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Synchronization of coupled mechanical oscillators

Przemysław Perlikowski and Andrzej Stefański
Technical University of Łódź, Division of Dynamics
Stefanowskiego 1/15, 90-924 Łódź, Poland

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Complete synchronization of coupled chaotic systems is usually a primary and crucial issue. Coupling in mechanical systems introduces mutual perturbation of their dynamics. In case of identical systems such perturbation can lead to the synchronization. We can predict the synchronization threshold of such systems using a concept called Master Stability Function (MSF). As a tool of MSF we use transverse Lyapunov exponents, which characterize the stability of synchronization state. We show areas of synchronization in coupling parameters space in typical nonlinear systems: Duffing and Duffing – Van der Pol oscillators.

Keywords: Complete synchronization, Duffing oscillators, Master stability function

1. Introduction

Analysis of synchronization phenomena in dynamical systems has been a subject of investigation for a very long time. Recently the idea of synchronization has been also adopted for chaotic systems. There is a wide variety of areas where this idea is applied [1, 3, 7], from methodological systems, twin-engine gas turbine [2], electronics till neural networks. It has been demonstrated that two or more chaotic systems can synchronize by linking them with mutual coupling or with a common signal or signals. In case of coupled, identical chaotic systems, i.e. the same set of ODEs and values of the system parameters, complete synchronization (CS) [5] can be obtain. The complete synchronization takes place when all trajectories converge to the same value and remain in step with another during further evolution (i.e. $\lim_{t\to\infty} |\mathbf{x}(t) - \mathbf{y}(t)| = 0$ for two arbitrarily chosen trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$). In this case, all subsystems of the augmented systems evolve on the same manifold on which one of these subsystems evolves (the phase space is reduced to the synchronization manifold).

In this paper we show the complete synchronization of identical mechanical systems with coupling. The main question in such systems is: *When synchronous* *behavior is stable?* Pecora and Carroll introduced a concept called Master Stability Function (MSF) [6], which allows us to determine the stability of synchronization by means of Lyapunov or Floquet exponents. In Section 2 we define the transversal Lyapunov exponents. In Section 3 we present the idea of MSF. Section 4 contains numerical results. We show areas of synchronization and their similarity in typical mechanical nonlinear systems: Duffing oscillator and Duffing – Van der Pol oscillator. The conclusions are started the last Section.

2. Transversal Lyapunov exponents

The stability problem of identical coupled systems can be formulated in a very general way by addressing a question of the stability of the CS synchronization manifold $\mathbf{x} \equiv \mathbf{y}$, or equivalently by studying the temporal evolution of the synchronization error $\mathbf{z} \equiv \mathbf{y} - \mathbf{x}$ ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbf{R}^m$, $\mathbf{x}$ and $\mathbf{y}$ being the state vectors of the coupled systems). The evolution of $\mathbf{z}$ is given by

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{x}, \mathbf{s}(t)) - \mathbf{f}(\mathbf{y}, \mathbf{s}(t)), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}$ represent the state vectors of the response of systems and its replica.

A CS regime exist when the synchronization manifold is asymptotically stable for all possible trajectories $\mathbf{s}(t)$of the driving system within the chaotic attractor. This property can be proved by using stability analysis of the linearized system for small $\mathbf{z}$

$$\dot{\mathbf{z}} = \mathbf{D_x}(\mathbf{s}(t))\,\mathbf{z}, \tag{2}$$

where $\mathbf{D}_x$ is a Jacobian of the vector field $\mathbf{f}$ evaluated onto driving trajectory $\mathbf{s}(t)$.

A possible solution is calculating the Lyapunov exponents of the system Eq. (2). These exponents could be defined for an initial condition of the driver signal $\mathbf{s}_0$ and initial orientation of the infinitesimal displacement $u_0 = \frac{z(0)}{|z(0)|}$ as

$$\lambda(\mathbf{s_0}, \mathbf{u_0}) = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{|\mathbf{z}(t)|}{|\mathbf{z_0}|}\right) = \lim_{t\to\infty} \frac{1}{t} \ln|\mathbf{Z}(\mathbf{s_0}, t) \cdot \mathbf{u_0}|, \tag{3}$$

where $\mathbf{Z}(s_0, t)$is a matrix solution of linearized equation

$$\dot{\mathbf{Z}} = \mathbf{D_x}(\mathbf{s}(t))\,\mathbf{Z}. \tag{4}$$

Subject to the initial condition $\mathbf{Z}(0)$=$\mathbf{I}$ ($\mathbf{I}$ is $m \times m$–dimensional identity matrix). The synchronization error $\mathbf{z}$ evolves according to $\mathbf{e}(t)$=$\mathbf{Z}(\mathbf{s_0}, t)\mathbf{e}_0$ and then the matrix$\mathbf{Z}$ determinate whether this error shrinks or groves in particular direction.

3. Master stability function

Consider the dynamical uncoupled system

$$\dot{x}_i = f(x_i). \tag{5}$$

$H : \mathbf{R}^m \to \mathbf{R}^m$ is output function of each oscillator's variables that is used in the coupling, $\mathbf{x}$= $(x_1, x_2, ..., x_N) \in \mathbf{R^m}$, $\mathbf{F(x)}$= $(f(x_1), ..., f(x_N))$, $\mathbf{G}$ is a matrix of coupling coefficient and σ is a coupling strength. We can create an arbitrary set of N coupled (linear coupling) identical systems:

$$\dot{\mathbf{x}} = \mathbf{F}(x_i) + (\sigma \mathbf{G} \otimes \mathbf{H}), \tag{6}$$

where $\otimes$ is a direct (Kronecker) product. Recall that the direct product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is given in form of:

$$\mathbf{A} \otimes \mathbf{B} = [a_{ij}\mathbf{B}],$$

where a_{ij} are the elements of matrix $\mathbf{A}$. Note also that manifold invariant requires $\sum_j G_{ij} = 0$. The variational equation of Eq. (5):

$$\dot{\mathbf{z}} = [\mathbf{1_N} \otimes \mathbf{DF} + \sigma \mathbf{G} \otimes \mathbf{DH}]\,\mathbf{z}, \tag{7}$$

where $z = \left(z^1, z^2, ..., z^N\right)$ is perturbations and $\mathbf{1}_N$ is $N \times N$ – dimensional identity matrix. After diagonalization, each block has the form

$$\dot{\mathbf{z}}_\mathbf{k} = [\mathbf{DF} + \sigma \gamma_\mathbf{k} \mathbf{DH}]\,\mathbf{z}_\mathbf{k}, \tag{8}$$

where: γ_k – eigenvalue of connectivity matrix $\mathbf{G}$, $(k = 0, 1, 2, ..., N-1)$. For $k = 0 \rightarrow \gamma_0$ we have variational equation for synchronization manifold. All k's correspond to transverse eigenvectors, so we have succeeded in separating the synchronization manifold from the other, transverse directions.

The Jacobian function **DF** and **DH** are the same for each block, since they are evaluated on the synchronized state. Thus, for each k the form of each block (Eq. (8)) in the same with only the scalar multiplier $\sigma\gamma_k$ differing for each. This leads to following formulation of the master stability equation and the associated MSF. We can calculate maximum transverse Lyapunov exponent λ_1 for generic variational equation

$$\dot{\mathbf{z}}_\mathbf{k} = [\mathbf{DF} + (\delta + \mathbf{i}\beta)\,\mathbf{DH}]\,\mathbf{z}_\mathbf{k}, \tag{9}$$

where $\sigma\gamma_k = \delta + i\beta$. In mechanical systems we have mutual interaction, hence $\sigma\gamma_k = \delta$ (real coupling $\beta = 0$).

4. Numerical simulation

In this section we present the examples of numerical analysis of coupled oscillators. All numerical simulations have been carried out with programs write in C++ [4].

Duffing oscillators working in chaotic regime. A single Duffing system is described by the equation of motion:

$$M\ddot{x} + c\dot{x} + kx^3 = F\sin(\omega t), \tag{10}$$

where M is the mass of oscillator, kx^2 is the non-linear stiffness of the spring, c is coefficient of viscous damping, F is the amplitude of the harmonic exciting force, ω is the frequency of forcing.

Introducing dimensionless form of Eq. (10) and substitution$x = x_1$, $\dot{x} = x_2$we obtain:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -hx_2 - ax_1^3 + q\sin(\eta\tau). \end{aligned} \tag{11}$$

According to Eq. (9) we can create MSF for the system under consideration (Eq. (11)). Supposed we choose systems coupled by spring and damper. Then

$$G = \begin{bmatrix} -d & d \\ d & -d \end{bmatrix},$$

it's the eigenvalues are: $\alpha_1 = 0$, $\alpha_2 = -2d$, and the matrix of connectivity is given by:

$$DH = \begin{bmatrix} 0 & 0 \\ \delta_1 & \delta_2 \end{bmatrix}.$$

Putting in Eq. (11) we obtain the augmented system in following form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -hz_2 - 3ax_1^2 z_1 - \delta_1 z_2 - \delta_2 z_1, \end{aligned} \tag{12}$$

where δ_1 and δ_2 are dimensionless coupling coefficients, first one by spring and second one by damper.

$$\dot{x}_i = f(x_i) + \sigma(x_{i-1} + x_{i+1} - 2x_i). \tag{13}$$

From Eq. (13) we can create **G** matrix

$$G = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \ddots & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}. \tag{14}$$

Eigenvalues of matrix **G** are

$$\gamma_k = -4\sin^2\left(\frac{k\pi}{N}\right). \tag{15}$$

From definition of MSF we can predict synchronization ranges for N-coupled oscillators. For chain coupled from synchronization of two coupled oscillators we get the following ranges of synchronization states: $0.95 < \delta_1 < 2.75$ and $\delta_1 > 3.58$ for coupling by springs and $\delta_2 > 0.2$ for coupling by dampers. We know that σ (coupling coefficient) is constant and defined by formula

$$\sigma = \frac{\delta}{\gamma_1}. \tag{16}$$

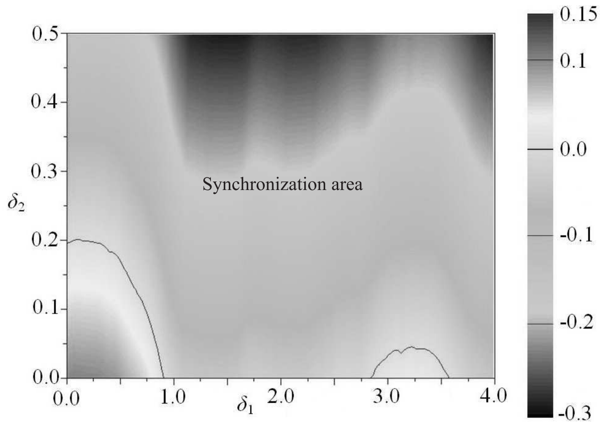

Figure 1 Diagram of maximal transversal Lyapunov exponent of Duffing oscillators as a function of coupling coefficients – MSF; $a = 1$, $h = 0.1$, $q = 5.6$, $\eta = 1$

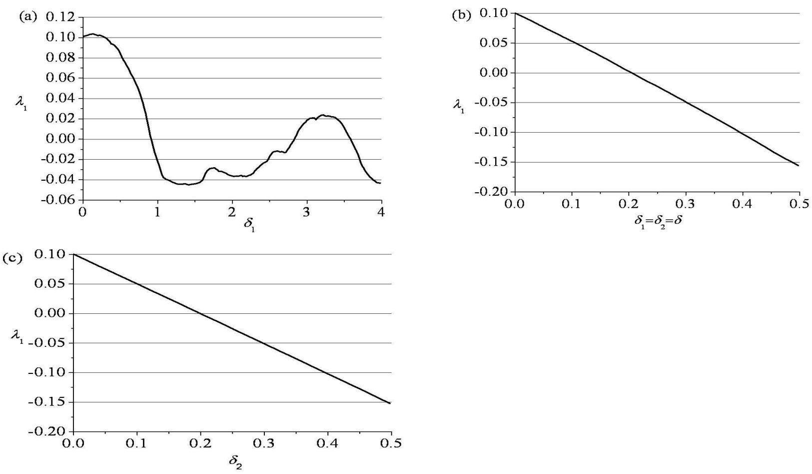

Figure 2 Cross-section of the diagram shown in Fig. 1, (a) - $\delta_1 = \delta, \delta_2 = 0$, (b) - $\delta_1 = \delta_2 = \delta$, (c) - $\delta_1 = 0$, $\delta_2 = \delta$.

For three oscillators coupled by spring in chain matrix **G** have form

$$G = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{17}$$

and the eigenvalues are $\gamma_1 = \gamma_2 = -3$, $\gamma_3 = 0$. We get the synchronization (for 3 coupled oscillators) when $\delta_{1-3} = 1.19$ for coupled by spring.

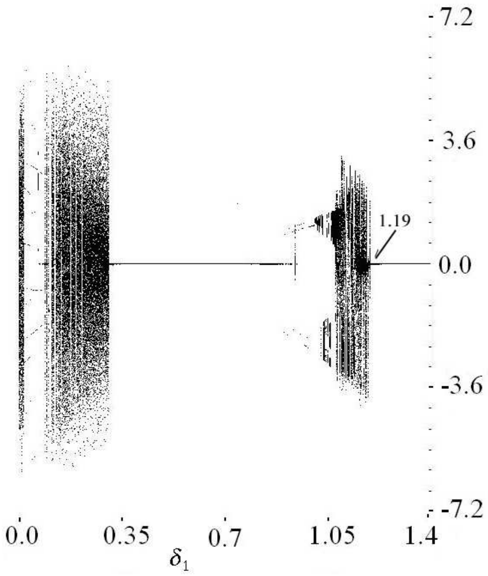

Figure 3 Bifurcation diagram of three coupled Duffing oscillators

5. Conclusions

We show in Fig. 3 that theory applying MSF concept for N-coupled systems allows us to calculate the synchronization threshold state in very easy way. In real coupling it is hard especially for multidimensional systems. In such systems we have problems with specify initial condition, because we are not sure if they evolve on the same attractor. For mechanical oscillators coupled by spring the synchronization manifold is very characteristic and unusual in other systems. In Fig. 2a and Fig. 3 we can see two areas of synchronization.

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References

[1] **Boccaletti S., Kurths J., Osipov G., Valladares D., Zhou C.**: The synchronization of chaotic systems. *Physics Reports* **366**, 2002.

[2] **Doroshko S. M., Sopulis J.**: On the effect of self-synchronization in the twin-engine gas turbine power plants. Computer Modelling & New Technologies, 3, 140-145 (1999).

[3] **Kapitaniak T.**: Chaos for Engineers, Springer (2000).

[4] **Parker T., Chua L.**: Practical numerical algorithms for chaotic systems, Springer-Verlag New York (1989).

[5] **Pecora L. and Carroll T.**:Synchronization in chaotic systems. Physical Review Letters, **64**, 821 (1990).

[6] **Pecora L. and Carroll T.**: Master Stability Functions for Synchronized Coupled Systems. Physical Review Letters, **80**, 2109 (1998).

[7] **Stefański A., Kapitaniak T.**: Synchronization of two chaotic oscillators via negative feedback mechanism. Int. Journal of Solids and Structures, **40**, 5175 (2003).