its value is so small. Table 5 shows the effect of the shear modulus R, here we calculate the lowest three frequencies setting $K = 0$ and three values for R. We realize that the increase of R from 0 to 10^2 N/m does not significantly affect the first frequency, but if R is increase to reach 10^6 N/m we can realize an increase in the first frequency.

In order to discuss the forced vibration response first we fix our attention on the mid point of the second sub-plate and calculate the forced vibration response along the time, see Figures 14 to 16. In each figure the free and forced vibration

Chang & Liang Ref. [39]	Perfect beam-plate	Present Work	Frequency ω
Not available	14.2994	13.9336	ω_1
Not available	52.0309	51.8729	ω_2
Not available	110.7878	109.4199	ω_3

Table 3 First three non-dimensional frequencies $\overline{h} = 0.08$, $\bar{a} = 0.10$, $\sigma_1 = 11.8412$ and $K = 0.029$

Chang & Liang Ref. [39]	Perfect beam-plate	Present Work	Frequency ω
6.6631	7.2016	6.8150	ω_1
46.3049	46.4093	46.1900	ω_2
104.5842	105.3597	103.9118	ω_3

Table 4 First three non-dimensional frequencies $\overline{h} = 0.08$, $\bar{a} = 0.10$, $\sigma_1 = 17.7629$ and $K = 0.029$

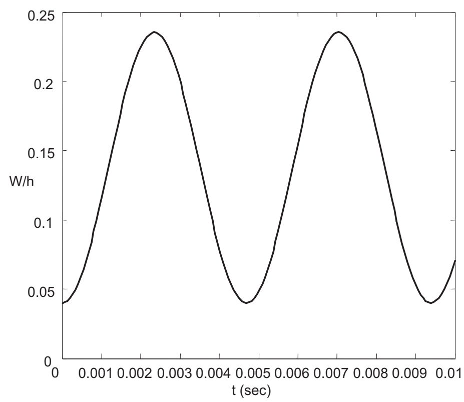

Figure 14 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.10$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

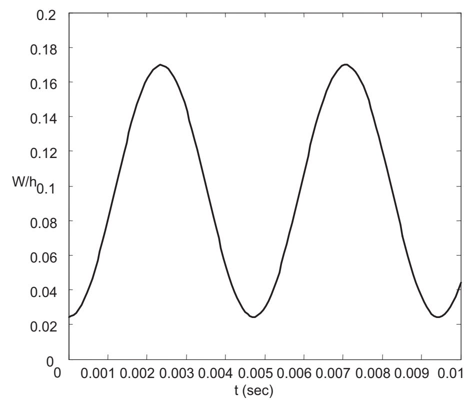

Figure 15 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.10$, $\overline{h} = 0.08$, $\overline{p} = 0.90$

responses are plotted versus time. It is clear that the response is periodic and the period depends mainly on the forced vibration frequency β_n and the deflection of the forced vibration is larger than that of the free vibration. From these figures we can realize that at $t = 0$ only the initial post-buckling deformations are provided. These figures give us the instants when the response is maximum and minimum.

Second the forced vibration response is plotted at the instant when the deflection is maximum, see Figures 17 to 19. From these figures we can realize that the behavior of the forced vibration of the delaminated beam-plate resting on a two parameter elastic foundation is the same that of the free vibration. In other words we can conclude that increasing the axial load increases the deflection, decreasing the length ratio decreases the deflection and at last increasing the depth ratio decreases the deflection.

ω_3	ω_2	ω_1	
104.8375	46.3517	6.8185	K= 0
104.4128	46.2689	6.8339	$K = 1.9316 \cdot 10^{-6}$
104.4123	46.2691	6.8353	$K = 0.0193$

Table 5 Effect of Foundation on the non-dimensional frequency $\overline{h} = 0.05$, $\bar{a} = 0.10$, $\sigma_1 = 17.7629$ (or $\bar{p} = 0.90$) and $R = 0.00$

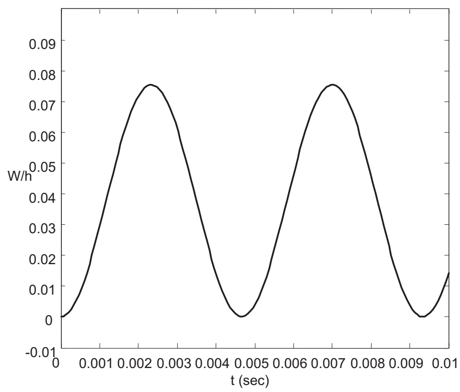

Figure 16 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.06$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

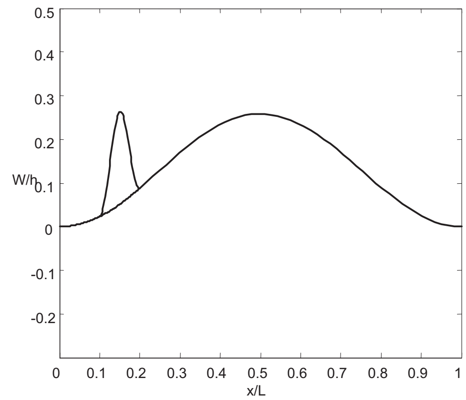

Figure 17 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.10$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

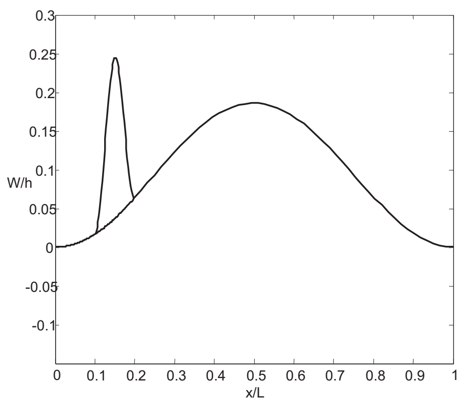

Figure 18 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.10$, $\overline{h} = 0.08$, $\overline{p} = 0.90$

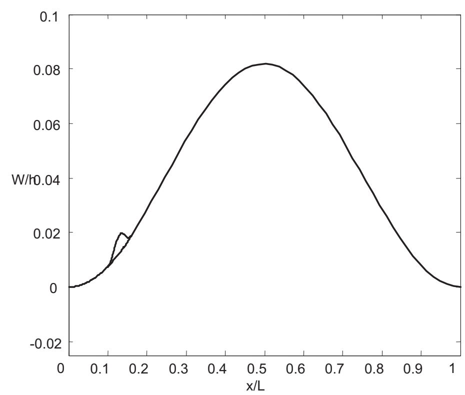

Figure 19 The forced vibrations response under the constant force 10^4 kN, $\overline{a} = 0.06$, $\overline{h} = 0.05$, $\overline{p} = 0.90$

7. Conclusions

The free and forced Vibrations of a delaminated beam-plate with clamped edges and resting on a two-parameter elastic foundation with respect to static postbuckling state have been investigated here. It was found that the mode shapes and natural frequencies of the vibrating beam plates are affected by the change in the delamination location, delamination length and delamination depth. It was also found that the natural frequencies and mode shapes are affected by the change in the foundation parameters.

Using the mode shapes and the natural frequencies we can detect the location of the delamination which may be considered as an internal imperfection in the material of the beam-plate. We used a one dimensional model in the present study but this model can be extended to become two dimensional or even three dimensional.

ω_3	ω_2	ω_1	
104.8375	46.3517	6.8339	R= 0
104.4128	46.2689	6.8339	$R = 10^2$ N/m
104.4123	46.2738	6.8431	$R = 10^6$ N/m

Table 6 Effect of Foundation on the non-dimensional frequency $\overline{h} = 0.05$, $\bar{a} = 0.10$, $\sigma_1 = 17.7629$ (or $\bar{p} = 0.90$) and $K = 0.00$

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