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Hydromagnetic Capillary Stability of Streaming Finite Hollow Cylinder with Azimuthal Field

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The hydro magnetic capillary stability of a streaming gas cylinder pervaded by an azimuthally varying magnetic field, surrounded by bounded fluid has been investigated for all kinds of perturbations. The problem is formulated, solved and upon applying appropriate boundary conditions, the dispersion relation is derived and discussed. The uniform magnetic field pervaded in the fluid is strongly stabilizing and that independently on the kind of perturbation. The azimuthally varying magnetic field in the gas region is purely destabilizing in the symmetric perturbation while it is so or otherwise in the asymmetric perturbation according to restriction. The radii ratio of the fluid-gas cylinder has a stabilizing tendency. The streaming has strong destabilizing influence for all short and long wavelengths. The capillary force is purely stabilizing in the asymmetric perturbation while it is so or not in the symmetric perturbation based on the wavelength is being shorter or longer than the gas cylinder circumference. Upon certain restrictions the destabilizing effect of the model could be suppressed and stability sets in.

Keywords: Magnetic field, capillary, stability.

1. Introduction

Here we investigate the magnetohydrodynamic capillary stability of a streaming gas cylinder penetrated by azimuthal varying magnetic field, surrounded by bounded fluid for all symmetric and asymmetric perturbations.

The response of discovering the ordinary hollow jet (a gas cylinder surrounded by infinite liquid) is being by Chandrasekhar (1981). That in discussing its stability in symmetric perturbation in comparing its stability results with its mirror case of a full liquid cylinder surrounded by a gas medium of negligible inertia (vacuum) documented by Lord Rayleigh (1945). Later on, the dispersion relation of this ordinary hollow cylinder is given by Drazin and Reid (1980). Cheng (1985) discussed the hydrodynamic capillary instability of ordinary hollow jet. In (1986) Kendall, starting the real realm of studying this model, performed very neat and accurate experiments for discussing the stability of hollow jet under the capillary force. Moreover, he (1986) attracted the attention for discussing the stability of this model for its important application that he did write about it. Following the same restrictions of Kendall (1986), the hydrodynamic stability of capillary viscous hollow jet has been analytically investigated for symmetric perturbation by Radwan and Elazab (1987). The effect of the inertia force on the capillary instability of an ordinary hollow cylinder is studied by Radwan (1990). Several works concerning the stability of this model under different acting forces have been carried out, e.g see Radwan, *et.al.* (2002).

2. Formulation of the problem

We consider a gas cylinder of radius R_0 surrounded by a finite fluid of radius qR_0 where $1 < q < \infty$. The bounded fluid is assumed to be incompressible, non-viscous and perfectly conducting. The fluid is assumed to be pervaded by homogeneous axial magnetic field

$$\underline{H}_0 = (0, 0, H_0)\,. \tag{1}$$

The gas cylinder of constant pressure P_0^g is pervaded by the azimuthally varying magnetic field

$$\underline{H}_0^g = \left(0, \frac{\beta H_0 r}{R_0}, 0\right)\,, \tag{2}$$

where β is parameter, H_0 is the intensity of the magnetic field in the fluid. The fluid is supposed to be streaming in the equilibrium state with

$$\underline{u}_0 = (0, 0, U)\,. \tag{3}$$

The components of $\underline{H}_0$, $\underline{H}_0^g$ and $\underline{u}_0$ are considered along the utilized cylindrical polar coordinates (r, φ, z) with z-axis coinciding with the axis of the coaxial cylinders. The fluid is acted by the capillary, pressure gradient, electromagnetic and inertia forces. The gas matter is acted by the capillary, electromagnetic and inertia forces, with the fluid inertia force is predominate over that of the gas matter.

Under these circumstances the magnetohydrodynamic basic equations required for present investigation are the combination of the ordinary hydrodynamic equations and those of Maxwell's concerning the electrodynamic theory. They are given by

$$\rho\left(\frac{\partial}{\partial t} + \underline{u}.\nabla\right)\underline{u} - \mu(\underline{H}.\nabla)\underline{H} = -\nabla P + \frac{\mu}{2}\nabla\left(\underline{H}.\underline{H}\right)\,, \tag{4}$$

$$\nabla.\underline{u} = 0\,, \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \underline{u}.\nabla\right)\underline{H} = \left(\underline{H}.\nabla\right)\underline{u}\,, \tag{6}$$

$$\nabla.\underline{H} = 0\,, \tag{7}$$

$$\nabla . \underline{H}^g = 0\,, \tag{8}$$

$$\nabla \wedge \underline{H}^g = 0\,, \tag{9}$$

$$P_{1s} = T(\nabla . \underline{N})\,, \tag{10}$$

$$\underline{N} = \nabla f / |\nabla f|\ . \tag{11}$$

Here ρ, $\underline{u}$ and P are the fluid density, velocity vector and kinetic pressure, $\underline{H}$ and $\underline{H}^g$ are magnetic field intensities in the fluid and gas regions, μ the magnetic field permeability, T the surface tension coefficient and $\underline{N}$ is the unit outward normal vector to the gas-fluid interface indicated as r does, where $f(r, \varphi, z, t) = 0$ is the equation of the fluid boundary surface.

3. Analysis of the problem

For small departures, from the initial state, based on the linear perturbation analysis, the variables of the problem may be expressed as

$$\begin{array}{rcll} \underline{u} & = & \underline{u}_0 + \varepsilon_0 \underline{u}_1 + \ldots & P_s = P_{os} + \varepsilon_0 P_{1s} + \ldots\,, \\ P & = & P_0 + \varepsilon_0 P_1 + \ldots & \underline{N} = \underline{N}_0 + \varepsilon_0 \underline{N} + \ldots\,, \\ H & = & \underline{H}_0 + \varepsilon_0 \underline{H}_1 + \ldots & \underline{H}^g = \underline{H}^g_0 + \varepsilon_0 \underline{H}^g_1 + \ldots\,. \end{array} \tag{12}$$

Here the symbols with subscripts 0 represent unperturbed quantities while those with index 1 are small increments due to perturbation. ε_0 is the initial amplitude of perturbation at $t = 0$ while the amplitude at time t is given by

$$\varepsilon = \varepsilon_0 \exp(\sigma t)\,. \tag{13}$$

where σ is the growth rate, note that $\varepsilon_0 = \varepsilon$ at $t = 0$.

By the use of expansions (12) for the basic equations (4)–(11) and equating the coefficients in both sides, two systems of equations are obtained, which are the perturbed and unperturbed systems.

The unperturbed system of equations is solved, and the required boundary conditions are applied at the boundary surfaces of the model. Consequently, the kinetic pressure of the fluid is given by

$$P_0 = P^g_0 - \frac{T}{R_0} + \frac{\mu H_0^2}{2}\left(\beta^2 - 1\right)\,. \tag{14}$$

It is remarkable as $\beta = 1$, that the constant gas pressure P^g_0 must be greater than the contribution of the surface tension (T/R_0) otherwise the hollow jet model collapses.

Based on the expansion (12), the radial distance of the gas-fluid interface may be written as

$$r = R_0 + \varepsilon_0 R_1 + \ldots \tag{15}$$

with

$$R_1 = \exp\left[i\left(kz + m\varphi\right) + \sigma t\right] \tag{16}$$

is the elevation of the surface wave measured from the unperturbed position where k and m are the longitudinal and azimuthal wavenumbers.

The perturbed system of equations is given by

$$\frac{\partial \underline{u}_1}{\partial t} - \left(\frac{\mu}{\rho}\right)(\underline{H}_0 . \nabla)\, \underline{H}_1 = -\nabla \Pi_1\,, \tag{17}$$

$$\rho \Pi_1 = P_1 + \frac{\mu}{2}\left(2 \underline{H}_0 . \underline{H}_1\right)\,, \tag{18}$$

$$\nabla . \underline{u}_1 = 0\,, \tag{19}$$

$$\nabla . \underline{H}_1 = 0\,, \tag{20}$$

$$\frac{\partial \underline{H}_1}{\partial t} + (\underline{u}_0 . \nabla)\, \underline{H}_1 + (\underline{u}_1 . \nabla)\, \underline{H}_0 = (\underline{H}_0 . \nabla)\, \underline{u}_1 + (\underline{H}_1 . \nabla)\, \underline{u}_0\,, \tag{21}$$

$$\nabla . \underline{H}_1^g = 0\,, \tag{22}$$

$$\nabla \wedge \underline{H}_1^g = 0\,, \tag{23}$$

$$P_{1s} = \frac{T}{R_0^2}\left(R_1 + \frac{\partial^2 R_1}{\partial \varphi^2} + R_0^2 \frac{\partial^2 R_1}{\partial z^2}\right)\,. \tag{24}$$

One has to refer here that $\rho \Pi_1 = P_1 + (\mu/2)\,(2 \underline{H}_0 . \underline{H}_1)$ is the total magneto-hydrodynamic pressure which is the sum of the kinetic pressure of the fluid and magnetic pressure.

Based on the linear perturbation technique, upon considering a single Fourier term, the perturbed terms $Q_1(r, \varphi, z, t)$ could be expressed as

$$Q_1(r, \varphi, z, t) = Q_1(r) \exp\left[i\,(kz + m\varphi) + \sigma t\right]\,, \tag{25}$$

where $Q_1(r, \varphi, z, t)$ stands for $\underline{u}_1$, Π_1, $\underline{H}_1$, $\underline{H}_1^g$ and P_1.

The linear perturbation equations (17)–(24) are combined and solved with taking into account the time-space dependence (25).

Apart from the singular solutions, we obtained

$$\underline{u}_1 = \frac{-(\sigma + ikU)}{\rho\left((\sigma + ikU)^2 + \Omega_A^2\right)} \nabla \Pi_1\,, \tag{26}$$

$$\underline{H}_1 = \frac{-H_0}{\rho\left((\sigma + ikU)^2 + \Omega_A^2\right)} \nabla\left(\frac{\partial \Pi_1}{\partial z}\right)\,, \tag{27}$$

$$\Pi_1 = \left(A I_m(kr) + B K_m(kr)\right) R_1\,, \tag{28}$$

$$\underline{H}_1^g = C \nabla I_m(kr) R_1\,, \tag{29}$$

$$P_{1s} = \frac{T}{R_0^2}(1 - m^2 - k^2 R_0^2) R_1\,. \tag{30}$$

Here $I_m(kr)$and $K_m(kr)$ are the modified Bessel functions of first and second kind of order m. The Alfven wave frequency Ω_A is defined by

$$\Omega_A = (\mu H_0^2 k^2 / \rho)^{\frac{1}{2}}\,. \tag{31}$$

A, B and C are constants of integration to be determined upon applying appropriate boundary conditions.

4. Dispersion relation

The solution of the basic equations (4)–(11) in the unperturbed and perturbed states given by (14) and (26)–(31) must satisfy appropriate boundary conditions across the fluids interfaces at $r = R_0$ and $r = qR_0$.

Under the present circumstances, these boundary conditions are given as follows.

(i) The normal component of the magnetic field across the gas-fluid interface must be continuous at $r = R_0$. Mathematically, this condition read

$$\underline{N}_0 . \underline{H}_1 + \underline{N}_1 . \underline{H}_0 = \underline{N}_0 . \underline{H}_1^g + \underline{N}_1 . \underline{H}_0^g \tag{32}$$

with

$$\underline{N}_0 = (1, 0, 0) \ , \tag{33}$$

$$\underline{N}_1 = \left(0, \frac{-im}{R_0}, -ik\right) R_1 \, . \tag{34}$$

By substituting from equations (1), (2), (27)–(29), (33) and (34) into the magnetic field continuity condition (equation (32)), we get

$$C = \frac{iH_0}{xI'_m(x)} (m\beta + \alpha . x) \, , \tag{35}$$

where

$$x = kR_0 \tag{36}$$

is the dimensionless longitudinal wavenumber.

(ii) The normal component of the fluid velocity must vanish at $r = qR_0$,

$$\underline{N}_0 . \underline{u}_1 + \underline{N}_1 . \underline{u}_0 = 0 \, . \tag{37}$$

This yields

$$AI'_m(y) + BK'_m(y) = 0 \, , \tag{38}$$

where

$$y = qx \tag{39}$$

is the dimensionless longitudinal wavenumber.

(iii) The normal component of the velocity must be compatible with the velocity of the boundary surface (15) at $r = R_0$

$$\underline{N}_0 . \underline{u}_1 + \underline{N}_1 . \underline{u}_0 = \frac{\partial r}{\partial t} \text{ at } r = R_0 \, . \tag{40}$$

This condition is given by

$$\underline{u}_{1r} = (\sigma + ikU) \, R_1 \, , \tag{41}$$

from which

$$AI'_m(x) + BK'_m(x) = -\frac{\rho_0 \left((\sigma + ikU)^2 + \Omega_A^2 \right) R_0}{x} \, . \tag{42}$$

By solving (38) and (41) for A and B, we get

$$B = -A\frac{I'_m(y)}{K'_m(y)} \tag{43}$$

and

$$A = \frac{R_0\left((\sigma + ikU)^2 + \Omega_A^2\right)K'_m(y)}{x\left[I'_m(x)K'_m(y) - I'_m(y)K'_m(x)\right]}\,. \tag{44}$$

Up to now we have determined the constants of integrations A , B and C so the solution (26)–(30) becomes more clear. We have to go one step more to apply some compatibility condition at $r = R_0$ as follows.

(iv) The normal component of the total stress tensor must be continuous across the perturbed gas–fluid interface (15) at $r = R_0$. Mathematically, this reads

$$\Pi_1 + R_1\frac{\partial \Pi_0}{\partial r} - \mu(\underline{H}_0^g.\underline{H}_1^g) - \frac{\mu}{2}\frac{\partial}{\partial r}\,(\underline{H}_0^g.\underline{H}_0^g) = P_{1s}\,. \tag{45}$$

Upon substitution in this equation about different variables with taking into account that $\Pi_0(= \frac{p_0}{\rho} + \frac{\mu}{2\rho}(\underline{H}_0.\underline{H}_0))$is constant, following dispersion relation is obtained

$$\begin{aligned}(\sigma + ikU)^2 \quad &= \quad \frac{-T}{\rho R_0^3}(1 - m^2 - x^2)\frac{xL^m_{x,y}}{L^m_y} \\ &= \quad +\frac{\mu H_0^2}{\rho R_0^2}\left\{-x^2 + \left[-\beta^2 + m^2\beta^2\frac{I_m(x)}{xI'_m(x)}\right]\right\}\frac{xL^m_{x,y}}{L^m_y}\end{aligned} \tag{46}$$

with

$$L^m_{x,y} = I'_m(x)K'_m(y) - I'_m(y)K'_m(x)\,, \tag{47}$$

$$L^m_y = I_m(x)K'_m(y) - I'_m(y)K_m(x)\,, \tag{48}$$

$$x = kR_0 \quad , \quad y = qx \text{ and } 1 < q < \infty\,. \tag{49}$$

5. General discussions

Equation (45) is the required dispersion of a gas cylinder pervaded by azimuthal varying magnetic field surrounded by bounded fluid of radius qR_0 subjected to the capillary and electromagnetic forces. It is a linear combination of dispersion relations of a bounded hollow jet subjected to the capillary force only and that one acted by the electromagnetic force. Equation (45) relates the growth rate σ with the wave numbers m, x and y, the modified Bessel functions of first and second kind, their derivatives and their combinations L^m_y and $L^m_{x,y}$; the magnetic field parameters α and β, and with the parameters T,ρ, R_0,H_0 and μ of the problem.

Since this problem is somewhat more general several reported works may recovered from the present work as limiting cases.

If we assume that $q \to \infty$, $U = 0$, $H_0 = 0$ and $m = 0$, the relation (45) reduces to

$$\sigma^2 = \frac{T}{\rho R_0^3}(1 - x^2)\frac{xK_1(x)}{K_0(x)},\ \text{where } K'_0(x) = -K_1(x)\,. \tag{50}$$

That coincides with the dispersion relation indicated by Chandrasekhar (1981).

If we assume that $H_0 = 0$, $U = 0$, $q \to \infty$, $m \geq 0$, the relation (45), yields

$$\sigma^2 = \frac{-T}{\rho R_0^3}(1 - m^2 - x^2)\frac{xK_m'(x)}{K_m(x)}\,, \tag{51}$$

which coincides with the dispersion relation given by Drazin and Reid (1980).

If $U = 0$, $m \geq 0$, $\beta = 0$ and $q \to \infty$, the relation (45) generates to

$$\sigma^2 = \frac{-T}{\rho R_0^3}(1 - m^2 - x^2)\frac{xK_m'(x)}{K_m(x)} + \frac{\mu H_0^2}{\rho R_0^2}\cdot\left\{-x^2\right\} \tag{52}$$

This relation represent the effect of electromagnetic force on the capillary instability of a hollow jet pervaded by an axial magnetic field.

If we suppose that $H_0 = 0$, the relation (45) becomes

$$(\sigma + ikU)^2 = \frac{-T}{\rho R_0^3}(1 - m^2 - x^2)\frac{xL_{x,y}^m}{L_y^m}\,. \tag{53}$$

This relation has been obtained earlier by Radwan (1990) in discussing the capillary stability of bounded hollow jet in streaming fluid.

If we assume that $T = 0$ and $U = 0$, the relation (45) gives

$$\sigma^2 = \frac{\mu H_0^2}{\rho R_0^2}\left\{-x^2 + \left[-\beta^2 + m^2\beta^2\frac{I_m(x)}{xI_m'(x)}\right]\frac{xL_{x,y}^m}{L_y^m}\right\}\,. \tag{54}$$

This recovers the dispersion relation established recently by Radwan *et.al.* (2002) as we put $\alpha = 0$ there.

6. Stability discussions

In order to discuss the stability of the present problem, we need to write down about the relations between the modified Bessel functions and their characters for different values of $x \neq 0$ and $y \neq 0$.

Consider the recurrence relations (cf. Abramowitz and Stegun 1970)

$$2I_m'(x) = I_{m-1}(x) + I_{m+1}(x)\,, \tag{55}$$

$$2K_m'(x) = -K_{m-1}(x) - K_{m+1}(x)\,. \tag{56}$$

Upon using the relations (55) and (56) and the fact for non-zero real values of x, that

$$I_m(x) > 0 \tag{57}$$

is positively definite and monotonic increasing while

$$K_m(x) > 0 \tag{58}$$

is monotonically decreasing but never negative, we may show that

$$I_m'(x) > 0 \tag{59}$$

and

$$K'_m(x) < 0\,. \tag{60}$$

Also since $y > x$ (see relation (49), we have

$$I_m(x) < I_m(y)\,, \tag{61}$$

$$K_m(x) > K_m(y)\,. \tag{62}$$

Therefore, for non-zero real values of x, we have

$$L_y^m = I_m(x)K'_m(y) - I'_m(y)K_m(x) < 0\,, \tag{63}$$

$$\begin{aligned} L_{x,y}^m &= I'_m(x)K'_m(y) - I'_m(y)K'_m(x) = \\ &-I'_m(x)\,|K'_m(y)| + I'_m(y)\,|K'_m(x)| > 0\,. \end{aligned} \tag{64}$$

Consequently, for $x \neq 0$, we obtain

$$\frac{xL_{x,y}^m}{L_y^m} < 0\,. \tag{65}$$

Now, let us return to our main task the investigation of the stability of the present model.

The capillary instability of the present model of a streaming bounded hollow cylinder may be discussed via the relation (53). From the viewpoint of the relations (53) and (54) and the inequalities (56)–(63), the sign of σ^2 in the relation (53) depends on the sign of $(1 - m^2 - x^2)$ for different values of m and x. In general, as $U = 0$, we have

$$\frac{\sigma^2}{(T/\rho R_0^3)} > 0 \qquad for \qquad m = 0 \qquad as \qquad 0 < x < 1\,, \tag{66}$$

$$\frac{\sigma^2}{(T/\rho R_0^3)} < 0 \qquad for \qquad m = 0 \qquad as \qquad 1 < x < \infty\,, \tag{67}$$

$$\frac{\sigma^2}{(T/\rho R_0^3)} = 0 \qquad for \qquad m = 0 \qquad as \qquad x = 1\,, \tag{68}$$

$$\frac{\sigma^2}{(T/\rho R_0^3)} < 0 \qquad for \qquad m \neq 0 \qquad as \qquad x \neq 0\,. \tag{69}$$

This means that the stationary ($U = 0$) capillary bounded hollow jet is purely stable in the asymmetric modes $m \neq 0$ of perturbation for all non-zero values of x and in the symmetric mode $m = 0$ as $1 < x < \infty$. It is marginally stable in $m = 0$ for $x = 1$, i.e. as the perturbed wavelength $\lambda = 2\pi R_0$. But it is unstable in small region $0 < x < 1$ for $m = 0$ i.e. as the perturbed wavelength is longer than the circumference $2\pi R_0$ of the gas cylinder.

As $U \neq 0$ it is found with increasing U values, that the unstable region $0 < x < 1$ for $m = 0$ is increasing while the stable regions ($1 \leq x < \infty$ for $m = 0$) and ($1 < x < \infty$ for $m > 0$) are decreasing. This means that the streaming has a destabilizing effect on the capillary stability of hollow jet.

It is found with increasing q-values $(1 < q < \infty)$ that the capillary stable regions are increasing while those of instability are decreasing. This means that the ratio q of the fluid-gas radii has stabilizing influence. Under certain restrictions the capillary instability could be suppressed and stability sets in.

The magneto dynamic instability of the present model of a streaming bounded hollow jet pervaded by azimuthally varying magnetic field could be identified upon investigating the dispersion relation (45) as $T = 0$. Here we have to get out the effect of different penetrating magnetic fields in the gas and fluid regions separately. Through which we could decide whether the electromagnetic forces are stabilizing or not. The effect of the axial magnetic field pervaded in the outer fluid is represented by the term $(-x^2)$ following the natural quantity $\left(\mu H_0^2/\rho R_0^2\right)$ in equation (54). It has always a stabilizing effect which is independent of the kind of perturbation. The azimuthally magnetic field pervaded into the interior gas cylinder of the hollow jet is represented by the terms including β following the natural quantity $\left(\mu H_0^2/\rho R_0^2\right)$ in equation (45) as $U = 0$, which are

$$\frac{\mu H_0^2}{\rho R_0^2}\left\{\left[-\beta^2 + m^2\beta^2\frac{I_m(x)}{xI'_m(x)}\right]\frac{xL^m_{x,y}}{L^m_y}\right\}. \tag{70}$$

In view of the inequality (65), in the symmetric mode $m = 0$, the azimuthally magnetic field effect represented by the term

$$\frac{\mu H_0^2}{\rho R_0^2}\left\{-\beta^2\frac{xL^o_{x,y}}{L^o_y}\right\} \tag{71}$$

has always a destabilizing effect for all short and long wavelengths. In the asymmetric mode of perturbation, the azimuthally magnetic field is destabilizing in the term $\left(\mu H_0^2/\rho R_0^2\right)\left(-\beta^2\right)\left(xL^m_{x,y}/L^m_y\right)$ while it is strongly stabilizing in the term $\frac{\mu H_0^2}{\rho R_0^2}\left\{m^2\beta^2\frac{I_m(x)}{xI'_m(x)}\right\}\frac{xL^m_{x,y}}{L^m_y}$ for all values of $m > 0$ for all short and long wavelengths.

To sum up, the azimuthally magnetic field pervaded in the gas region is purely destabilizing in the symmetric mode $m = 0$ while it is stabilizing or not in the asymmetric mode $m > 0$ under some restrictions.

In examining the effect of the streaming on the magneto dynamic stability of the present model of hollow jet, it is found as U values are increasing, the magneto dynamic stable regions are decreasing while the unstable regions are increasing. This means that the streaming has a strong destabilizing effect on the magneto dynamic stability of the present model of bounded hollow jet.

Also as q-values (q is the fluid-gas radii ratio) are increased it is found that the stable regions are increasing but the unstable regions are decreasing. From which we deduce that the radii ratio of the fluid-gas cylinders has strong stabilizing effect.

Under certain restrictions the magneto dynamic instability behavior of the bounded hollow jet may be suppressed and stability sets in.

7. Numerical discussions

The dispersion relation (45) has been formulated for $m = 0$ in the non-dimension form

$$
\begin{aligned}
(\sigma^* + U^*)^2 \quad = \quad & \left(1 - x^2\right) \frac{x\left[I_1(x)K_1(y) - I_1(y)K_1(x)\right]}{\left[I_0(x)K_1(y) + I_1(y)K_0(x)\right]} \\
& + \left(\frac{Ho}{H_s}\right)^2 \left\{-x^2 + \beta^2 \frac{x\left[I_1(x)\,K_1(y) - I_1(y)K_1(x)\right]}{(I_0(x)K_1(y) + I_1(y)K_0)}\right\}
\end{aligned} \tag{72}
$$

with

$$
\sigma^* = \sigma/(T/\rho R_0^3)^{1/2}, \quad U^* = ikU/(T/\rho R_0^3)^{1/2} \text{ and } H_s = (\mu R_0/T)^{1/2}
$$

where use has been made of $I_0' = I_1$, $K_0' = -K_1$ and $H_0^2 = (\mu R_0/T)$ noting that $H_s(MR_0/T)^{1/2}$ has a unit of magnetic field.

The numerical analysis has been carried out by utilizing the normalized dimensionless relation (72) in the computer simulation. This is to verify the analytical discussions and in order to identify the effect of the streaming speed U^* and also the influence of the basic magnetic field H_0. Relative to H_s on the capillary instability of the bounded hollow cylinder. The numerical investigations of the relation (71) has been carried out in the axisymmetric disturbance mode $m = 0$ for all short and long wavelengths in the range $0 \leq x \leq 3.0$. The corresponding values of the temporal amplification σ^* and the oscillation frequency ω^* (with $\omega = i\sigma$ where $i = \sqrt{-1}$ is the imaginary factor) are collected, tabulated and represented graphically. Such calculations have been elaborated for different values of

$$
U^* = 0,\ 0.1,\ 0.4,\ 0.7,\ 1.0,\ 1.3,\ 1.5 \text{ and } 2.0\,,
$$

$$
H_0/H = 0,\ 0.3,\ 0.5,\ 0.8,\ 1.0,\ 3.0,\ \text{and } 5.0\,,
$$

$$
\beta = 0.0,\ 0.25,\ 0.50,\ 0.75,\ 1.0,\ 1.5 \text{ and } 2.0\,,
$$

and

$$
q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0
$$

for regular values of x. The numerical data are plotted, see Figures 1–10 from which we may deduce the following.

For ($U^* = 0$, $(\mathrm{H}_0/H_s) = 0.3$ and $\beta = 0.25$) corresponding to

$$
q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0
$$

it is found that the unstable domains are given by

$$
\begin{aligned}
0 \leq x < 1.35300\,, \\
0 \leq x < 1.12614\,, \\
0 \leq x < 1.07631\,, \\
0 \leq x < 1.04187\,, \\
0 \leq x < 1.03322\,.
\end{aligned}
$$

While the stable domains are

$$\begin{aligned}1.35300 &\le x < \infty\,,\\ 1.12614 &\le x < \infty\,,\\ 1.07631 &\le x < \infty\,,\\ 1.04187 &\le x < \infty\,,\\ 1.03322 &\le x < \infty\,,\end{aligned}$$

where the equalities correspond to marginal stability. See Figure 1.

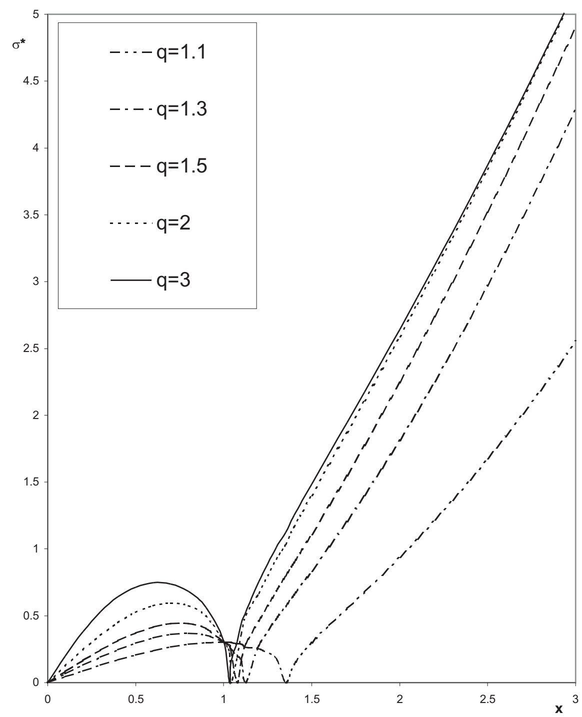

Figure 1 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $H_0/H_s = 0.5$ and $\beta = 0.25$ corresponding to

$$q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned}0 &\le x < 1.84733\,,\\ 0 &\le x < 1.33400\,,\\ 0 &\le x < 1.27200\,,\\ 0 &\le x < 1.14900\,,\\ 0 &\le x < 1.10361\,,\end{aligned}$$

with maximum modes of instability

$$\sigma^*_{max} = 0.55091,\ 0.50834,\ 0.548726,\ 0.662741 \text{ and } 0.794921$$

at $x_{max} = 1.3$, 0.9, 0.8, 0.8 and 0.7. While the neighboring stable domains are given by

$$\begin{aligned} 1.84733 &\le x < \infty\,, \\ 1.33400 &\le x < \infty\,, \\ 1.27200 &\le x < \infty\,, \\ 1.14900 &\le x < \infty\,, \\ 1.10361 &\le x < \infty\,, \end{aligned}$$

where the equalities correspond to marginal stability. See Figure 2.

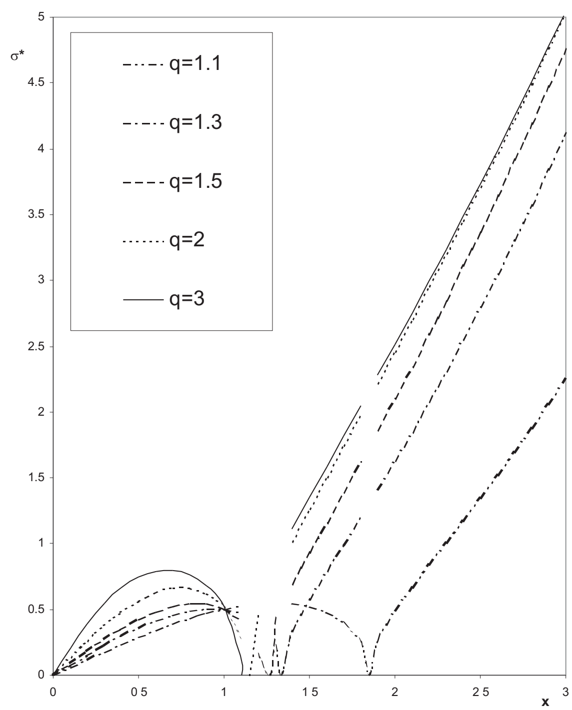

Figure 2 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.5$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $H_0/H_s = 0.8$ and $\beta = 0.25$ corresponding to

$$q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned} 0 &\le x < 2.699700\,, \\ 0 &\le x < 1.753920\,, \\ 0 &\le x < 1.513540\,, \\ 0 &\le x < 1.336269\,, \\ 0 &\le x < 1.288330\,, \end{aligned}$$

with maximum modes of instability

$$\sigma^*_{max} = 1.161189,\ 0.855652,\ 0.81407,\ 0.85034 \text{ and } 0.924651$$

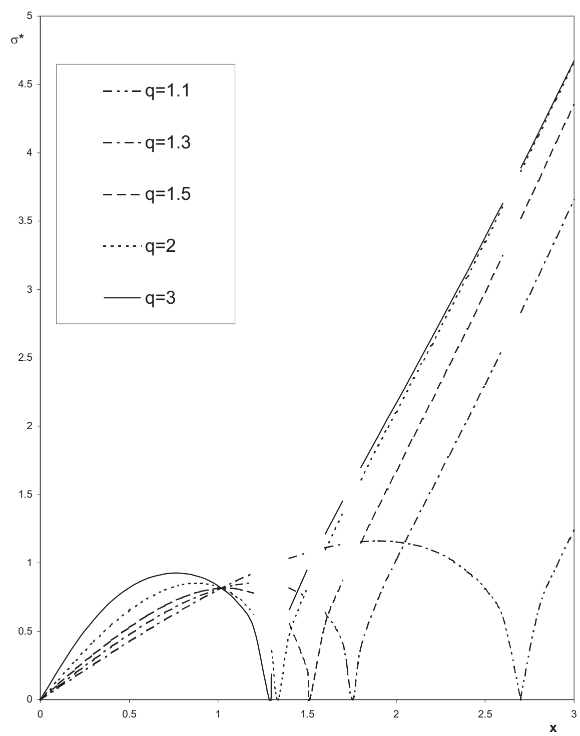

Figure 3 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.8$, $\beta = 0.25$, $U^* = 0$

while the neighboring stable domains are

$$\begin{aligned} 2.699700 &\le x < \infty\,, \\ 1.753920 &\le x < \infty\,, \\ 1.513540 &\le x < \infty\,, \\ 1.336269 &\le x < \infty\,, \\ 1.288330 &\le x < \infty\,, \end{aligned}$$

where the equalities correspond to marginal stability states. See Figure 3.

For $U^* = 0$, $H_0/H_s = 3.0$ and $\beta = 0.25$ corresponding to

$$q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0$$

it found that the model is completely stable not only for short wavelengths but also for long perturbed wavelengths. All the stable domains ranged in $0 \le x < \infty$. See Figure 4.

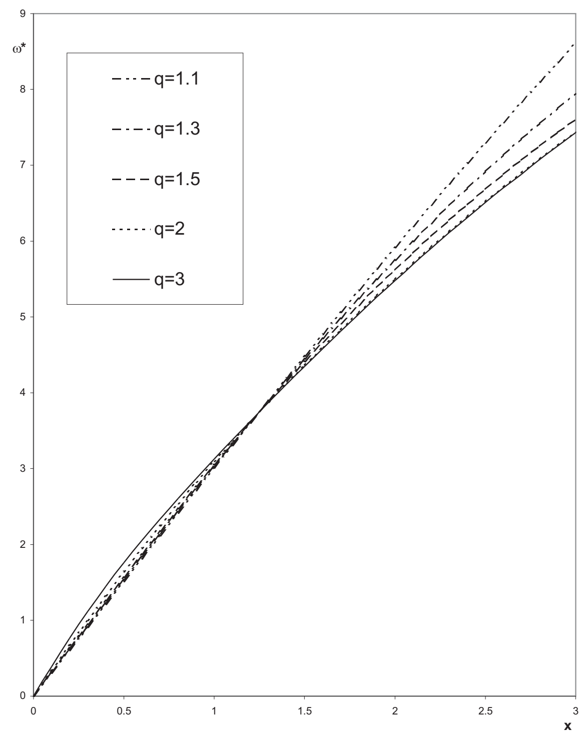

Figure 4 Relation between the dimensionless wavenumber x and the oscillation frequency ω^*) for $H_0/H_s = 3$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $H_0/H_s = 0.3$ and $\beta = 0.5$ corresponding to

$$q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned} &0 \le x < 1.369280\,, \\ &0 \le x < 1.133103\,, \\ &0 \le x < 1.085419\,, \\ &0 \le x < 1.050069\,, \\ &0 \le x < 1.041380\,, \end{aligned}$$

with maximum modes of instability

$$\sigma^*_{max} = 0.303891,\ 0.37450,\ 0.448642,\ 0.602968 \text{ and } 0.755831$$

while the neighboring stable domains are

$$\begin{aligned} 1.369280 &\leq x < \infty\,, \\ 1.133103 &\leq x < \infty\,, \\ 1.085419 &\leq x < \infty\,, \\ 1.050069 &\leq x < \infty\,, \\ 1.041380 &\leq x < \infty\,. \end{aligned}$$

See Figure 5.

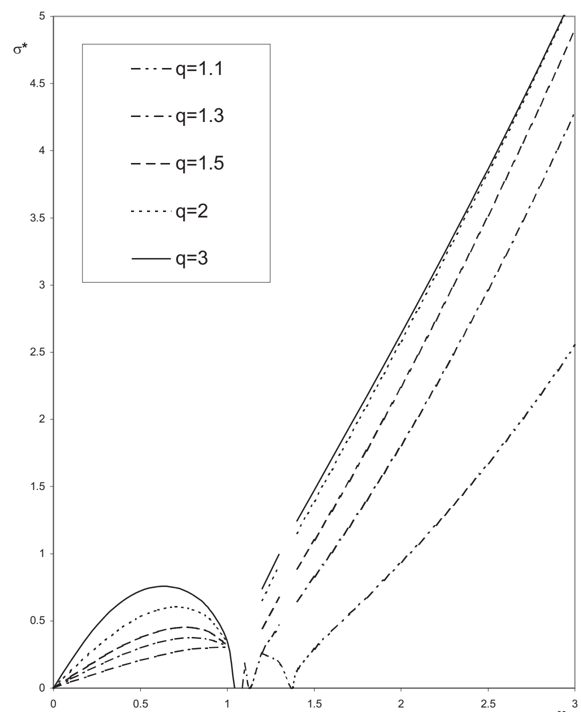

Figure 5 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.5$, $U^* = 0$

For $U^* = 0$, $H_0/H_s = 0.3$ and $\beta = 0.75$ corresponding to

$$q = 1.1,\ 1.3,\ 1.5,\ 2.0 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned} 0 \le x &< 1.380570\,, \\ 0 \le x &< 1.145078\,, \\ 0 \le x &< 1.100798\,, \\ 0 \le x &< 1.064120\,, \\ 0 \le x &< 1.055259\,, \end{aligned}$$

with maximum modes of instability

$$\sigma^*_{max} = 0.308702,\ 0.382518,\ 0.460272,\ 0.616798 \text{ and } 0.770818\,.$$

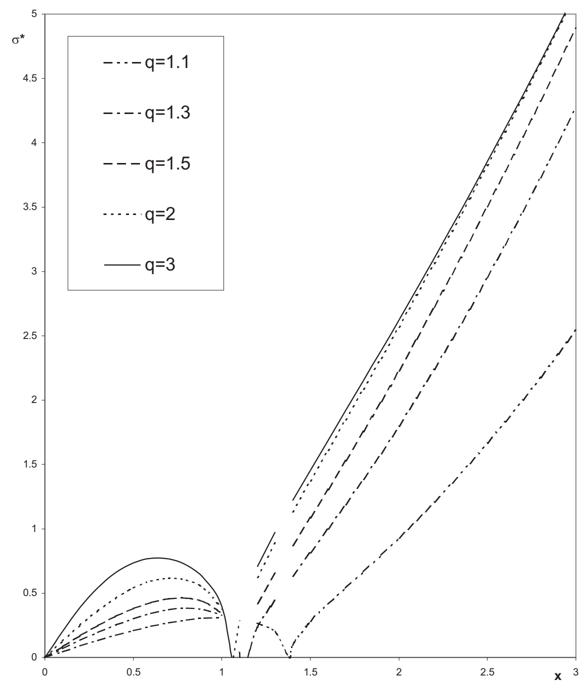

Figure 6 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.75$, $U^* = 0$

While the stable domains are

$$\begin{aligned} 1.380570 \le x &< \infty\,, \\ 1.145078 \le x &< \infty\,, \\ 1.100798 \le x &< \infty\,, \\ 1.064120 \le x &< \infty\,, \\ 1.055259 \le x &< \infty\,, \end{aligned}$$

where the equalities correspond to marginal stability states. See Figure 6.

For $U^* = 0$, $H_0/H_s = 0.3$ and $\beta = 0$ corresponding to

$$q = 1.1, 1.3, 1.5, 2.0 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned} 0 \le x &< 1.360510\,, \\ 0 \le x &< 1.123860\,, \\ 0 \le x &< 1.073339\,, \\ 0 \le x &< 1.060540\,, \\ 0 \le x &< 1.043940\,, \end{aligned}$$

with maximum modes of instability

$$\sigma^*_{max} = 0.300,\ 0.36794,\ 0.439181,\ 0.59166714 \text{ and } 0.743626$$

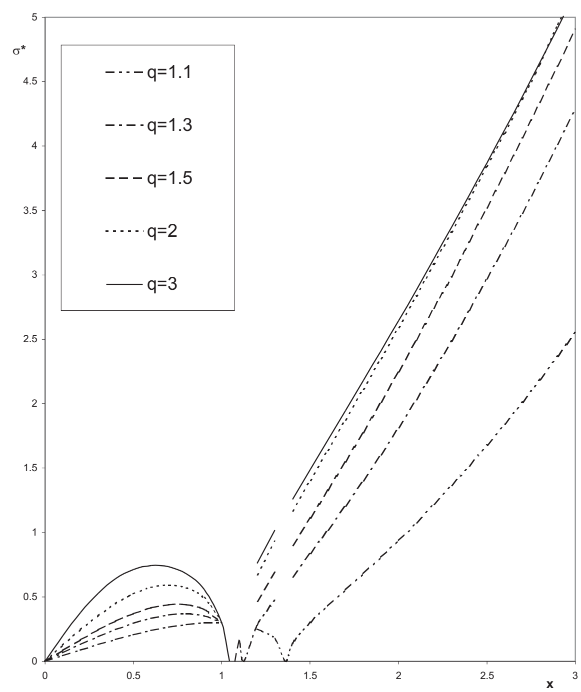

Figure 7 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.0$, $U^* = 0$

while the neighboring stable domains are

$$\begin{aligned} 1.360510 \le x &< \infty\,, \\ 1.123860 \le x &< \infty\,, \\ 1.073339 \le x &< \infty\,, \\ 1.060540 \le x &< \infty\,, \\ 1.043940 \le x &< \infty\,, \end{aligned}$$

where the equalities correspond to marginal stability states. See Figure 7.

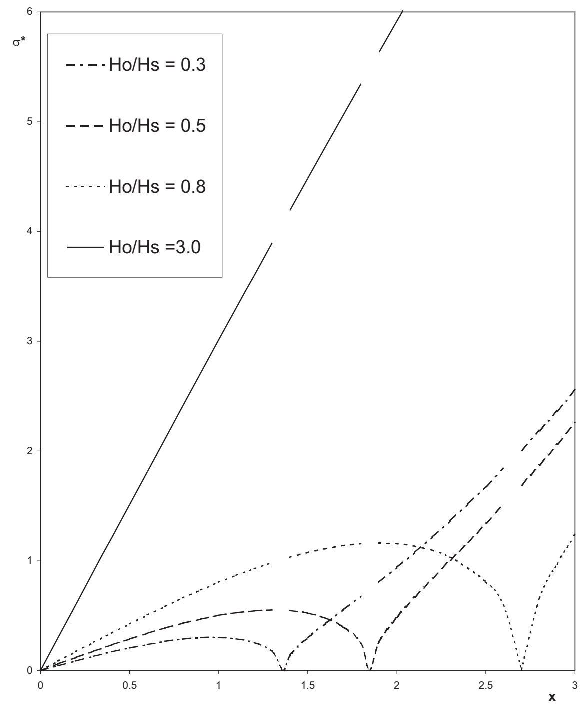

Figure 8 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $q = 1.1$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $q = 1.1$ and $\beta = 0.25$ corresponding to

$$H_0/H = 0.3,\ 0.5,\ 0.8 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{gathered} 0 \leq x < 1.36260\,, \\ 0 \leq x < 1.84733\,, \\ 0 \leq x < 2.69990\,, \\ 0 \leq x < \infty\,. \end{gathered}$$

Here for $U^* = 0$, $q = 1.1$, $\beta = 0.25$ and $(H_0/H_s) = 0.3$ the model is completely unstable for all short and long wavelengths. The maximum modes of instability as $H_0/H_s = 0.3$, 0.5 and 0.8 are

$$\sigma^*_{max} = 0.300965,\ 0.550908 \text{ and } 1.161189\,.$$

While the stable domains are given by

$$\begin{aligned} 1.36260 &\le x < \infty\,, \\ 1.84733 &\le x < \infty\,, \\ 2.69997 &\le x < \infty\,, \end{aligned}$$

where the equalities correspond to marginal stability states. See Figure 8.

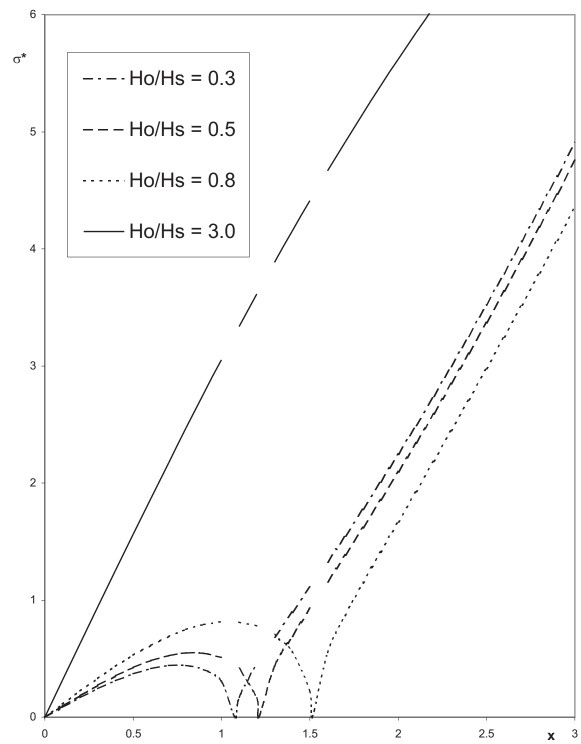

Figure 9 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $q = 1.5$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $q = 1.5$ and $\beta = 0.25$ corresponding to

$$H_0/H = 0.3,\ 0.5,\ 0.8 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{aligned} 0 \le x &< 1.07631\,, \\ 0 \le x &< 1.21225\,, \\ 0 \le x &< 1.51354\,, \\ 0 \le x &< \infty\,. \end{aligned}$$

The maximum modes of instability for the cases with ($H_0/H = 0.3$, 0.5 and 0.8 are given by

$$\sigma^*_{max} = 0.44152,\ 0.548726 \text{ and } 0.81407$$

while that as $H_0/H_s = 0.3$ is very large and may be tends to infinity. The model is completely unstable as $H_0/H_s = 0.3$ while there are stable domains:

$$\begin{array}{l} 1.07631 \leq x < \infty\,, \\ 1.21225 \leq x < \infty\,, \\ 1.51354 \leq x < \infty\,, \end{array}$$

as $H_0/H_s = 0.3$, 0.5 and 0.8 where the equalities correspond to marginal stability states. See Figure 9.

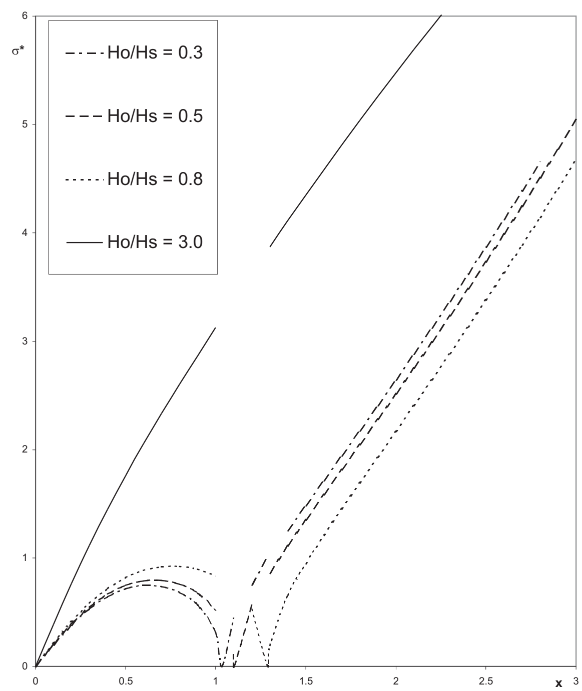

Figure 10 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $q = 3.0$, $\beta = 0.25$, $U^* = 0$

For $U^* = 0$, $q = 3.0$ and $\beta = 0.25$ corresponding to

$$H_0/H_s = 0.3,\ 0.5,\ 0.8 \text{ and } 3.0$$

it is found that the unstable domains are

$$\begin{gathered} 0 \le x < 1.03322\,, \\ 0 \le x < 1.10361\,, \\ 0 \le x < 1.28833\,, \\ 0 \le x < \infty\,. \end{gathered}$$

The maximum mode of instability as $H_0/H_s = 0.3$ in this case is much large and may be tends to infinity while the maximum modes of instability as

$$H_0/H_s = 0.3,\ 0.5 \text{ and } 0.8$$

are found to be

$$\sigma^*_{max} = 0.746693,\ 0.794921 \text{ and } 0.924651\,.$$

The stable domains are given by

$$\begin{gathered} 1.03322 \le x < \infty\,, \\ 1.10361 \le x < \infty\,, \\ 1.28833 \le x < \infty\,, \end{gathered}$$

where the equalities correspond to the marginal stability states. See Figure 10.

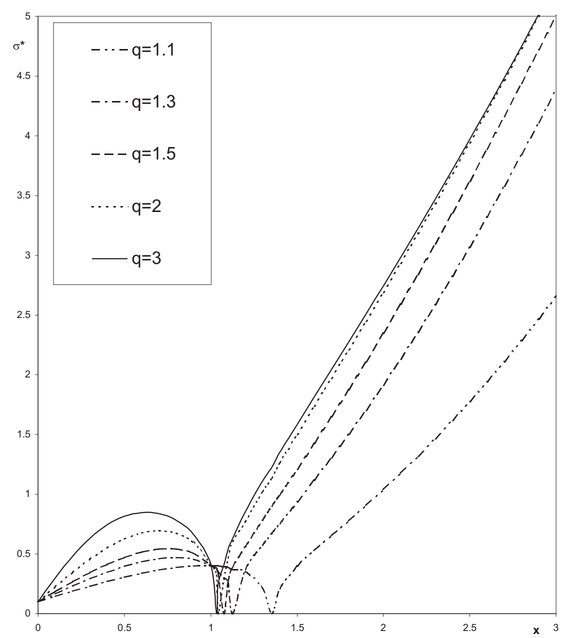

Figure 11 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 0.1$

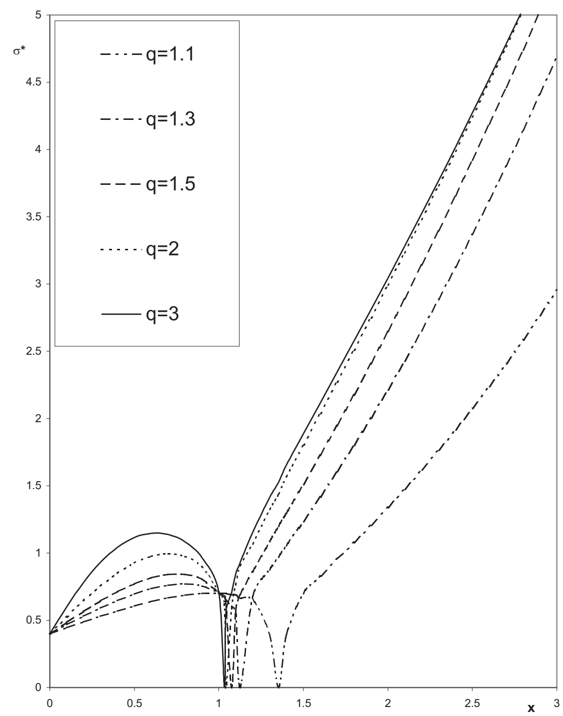

Figure 12 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 0.4$

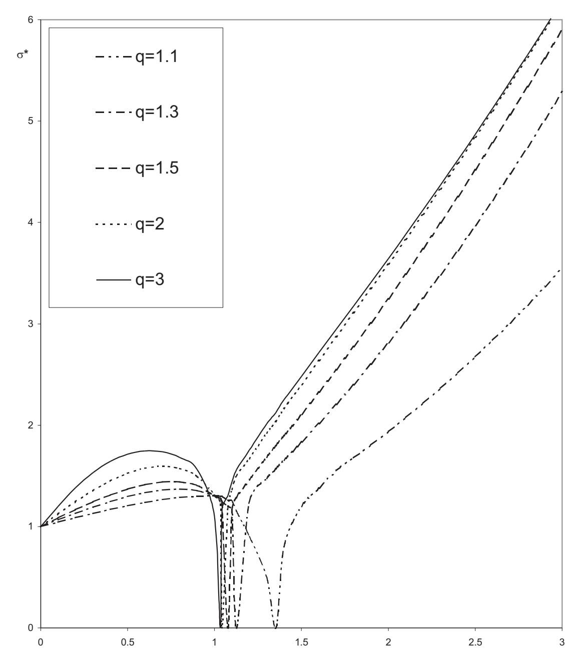

Figure 13 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 1.0$

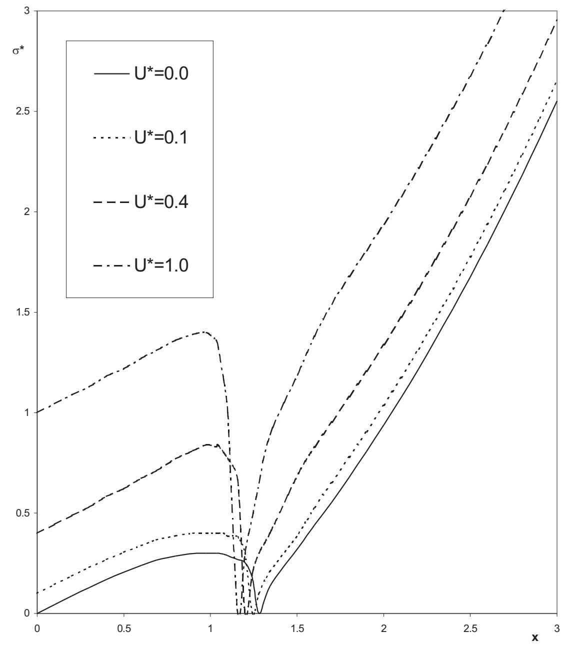

Figure 14 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 1.1$

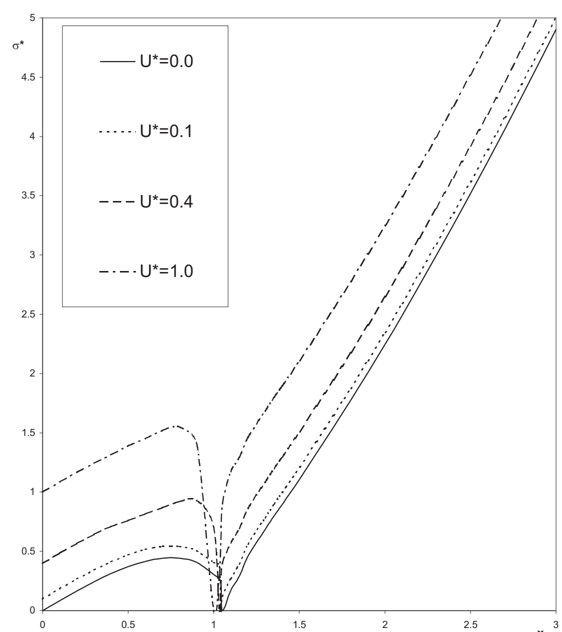

Figure 15 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 1.5$

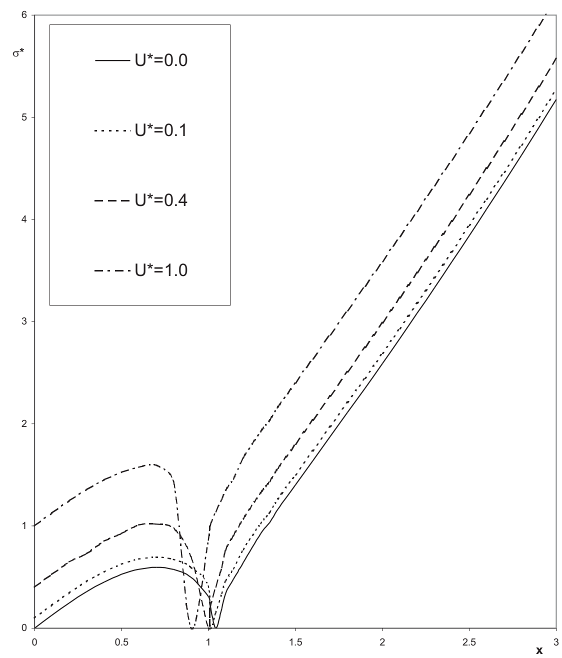

Figure 16 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 2.0$

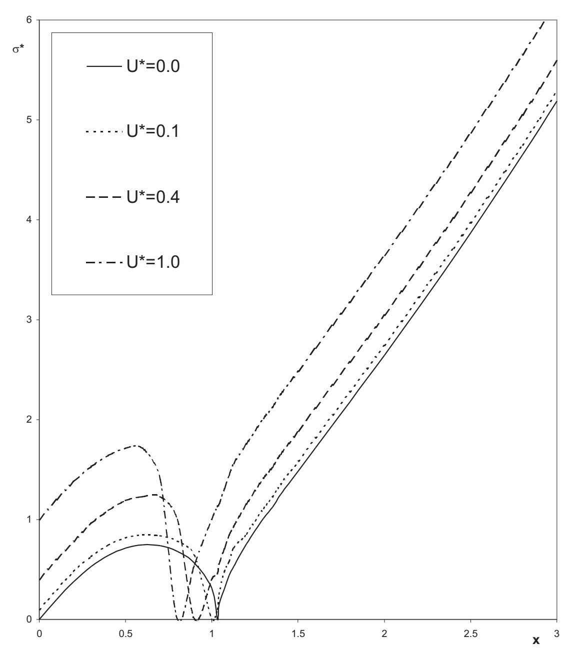

Figure 17 Relation between the dimensionless wavenumber x and the temporal amplification σ^* (or the oscillation frequency ω^*) for $H_0/H_s = 0.3$, $\beta = 0.25$, $U^* = 3.0$

8. Conclusions

From the discussions of the foregoing numerical analysis, see Figures 1–17 we may write down the following conclusions which verify the analytical results.

(i) The streaming has a destabilizing effect on the present model.

(ii) The magnetic field intensity has a strong stabilizing effect on the capillary instability of the present model.

(iii) The geometrical factor q which is the ratio of the liquid- gas radii has strong stabilizing effect. In fact as q tends to infinity i.e., we have an infinite liquid (radically) and as $H_0 = -0$ we found the same results which are given by Kendall (1986) experimentally.

(iv) The azimuthally varying magnetic field as $m = 0$ has strong stabilizing influence.

(v) In the axisymmetric perturbations, the capillary force is stabilizing for all short wavelength, but it is destabilizing for the long wavelengths λ as:

$$\lambda > 2\pi R_0 \,.$$

While it is stabilizing in the non-axisymmetric modes $m > 0$ for all short and long wavelengths.

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