Mechanics and Mechanical Engineering
Vol. 9, No. 2 (2005) 111–114

Viscous Damping of Surface Waves

Ismail A. Kholief
Department of Engineering Mathematics and Physics
Cairo University
Giza, Egypt

Received (15 May 2005)
Revised (29 September 2005)
Accepted (21 November 2005)

In this article, the viscous damping of surface waves in a layer of constant depth is determined. A viscous term is included in Bernoulli equation, which agrees with Navier-Stoke's equation as proved here. The X dependence of the amplitude of the damped wave is determined by similarity and Fourier transforms.

Keywords: Viscous damping, surface waves, Bernoulli equation, Fourier transform.

1. Introduction

Surface waves are extensively treated in fluid mechanics [1–4], but no attempt has been made to include the effect of viscosity, since the solution is based on Bernoulli equation, which is classically derived for inviscid flow. It, is our intention in this paper to prove that the viscosity term can be introduced in Bernoulli equation contrary to the conception that viscous flow is not irrotational and a velocity potential cannot be defined. This can be explained in the following analysis.

2. Analysis

Consider the one-dimensional incompressible viscous flow represented by,

$$W = 0\,, \quad u = u(t, x, z)\,.$$

The continuity equation gives

$$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad u = u_0(t, z)\,,$$

and Navier-Stoke's equation will give,

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 u\,.$$

We can define the velocity potential ϕ such that,

$$u = \frac{\partial \phi}{\partial x} = u_0(t, z)\,,$$

so that i.e.,

$$\phi = x u_0(t, y) + \phi_0\,.$$

ϕ_0 is an arbitrary constant. Accordingly

$$\rho\left[\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\frac{p}{\rho} - \frac{\mu}{\rho}\frac{\partial}{\partial x}\nabla^2\phi\right] = 0\,.$$

Accordingly,

$$\frac{\partial \phi}{\partial x} + \frac{p}{\rho} - \nu\nabla^2\phi = F(t)\,,$$

which can be chosen as 0. Here, ν is the kinematics viscosity. This simple derivation can be generalized in cases when $w \neq 0$ in the existence of gravity, the term gz must be added accordingly.

2.1. *The Model Problem*

Consider the vertical layer of fluid with density ρ and viscosity μ extending along the positive direction from zero to ∞. Bernoulli equation gives for the elevation ?δ of the free surface.

$$\delta = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} - \nu\nabla^2\phi\right]\,.$$

For the vertical velocity at the free surface we have,

$$\frac{\partial \phi}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0}$$

at the free surface. Accordingly

$$\frac{\partial^2 \phi}{\partial t^2} - \nu\frac{\partial}{\partial t}\nabla^2\phi + g\frac{\partial \phi}{\partial z} = 0\,, \quad z \Rightarrow 0\,.$$

We can restrict ourselves to the case: $\frac{\partial \phi}{\partial z} = 0$.

$$(\phi_t - \nu\phi_{xx})_t = 0\,,$$

i.e.,

$$(\phi_t - \nu\phi_{xx})_t = F(x)\,,$$

when $F(x) = 0$, we have $\phi_t = \nu\phi_{xx}$. Similarity based solution gives

$$\phi = \phi_0 \operatorname{erfc} \frac{x}{2\sqrt{\nu t}}\,,$$

$$u = \frac{\partial \phi}{\partial x} = \phi_0\frac{\exp\left(-\frac{x^2}{4\nu t}\right)}{2\sqrt{\nu t}}\,.$$

Subjected to

$$\lim_{x,t\to 0} \Rightarrow 1 \quad \text{therefore } \phi_0 = u_0\,,$$

$$u' = u_0 \frac{\partial \phi}{\partial x} = \phi_0 \frac{\exp\left(-\frac{x^2}{4\nu t}\right)}{2\sqrt{\nu t}}\,.$$

For periodic flow at $x = 0$

$$u_{x=0} = u_0 \cos \Omega t = \left.\frac{\partial \phi}{\partial x}\right|_{x=0}$$

and

$$\left.\phi\right|_{x=\infty} = 0\,.$$

Taking Fourier cosine transfer in $x \to \infty$

$$\hat{\phi}_t + \nu\omega^2 \hat{\phi} = -\nu u_0 \cos \Omega t\,.$$

The solution satisfying $\left.\hat{\phi}\right|_{t=0} = 0$ is

$$\hat{\phi} = \frac{\nu u_0}{\nu^2\omega^4 + \Omega^2}\left[\nu\omega^2 \exp(-\nu\omega t) - \nu\omega^2 \cos \Omega t - \Omega \sin \Omega t\right]\,.$$

For sufficiently large νt, the exponential terms vanishes.

$$\hat{\phi} = -\frac{\nu u_0}{\nu^2\omega^4 + \Omega^2}\left[-\nu\omega^2 \cos \Omega t - \Omega \sin \Omega t\right]\,, \quad \nu \neq 0\,.$$

Inverting Fourier transform

$$\phi(x,t) = -\frac{2}{\pi} u_0 \left[\nu \cos \Omega t \int_0^\infty \frac{\omega^2 \cos \omega x d\omega}{\nu^2\omega^4 + \Omega^2} + \Omega \sin \Omega t\right]\,,$$

The integral can be evaluated by residue's [5].

$$\begin{aligned}
\int_0^\infty \frac{\omega^2 \cos \omega x d\omega}{\nu^2\omega^4 + \Omega^2} &= -\frac{pi}{8\nu^2}\sqrt{\frac{2\nu}{\Omega}} \exp\left(-\sqrt{\frac{\Omega}{2\nu}}\right)\left[\sin \sqrt{\frac{\Omega}{2\nu}} x + \cos \sqrt{\frac{\Omega}{2\nu}} x\right]\,,\\
\int_0^\infty \frac{\cos \omega x d\omega}{\nu^2\omega^4 + \Omega^2} &= \frac{pi}{8\nu^2}\left(\frac{2\nu}{\Omega}\right)^{\frac{3}{2}} \exp\left(-\sqrt{\frac{\Omega}{2\nu}}\right) \sin \sqrt{\frac{\Omega}{2\nu}} x\,.
\end{aligned}$$

3. Comment

In this proceeding analysis, we have obtained a solution over the free surface for the (x,t) amplitude of the viscous wave in the case of sufficiently large $\frac{\nu t}{x^2}$ portraying the viscous damaging. The relative amplitude is equal to unity for $\frac{\nu}{\Omega} = 8$ unit length, this is due to the approximation mode. Two waves appear and the wave functions are:

$$\cos \Omega t \left[\cos \sqrt{\frac{\Omega}{2\nu}} x + \sin \sqrt{\frac{\Omega}{2\nu}} x\right]$$

and

$$\sin \Omega t \sin \sqrt{\frac{\Omega}{2\nu}} x \, ,$$

And the damping factor is

$$\exp\left(-\sqrt{\frac{n}{2\nu}} x\right) \, .$$

References

[1] **Kundu, K** and **Cohen, I**: *Fluid Mechanics*, (2004), Elsevier Acad. Press, London, 199-400.

[2] **Blevins, RD**: *Flow Induced Vibration*, (1977), Von Nostrand, NY.

[3] **Davis, SH**: The Stability of time periodic flows, *Ann. Rev.*, (1976).

[4] **Obvemsbi, HJ** and **Feger, AA**: Transition in Oscillating Boundary Layer Flow, *Fluid Mechanics*, (1967), **29**, 93-111.

[5] **Gupta, BD**: *Mathematical Physics*, (1982), Vikas Pub. India, 528.