Mechanics and Mechanical Engineering
Vol. 10, No 1 (2006) 138–159

Determination of inertia tensor co-ordinates of manipulators on the basis of local matrices of homogeneous transformations

Przemysław Szumiński, Tomasz Kapitaniak
Division of Dynamics, Technical University of Lodz
Stefanowskiego 1/15, 90-924 Lodz, Poland
e-mail: przeszum@p.lodz.pl

Received (20 April 2006)
Revised (31 May 2006)
Accepted (8 June 2006)

The aim of this paper is to present an algorithm for the determination of co-ordinates of inertia tensors of manipulator links and objects manipulated, which employs homogeneous transformations. A division of a manipulator link (an object manipulated) and local systems of reference, which correspond to this division, have been introduced. An introduction of additional local systems of reference allows for their most advantageous orientation in order to calculate the co-ordinates of inertia tensors of parts and elements, whose forms and shapes often cause serious problems during the calculation of inertia tensors with the traditional method. In order to transform geometrical dimensions of solid bodies and moments of inertia, an application of matrices of homogenous transformations, typical of the description of manipulator kinematics, has been suggested. Owing to the employment of homogenous transformations and their mappings, the proposed algorithm makes the determination of co-ordinates of inertia tensors much easier, especially in the case their shape is complex.

Keywords: Inertia tensor, homogeneous transformations, manipulators

1. Introduction

If a rigid solid body that rotates around a fixed axis of rotation is considered, then the notion of a moment of inertia is used. For a rigid body that performs a spatial motion, the notion of a tensor of inertia is introduced as a scalar generalisation of the moment of inertia. Among fundamental properties of inertia tensors of solids, one can mention their dependence on the position and orientation of the system of reference with respect to which the tensor of inertia is defined. A classical way of the determination of inertia tensor parameters of the solid body, in the case of manipulators - usually of a link or an object manipulated, consists in the calculation of its co-ordinates with respect to the local system of reference of the solid that is fixed it the solid centre of gravity [1, 4, 6, 7, 11]. This procedure results from the notation of generally accepted equations of manipulator dynamics. In the developed algorithm for the determination of inertia tensor parameters, a method that introduces a subdivision of the manipulator link (manipulated object) and local systems of reference is proposed. The link (manipulated object) is structurally divided into elements, and they in turn, are composed of parts. This division is each time determined by the design structure of the manipulator link or the manipulated object under consideration. This method enables the determination of the co-ordinates of inertia tensors of links (manipulated objects), the elements and parts that are specified in them, with respect to local systems of reference that are arbitrarily orientated in space. These systems of reference have their beginnings in the centres of gravity of respective elements and parts of the link (object). In order to transform the geometrical dimensions of solids and their moments of inertia, a usage of matrices of homogeneous transformations is suggested [5, 2, 4, 10]. The proposed algorithm simplifies significantly the determination of the co-ordinates of inertia tensors of the link and the manipulated object, especially when their shape is complex. Matrices of homogeneous transformations, well known in the manipulator kinematics, are employed here. An exemplary analysis of the parameters of inertia tensors of links and a manipulated object is presented.

2. Structural division of manipulator links. Local systems of reference of links

Let us consider an arbitrary link i of the manipulator, which we subdivide into j_i elements, Fig. 1. Forms and number of elements depend on the design of the link under consideration. In turn, each element has individual parts, which are for instance housings of bearings, bearing journals, drive transfer elements or gripping device elements. We assume that the volumes of the parts of the bearing systems and drive transfer systems connected with the link for both the kinematic pairs and the link body, as well as respective material properties, are known. Let us specify elements k, r, l from the total number of elements j_i of the link i, where p denotes any element of the link i, r refers to the element of the link i that includes the local system of reference of the link i, and l stands for the element of the link i that includes the local system of reference of the link $i-1/i+1$. Further, let us assume thus that each element of the link i has in turn the specified parts q_k, q_r, q_l, correspondingly. In general, any p element of the link i has specified parts q_p.
The system of co-ordinates $x_r y_r z_r 0_r$ is the local system of reference of the element r, which includes the local system of reference $x_i y_i z_i 0_i$ at the same time.
In Fig. 1: L_i - vector defining the distance of centres of local systems of reference of the link i and $i+1/i-1$ along the straight line that connects them; α_i, β_i - design data of the link (geometrical distance of the point 0_p from the element of the link body); V_p^i - sum of volumes of specified parts of the p-th link element; c_r, c_k, c_l - positions of resultants of centres of gravity of the elements r, k, l, respectively. These points are resultant positions of centres of gravity of bearing systems and systems of drive transfer related to the link; 0_r, 0_l - centres of local systems of reference of the link i and $i+1/i-1$; S_i - position of the link centre of gravity. Let us introduce additional local systems of reference for elements k and l, respectively. The systems of co-ordinates $x_k y_k z_k 0_k$ and $x_l y_l z_l 0_l$ are local systems of reference of the k-th and l-th element of the link i, correspondingly.

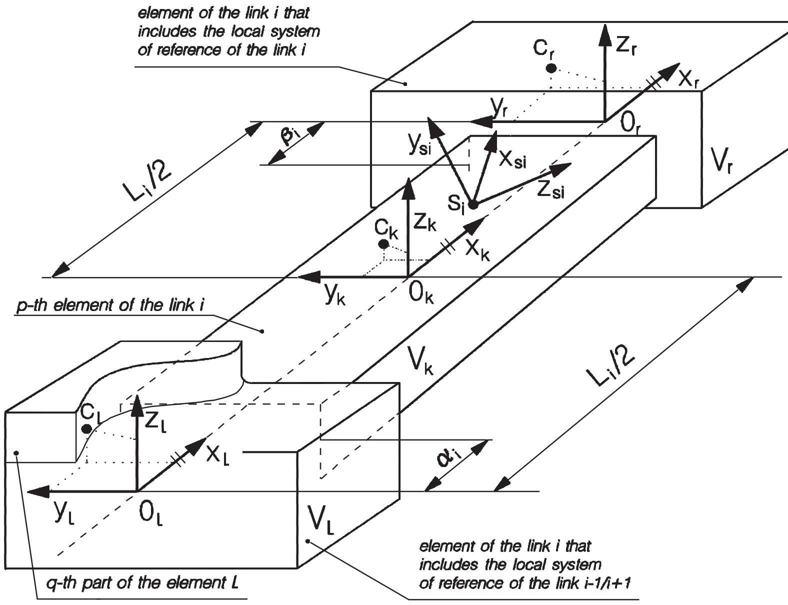

Figure 1 Structural division of the i-th link of the manipulator

The most advantageous way of the selection of local systems of link elements is such when one axis of each system lies on the straight line that connects centres of these systems. The local system of the link body should be selected in the half-length of L_i (the straight line that connects $0_r 0_l$). Furthermore, it is advantageous to assume local systems of reference of individual elements of the link in such a way that they have respective axes parallel and senses consistent with each other. The element that includes the local system of reference of the link has also a local system of reference with its beginning fixed in the centre of the local system of reference of this link. A selection of local systems of reference of individual parts and elements of the link is determined by the most advantageous determination of corresponding tensors of inertia (determination of geometrical dimensions and limits of integration). Employing matrix of rotation (17), we can transform the geometrical dimensions and the limits of integration (Section 6) between the local systems of reference of parts and elements and the system of co-ordinates of the link i.

The position of the resultant centre of gravity of the p-th element has been described by the vector:

$$\overline{r}^i_{cp} = [r^x_{cp}, r^y_{cp}, r^z_{cp}]^T \tag{1}$$

in the local system of reference of the p-th element of the link i. The positions of resultant centres of gravity of the elements r, k, l are described by the following vectors:

$$\overline{r}^i_{cr} = [r^x_{cr}, r^y_{cr}, r^z_{cr}]^T,\ \overline{r}^i_{ck} = [r^x_{ck}, r^y_{ck}, r^z_{ck}]^T,\ \overline{r}^i_{cl} = [r^x_{cl}, r^y_{cl}, r^z_{cl}]^T \tag{2}$$

in the corresponding local systems of reference of these elements.

Generally, the position of the centre of gravity of the link i with respect to the local system of reference of the element that includes the local system of reference of the link i (that is to say, of the element r), can be determined from the relationship:

$$x_{si}^r = \frac{\sum\limits_p^{j_i} \rho_p^i V_p^i x_{cp}^i}{\sum\limits_p^{j_i} \rho_p^i V_p^i}, y_{si}^r = \frac{\sum\limits_p^{j_i} \rho_p^i V_p^i y_{cp}^i}{\sum\limits_p^{j_i} \rho_p^i V_p^i}, z_{si}^r = \frac{\sum\limits_p^{j_i} \rho_p^i V_p^i z_{cp}^i}{\sum\limits_p^{j_i} \rho_p^i V_p^i} \tag{3}$$

where: $p = 1, 2, \ldots, r, \ldots, k, \ldots, l, \ldots, j_i$, j_i - number of elements of the model of the link i; ρ_p^i - material density of the p-th element of the link i; V_p^i - volume of the p-th element of the link i; $\left[x_{cp}^i,\ y_{cp}^i,\ z_{cp}^i\right]^T$ - vector of the position of the resultant centre of gravity of the p-th element in the local system of reference of the element r (that includes the local system of reference of the link i), and:

$$\sum_p^{j_i} V_p^i = \sum_p^{j_i} \sum_q^{q_p} \frac{m_{ip,q}}{\rho_{q,p}^i} \tag{4}$$

where: $m_{ip,q}$ - mass of the q-th part of the p-th element of the link i; $\rho_{q,p}^i$ - material density of the q-th part of the p-th element of the link i. Assuming the notations as in Fig. 1, we can express the volume of the link i as follows:

$$\sum_p^{j_i} V_p^i = V_r^i + V_k^i + V_l^i = \sum_{q=1}^{q_r} \frac{m_{ir,q}}{\rho_{q,r}^i} + \sum_{q=1}^{q_k} \frac{m_{ik,q}}{\rho_{q,k}^i} + \sum_{q=1}^{q_l} \frac{m_{il,q}}{\rho_{q,l}^i} \tag{5}$$

where: $j_i = 3$;

V_r^i, V_l^i - volume of the housing and the bearing system and elements of the drive related to the kinematic pair i and $i+1/i-1$, respectively, V_k^i - volume of the body of the link i.

Choosing local systems of reference of the link i as in Fig. 1 and Fig. 5, for a constant value of the material density of the link, we can express equations of co-ordinates (3) in the form:

co-ordinate x_{si}^r:

$$\begin{gathered} x_{cr}^i = r_{cr}^x,\ x_{ck}^i = -\tfrac{L_i}{2} + r_{ck}^x,\ x_{cl}^i = -L_i + r_{cl}^x \\ x_{si}^r = \tfrac{V_r^i r_{cr}^x - \frac{1}{2} V_k^i L_i + V_k^i r_{ck}^x + V_l^i r_{cl}^x - V_l^i L_i}{V_r^i + V_k^i + V_l^i} \end{gathered} \tag{6}$$

whereas:

$$x_{si}^r = -\frac{1}{2} L_i <=> r_{cr}^x + r_{ck}^x + r_{cl}^x = 0 \ \wedge \ V_r^i = V_k^i = V_l^i \tag{7}$$

co-ordinate y_{si}^r:

$$\begin{gathered} y_{cr}^i = r_{cr}^y,\ y_{ck}^i = r_{ck}^y,\ y_{cl}^i = r_{cl}^y \\ y_{si}^r = \tfrac{V_r^i r_{cr}^y + V_k^i r_{ck}^y + V_l^i r_{cl}^y}{V_r^i + V_k^i + V_l^i} \end{gathered} \tag{8}$$

whereas:

$$
\begin{gathered}
y^r_{si} = 0 <=> V^i_r r^y_{cr} + V^i_k r^y_{ck} + V^i_l r^y_{cl} = 0: \\
r^y_{cr} = r^y_{ck} = r^y_{cl} = 0 \ \vee \ r^y_{cr} + r^y_{ck} + r^y_{cl} = 0 \ \wedge \ V^i_r = V^i_k = V^i_l \ \vee \\
r^y_{cr} = 0 \ \wedge \ \frac{V^i_k}{V^i_l} = -\frac{r^y_{cl}}{r^y_{ck}} \ \vee \\
r^y_{ck} = 0 \ \wedge \ \frac{V^i_r}{V^i_l} = -\frac{r^y_{cl}}{r^y_{cr}} \ \vee \\
r^y_{cl} = 0 \ \wedge \ \frac{V^i_r}{V^i_k} = -\frac{r^y_{ck}}{r^y_{cr}}
\end{gathered}
\tag{9}
$$

co-ordinate z^r_{si}:

$$
\begin{gathered}
z^i_{cr} = r^z_{cr}, \ z^i_{ck} = r^z_{ck}, \ z^i_{cl} = r^z_{cl} \\
z^r_{si} = \frac{V^i_r r^z_{cr} + V^i_k r^z_{ck} + V^i_l r^z_{cl}}{V^i_r + V^i_k + V^i_l}
\end{gathered}
\tag{10}
$$

whereas:

$$
\begin{gathered}
z^r_{si} = 0 <=> V^i_r r^z_{cr} + V^i_k r^z_{ck} + V^i_l r^z_{cl} = 0: \\
r^z_{cr} = r^z_{ck} = r^z_{cl} = 0 \ \vee \ r^z_{cr} + r^z_{ck} + r^z_{cl} = 0 \ \wedge \ V^i_r = V^i_k = V^i_l \ \vee \\
r^z_{cr} = 0 \ \wedge \ \frac{V^i_k}{V^i_l} = -\frac{r^z_{cl}}{r^z_{ck}} \ \vee \\
r^z_{ck} = 0 \ \wedge \ \frac{V^i_r}{V^i_l} = -\frac{r^z_{cl}}{r^z_{cr}} \ \vee \\
r^z_{cl} = 0 \ \wedge \ \frac{V^i_r}{V^i_k} = -\frac{r^z_{ck}}{r^z_{cr}}
\end{gathered}
\tag{11}
$$

The position of the centre of gravity of the link i with respect to the local system of reference of the element r can be expressed as:

$$
\overline{r}^r_{si} = \begin{bmatrix} x^r_{si} \\ y^r_{si} \\ z^r_{si} \end{bmatrix} = \begin{bmatrix} \frac{V^i_r r^x_{cr} + V^i_k (r^x_{ck} - \frac{L_i}{2}) + V^i_l (r^x_{cl} - L_i)}{V^i_r + V^i_k + V^i_l} \\ \\ \frac{V^i_r r^y_{cr} + V^i_k r^y_{ck} + V^i_l r^y_{cl}}{V^i_r + V^i_k + V^i_l} \\ \\ \frac{V^i_r r^z_{cr} + V^i_k r^z_{ck} + V^i_l r^z_{cl}}{V^i_r + V^i_k + V^i_l} \end{bmatrix}
\tag{12}
$$

In general, the position of the centre of gravity of the link i with respect to the local system of reference of the link i $(x_i y_i z_i 0_i)$, Fig. 2a, can be written as:

$$
\overline{r}_{si} = A_o \overline{r}^r_{si}
\tag{13}
$$

where: A_o - matrix of rotation of the co-ordinate system of the link, $x_i y_i z_i 0_i$, with respect to the position of the local system of the element r.

Knowing the description of an arbitrary vector with respect to the local system of the element r, we intend to describe this vector in the system of reference of the link i, as the beginnings of both the systems coincide. This transformation is possible provided the description of the orientation of the local system r with respect to the system of co-ordinates of the link i is known. This orientation is defined by the matrix of rotation A_o , whose columns are versors of the system of reference of the element r described in the system of reference of the link, and rows are versors of the system of the link r described in the system of the element r. Owing to the matrix of rotation A_o, a description of the orientation of the system of the element r with respect to the system of reference of the link i is possible, i.e. a transformation of the orientation of any arbitrary vector described with respect to the local system of reference of the element r to the local system of reference of the link i. In order to describe the orientation of the rotation in the three-dimensional space, one should

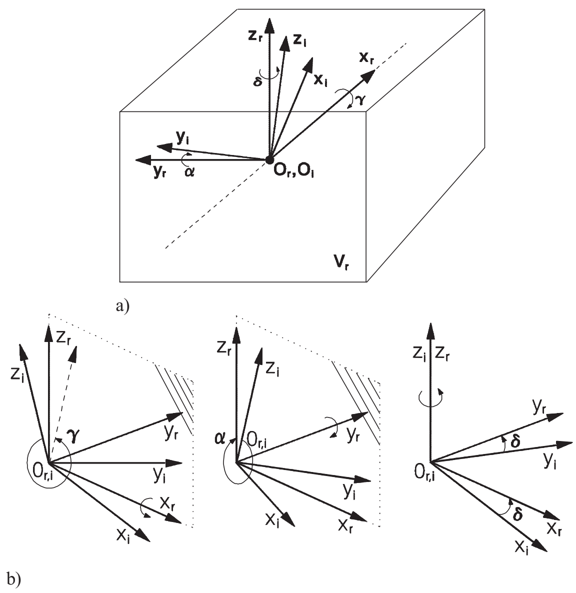

Figure 2 Transformations of local systems of reference

remember that rotations in general are not commutative (multiplication of matrices does not exhibit the property of commutation). Therefore, in order to describe the orientation of the local system of reference of the element r with respect to the local system of reference of the link i, the local system of the reference of the link i should be rotated with respect to the respective axes of the local system of the element r of this link.

In order to transform (rotate) the local system of reference of the link i to the local system of the reference of the element r, the following rotations around the corresponding axes of the local system, Fig. 2b, have been performed in the following sequence:

$$\begin{array}{l} \text{- rotation around the axis } x_r \text{ by an angle} \pm \gamma \\ \text{- rotation around the axis } y_r \text{ by an angle} \pm \alpha \\ \text{- rotation around the axis } z_r \text{ by an angle} \pm \delta \end{array} \tag{14}$$

where, depending on the positions of systems of reference with respect to each other, the angles α, δ and γ can assume positive or negative values. Angles consistent with the trigonometric direction of the rotation will be considered positive, if we assume the axis of rotation of the system directed at the person looking at it. Each of specified rotations is performed along respective axes of the defined system of reference of the element r.

The matrix of rotation of the local system of reference of the link i with respect to the local system of reference of the element r can be written in the general form as follows:

$$A_o = A_\delta A_\alpha A_\gamma \tag{15}$$

The matrices of elementary rotating transformations are as follows:

$$A_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix},$$
$$A_\delta = \begin{bmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

If we treat rotations as operators [2], we will perform rotations A_γ, A_α and A_δ. After multiplications according to Eq. (15), we obtain:

$$A_o = \begin{bmatrix} \cos(\alpha)\cos(\delta) & \sin(\alpha)\sin(\gamma)\cos(\delta) - \sin(\delta)\cos(\gamma) & \sin(\alpha)\cos(\delta)\cos(\gamma) + \sin(\delta)\sin(\gamma) \\ \sin(\delta)\cos(\alpha) & \sin(\alpha)\sin(\delta)\sin(\gamma) + \cos(\delta)\cos(\gamma) & \sin(\alpha)\sin(\delta)\cos(\gamma) - \sin(\gamma)\cos(\delta) \\ -\sin(\alpha) & \sin(\gamma)\cos(\alpha) & \cos(\alpha)\cos(\gamma) \end{bmatrix} \tag{17}$$

Matrix (17) is correct only for rotations performed in the sequence given in (14). If we employ matrix (17), then Eq. (13) takes the form:

$$\bar{r}_{si} = \begin{bmatrix} \cos(\alpha)\cos(\delta) & \sin(\alpha)\sin(\gamma)\cos(\delta) - \sin(\delta)\cos(\gamma) & \sin(\alpha)\cos(\delta)\cos(\gamma) + \sin(\delta)\sin(\gamma) \\ \sin(\delta)\cos(\alpha) & \sin(\alpha)\sin(\delta)\sin(\gamma) + \cos(\delta)\cos(\gamma) & \sin(\alpha)\sin(\delta)\cos(\gamma) - \sin(\gamma)\cos(\delta) \\ -\sin(\alpha) & \sin(\gamma)\cos(\alpha) & \cos(\alpha)\cos(\gamma) \end{bmatrix} \begin{bmatrix} x_{si}^r \\ y_{si}^r \\ z_{si}^r \end{bmatrix} \tag{18}$$

Assuming the equality $\alpha = \delta = \gamma = 0$ the matrix of rotation has the following form:

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

then

$$\bar{r}_{si} = \bar{r}^r_{si} \tag{20}$$

This is the case when the local system of reference of the link is the local system of reference of the link element within which there is the local system of reference of this link.

3. Algorithm for the determination of parameters of inertia tensors of manipulator links

As we know, determination of the tensor of inertia of the link i consists in determination of co-ordinates of the tensor of inertia in the system of co-ordinates, whose beginning lies in the centre of mass of the link and whose axes have consistent senses and are parallel to the corresponding axes of the local system of co-ordinates related to this link. The general form of the inertia tensor of the link i is as follows [5, 8, 9]:

$$I_{si} = \begin{bmatrix} B_{x_{si}} & -B_{xy_{si}} & -B_{xz_{si}} \\ -B_{xy_{si}} & B_{y_{si}} & -B_{yz_{si}} \\ -B_{xz_{si}} & -B_{yz_{si}} & B_{z_{si}} \end{bmatrix} \tag{21}$$

where: $B_{\alpha_{si}}$ - mass moment of inertia of the i-th link with respect to the planes perpendicular to the axis α, $\alpha = \{x, y, z\}$; $B_{\alpha\beta_{si}}$ - mass centrifugal moment of inertia of the i-th link with respect to the planes perpendicular to the axis α and β, $\alpha = \{x, y, z\}$ and $\beta = \{x, y, z\}$ and $\alpha \neq \beta$.

The system of reference $(xyz)_{si}$ has its beginning in the centre of mass of the link i and axes parallel and with consistent senses to the respective axes of the system related to the link (that is to say, parallel to the axis of the system $x_i y_i z_i 0_i$).

Each link as a material body can be divided in mind into indefinitely small elements. Let us denote the mass of one of such elements of this body by dm and let us assume that it has the material density ρ and the volume dV, then $dm = \rho dV$. The tensor of inertia of any q-th part of the p-th element of the link i assumes the form:

$$I^i_{q,p} = \rho^i_{q,p} \int\limits_{V^i_{q,p}} \begin{bmatrix} y^2_m + z^2_m & -x_m y_m & -x_m z_m \\ -y_m x_m & x^2_m + z^2_m & -y_m z_m \\ -z_m x_m & -z_m y_m & x^2_m + y^2_m \end{bmatrix} dV \tag{22}$$

where: m - any point of the link i; $\rho^i_{q,p}$ - material density of the q-th part of the p-th element of the link i; $V^i_{q,p}$ - volume of the q-th part of the p-th element of the link i, $dV = dxdydz$ - elementary volume; m_i, V_i - mass and volume of the i-th link, respectively.

Let us denote the vector of the position of an arbitrary point m of the link in the system of co-ordinates whose beginning is in the centre of gravity of the part q of the element p and whose axes have consistent sensors and are parallel to the corresponding axes of the local system of reference of the link i (system of co-ordinates of the link i) by $\bar{r}_m = [x_m, y_m, z_m]^T$.

The vector $\bar{r}_m$ can be written if we know the position of the point m in the local system of reference of the q-th part of the p-th element. The relationship between vectors $\bar{r}_m$ and $\bar{r}_{ml}$ has been defined as:

$$\bar{r}_m = A_o \bar{r}_{ml} \tag{23}$$

where: A_o – matrix of rotation of the local system of reference of the link i, transformed to the centre of gravity of the considered (specified) element or part of the link, to the local system of reference of this element or part; $\bar{r}_{ml}$ - vector of the position of the point m of the link in the local system of co-ordinates of the part q of the p-th element, ($\bar{r}_{ml} = [x_{ml}, y_{ml}, z_{ml}]^T$).
Substituting relationship (17) into Eq. (23) we obtain:

$$\bar{r}_m = A_o \bar{r}_{ml} \Rightarrow$$

$$\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\delta)x_{ml} + [\sin(\alpha)\sin(\gamma)\cos(\delta) - \sin(\delta)\cos(\gamma)]y_{ml} \\ \sin(\delta)\cos(\alpha)x_{ml} + [\sin(\alpha)\sin(\delta)\sin(\gamma) + \cos(\delta)\cos(\gamma)]y_{ml} \\ -\sin(\alpha)x_{ml} + \sin(\gamma)\cos(\alpha)y_{ml} + \cos(\alpha)\cos(\gamma)z_{ml} \end{bmatrix.} \tag{24}$$

$$\begin{matrix} +[\sin(\alpha)\cos(\delta)\cos(\gamma) + \sin(\delta)\sin(\gamma)]z_{ml} \\ +[\sin(\alpha)\sin(\delta)\cos(\gamma) - \sin(\gamma)\cos(\delta)]z_{ml} \\ \end{matrix} \Bigg]$$

Having substituted (24) into (22), for the case when $\alpha = 0$ and remembering to transform the limits of integration to the local system of reference of the part in order to simplify the calculations, we obtain as a result of these transformations what follows:

$$I^i_{q,p} = \rho^i_{q,p} \int\limits_{V^i_{q,p}} \begin{bmatrix} a & & sym. \\ b & c & \\ d & e & f \end{bmatrix} dV \tag{25}$$

where:

$$
\begin{aligned}
&a = \sin(\delta)[\sin(\delta)x_{ml} + 2\cos(\delta)(\cos(\gamma)y_{ml} - \sin(\gamma)z_{ml})]x_{ml}+\\
&\cos^2(\delta)[\cos(\gamma)y_{ml} - \sin(\gamma)z_{ml}]^2 + [\sin(\gamma)y_{ml} + \cos(\gamma)z_{ml}]^2\\
&b = -\sin(\delta)\cos(\delta)[x_{ml}^2 - (\sin(\gamma)z_{ml} - \cos(\gamma)y_{ml})^2]+\\
&x_{ml}(\sin^2(\delta) - \cos^2(\delta))[\sin(\gamma)z_{ml} - \cos(\gamma)y_{ml}]\\
&c = \cos(\delta)[\cos(\delta)x_{ml} + 2\sin(\delta)(\sin(\gamma)z_{ml} - \cos(\gamma)y_{ml})]x_{ml}+\\
&\sin^2(\delta)[\cos(\gamma)y_{ml} - \sin(\gamma)z_{ml}]^2 + [\sin(\gamma)y_{ml} + \cos(\gamma)z_{ml}]^2\\
&d = -\sin(\delta)[\sin(\gamma)\cos(\gamma)(z_{ml}^2 - y_{ml}^2) + y_{ml}z_{ml}(\sin^2(\gamma) - \cos^2(\gamma))]+\\
&\cos(\delta)[\sin(\gamma)y_{ml} + \cos(\gamma)z_{ml}]x_{ml}\\
&e = -\cos(\delta)[\sin(\gamma)\cos(\gamma)(y_{ml}^2 - z_{ml}^2) + y_{ml}z_{ml}(\cos^2(\gamma) - \sin^2(\gamma))]+\\
&\sin(\delta)[\sin(\gamma)y_{ml} + \cos(\gamma)z_{ml}]x_{ml}\\
&f = [\cos(\delta)x_{ml} - \sin(\delta\cos(\gamma)y_{ml} + \sin(\delta\sin(\gamma)z_{ml}]^2+\\
&[\sin(\delta)x_{ml} + \cos(\delta)\cos(\gamma)y_{ml} - \cos(\delta)\sin(\gamma)z_{ml}]^2
\end{aligned}
\tag{26}
$$

In turn, the matrix $I^i_{q,p,s}$ is calculated with respect to the centre of gravity of the link i, whereas the matrix $I^i_{q,ps}$ - with respect to the centre of gravity of the element p of the link i, using the Huygens-Steiner theorem for this aim [3]. Next, the tensor of inertia of the link i (I_{is}) is determined as a sum of the matrix $I^i_{q,p,s}$ (summation of the corresponding terms of the matrix) of the subsequent parts of the respective elements of the link i (or the matrix $I^i_{p,s}$ of the subsequent elements of the link i) and the tensor of inertia of the element $p(I^i_{sp})$ as a sum of the matrix $I^i_{q,ps}$ of the subsequent parts of the p-th element with respect to the centre of gravity of the element p. The determination of tensors of inertia of p-th elements of the link i allows for an analysis of the factors that affect the relations of inertia tensors of individual elements of the link. An exemplary analysis, based on the presented algorithm, is carried out in Section 5.

4. Algorithm for the determination of parameters of inertia tensors of manipulated objects

Let us consider the manipulated object shown in Fig. 3. Let us assume that the position of the centre of gravity of the manipulated object is displaced with respect to the beginning of the local system of reference of the last link of the kinematic manipulator chain in the form of the vector $\bar{r}_p$. We want to describe the tensor of inertia of the manipulated object with respect to the system of reference with the beginning in the centre of gravity of this object and with axes that have consistent senses and are parallel to the corresponding axes of the local system of reference of the last link of the manipulator kinematic chain (link n). It allows for the determination of the kinematic energy of the manipulated object, employing the knowledge of kinematics of the last link of the manipulator kinematic chain thus.

In order to simplify the calculations of the co-ordinates of the inertia tensor of the manipulated object, it is advantageous to introduce a local system of reference of the object with the beginning in the centre of gravity of the manipulated object

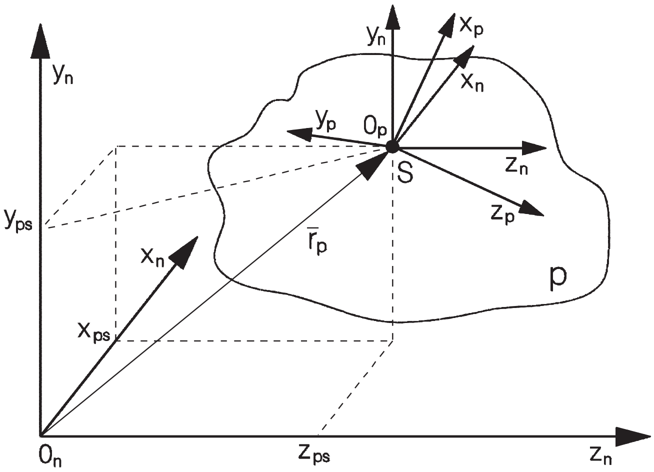

Figure 3 Local systems of reference related to the object manipulated

$(x_p y_p z_p 0_p\ )$. The orientation of the local system of reference of the manipulated object should be selected in the most advantageous way from the viewpoint of the calculations of the inertia tensor of the manipulated object with respect to its own system of reference.

Let us denote the vector of the position of the centre of gravity of the manipulated object in the local system of reference of the n-th link of the manipulator kinematic chain, Fig. 3, by $\bar{r}_p = [x_{ps}, y_{ps}, z_{ps}]^T$. The tensor of inertia of the manipulated object with respect of the object centre of gravity, in its local system of reference $(x_p y_p z_p 0_p)$, assumes the following form in general:

$$I_p^l = \begin{bmatrix} B_x^l & -B_{xy}^l & -B_{xz}^l \\ -B_{yx}^l & B_y^l & -B_{yz}^l \\ -B_{zx}^l & -B_{zy}^l & B_z^l \end{bmatrix} = \begin{bmatrix} m_p i_{px}^2 & -B_{xy}^l & -B_{xz}^l \\ -B_{yx}^l & m_p i_{py}^2 & -B_{yz}^l \\ -B_{zx}^l & -B_{zy}^l & m_p i_{pz}^2 \end{bmatrix} \tag{27}$$

where: $B_\alpha, B_{\alpha\beta}, \alpha = \{x, y, z\}$, $\beta = \{x, y, z\}$, $\alpha \neq \beta$ - mass moments of inertia of the object manipulated with respect to the local system of reference of the object located in its centre of gravity; they depend on the shape of the object; m_p - mass of the object manipulated; $i_{p\alpha}^2$ - respective radii of inertia.

Then, after transformations (17) of geometrical dimensions and limits of integration (Section 6), we can transform the quantities written in the local system of reference of the object to the local system of reference located in the centre of gravity of the manipulated object and with axes that have consistent senses and are parallel to the respective axes of the local system of co-ordinates of the last link of the manipulator kinematic chain (see Eq. (25) as an example). As a result, we obtain the tensor of inertia of the manipulated object I_p^p with respect to the system of reference fixed in the centre of gravity of the object and with axes of consistent senses and parallel to the corresponding axes of the last link of the kinematic chain:

$$I_p^p = \begin{bmatrix} B_x^p & -B_{xy}^p & -B_{xz}^p \\ -B_{yx}^p & B_y^p & -B_{yz}^p \\ -B_{zx}^p & -B_{zy}^p & B_z^p \end{bmatrix} \tag{28}$$

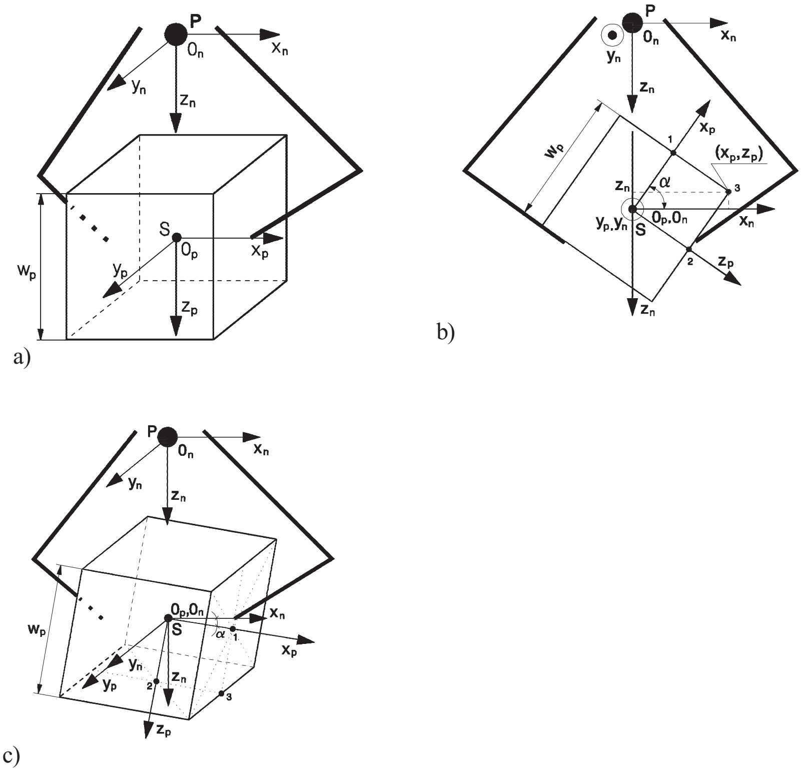

Figure 4 The manipulator gripping device - manipulated object system. Exemplary selection of local systems of reference for the case of the object manipulated in the form of a homogeneous cube: a) matrix of transformation of local systems of reference according to Eq. (19); b), c) matrix of transformation of local systems of reference according to Eq. (30). The local system of reference of the manipulated object $(x_p y_p z_p 0_p)$ is its main central system of inertia

In the case when the shape of the object manipulated is such that a selection of the local system of reference - the most advantageous from the viewpoint of the determination of its mass moments of inertia - is such that the corresponding axes of the local system of reference of the object and the local system of reference of the last link of the manipulator kinematic chain are parallel and have consistent senses, matrix (17) of transformations of these systems assumes form (19). Then, the equality of tensors of inertia takes place:

$$I_p^p = I_p^l \tag{29}$$

Let us consider a homogeneous hexagon whose side length is w_p as the manipulated object. Assuming the local systems of reference as in Figs. 4a and 4b,c, matrices A_o assume the forms (19) and (30), respectively. Considering the case presented in Figs. 4b,c, we perform a rotation by an angle α around the axis y_p of the local system of reference of the last link of the kinematic chain with respect to the position of the local system of reference of the object manipulated. Employing relationships (14) and (15), the matrix of rotation assumes then the form:

$$A_o = A_\alpha = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \tag{30}$$

In the case under consideration, we will write Eq. (23) as follows:

$$\bar{r}_n = A_o \bar{r}_p \tag{31}$$

After transformations of Eq. (31), the components of the vector $\bar{r}_p$ assume the following forms, correspondingly:

$$\begin{gathered} x_n = x_p \cos(\alpha) + z_p \sin(\alpha) \\ y_n = y_p \\ z_n = z_p \cos(\alpha) - x_p \sin(\alpha) \end{gathered} \tag{32}$$

As an example, we will transform the geometrical dimensions of the position of some selected, exemplary points of the object under manipulation in the local system of reference of the last link of the manipulator chain:

$$\begin{gathered} \textit{point } 1: \\ x_p = a, z_p = 0 => x_n = x_p \cos(\alpha), z_n = -x_p \sin(\alpha) \\[1em] \textit{point } 2: \\ x_p = 0, z_p = b => x_n = z_p \sin(\alpha), z_n = z_p \cos(\alpha) \\[1em] \textit{point } 3: \\ x_p = a, z_p = b => x_n = a\cos(\alpha) + b\sin(\alpha),\ z_n = b\cos(\alpha) - a\sin(\alpha) \end{gathered} \tag{33}$$

In the case of the object manipulated under consideration $a = b = w_p/2$. In turn, the tensor of inertia of the manipulated object with respect to the local system of reference placed in the centre of gravity of the object and with axes that have consistent senses and are parallel to the respective axes of the local system of the last link of the manipulator kinematic chain, for the case presented in Fig. 4a, assumes the following form:

$$I_p^p = I_p^l = \begin{bmatrix} B_x^p & 0 & 0 \\ 0 & B_y^p & 0 \\ 0 & 0 & B_z^p \end{bmatrix} \tag{34}$$

where $B_x^p = B_y^p = B_z^p = \frac{m_p w_p^2}{6}$.

5. Example of the analysis of manipulator inertia tensors. Criteria for the selection of positions of centres of gravity of manipulator links

Let us focus on an exemplary analysis of tensors of inertia. Let us take an industrial manipulator *NM7MAR*, whose kinematic schematic view is presented in Fig. 5, due to the simplicity of its design and clearness of the analysis.

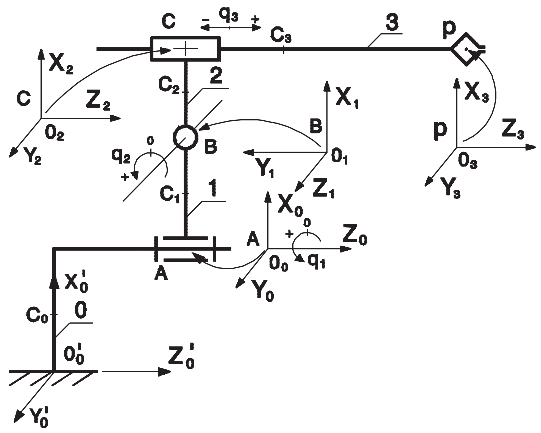

Figure 5 Kinematic scheme of the *NM7MAR* manipulator. Selection of systems of co-ordinates of links

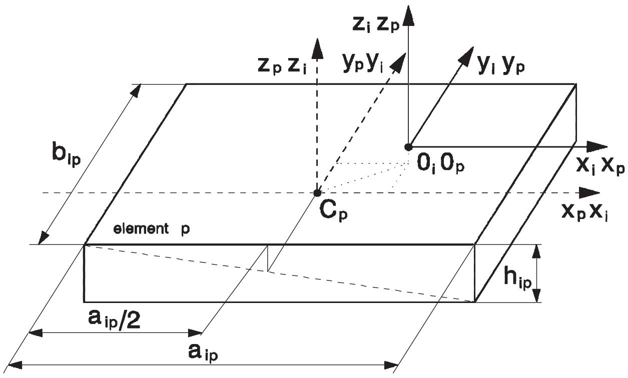

Figure 6 Shape of the p-th element of the i-th link of the exemplary manipulator

Considering the structure of links of this manipulator, three elements in the form of homogeneous cubicoids (taking into account parts that form bearing systems and their housings and systems of drive transfer) have been specified in each link. Let us assume the general case of the p-th element of the link i. Let us assume the geometrical notations of the p-th element as in Fig. 6. Thus, having in mind the assumptions concerning links of the manipulator under analysis, we have $i = 1, 2, 3$; $p = 1(r), 2(k), 3(l)$ - p-th element of the link, m_{ip} - mass of the p-th element of the link, and analysing Figs. 1, 6:

$$\begin{gathered} m_{ip} = \rho_p^i a_{ip} b_{ip} h_{ip} \\ for\ i = 1,2\ and\ p = k => a_{ip} = L_i - \alpha_i - \beta_i \\ for\ i = 3\ and\ p = k => h_{ip} = L_i - \alpha_i - \beta_i \end{gathered} \tag{35}$$

Employing relationship (22), the tensor of inertia of the p-th element with respect to the local system of co-ordinates of the link i that is located in the centre of gravity of the element p, Fig. 6, assumes the form in general:

$$I_{sp}^i = \int\limits_{V_p^i} \begin{bmatrix} (y_m^2 + z_m^2)dm & -x_m y_m dm & -x_m z_m dm \\ -x_m y_m dm & (x_m^2 + z_m^2)dm & -y_m z_m dm \\ -x_m z_m dm & -y_m z_m dm & (x_m^2 + y_m^2)dm \end{bmatrix} \tag{36}$$

In the manipulator under consideration, owing to the assumed positions of local systems of reference, the following equality holds (from Eq. (23)):

$$\overline{r}_m = \overline{r}_{ml} \tag{37}$$

Assuming that the vector $\overline{r}_{cp}^i = [r_{cp}^x, r_{cp}^y, r_{cp}^z]^T$ defines the position of the centre of gravity C_p of the element p in its local system of reference, then the tensor of inertia I_{sp}^i with respect to the local system of reference $x_p y_p z_p 0_p$ assumes the form:

$$I_{sp}^i = \begin{bmatrix} \frac{m_{ip}}{12}(b_{ip}^2 + h_{ip}^2) + m_{ip}\left((r_{cp}^y)^2 + (r_{cp}^z)^2\right) & -m_{ip} r_{cp}^x r_{cp}^y & -m_{ip} r_{cp}^x r_{cp}^z \\ -m_{ip} r_{cp}^x r_{cp}^y & \frac{m_{ip}}{12}(a_{ip}^2 + h_{ip}^2) + m_{ip}\left((r_{cp}^x)^2 + (r_{cp}^z)^2\right) & -m_{ip} r_{cp}^y r_{cp}^z \\ -m_{ip} r_{cp}^x r_{cp}^z & -m_{ip} r_{cp}^y r_{cp}^z & \frac{m_{ip}}{12}(a_{ip}^2 + b_{ip}^2) + m_{ip}\left((r_{cp}^x)^2 + (r_{cp}^y)^2\right) \end{bmatrix} \tag{38}$$

where: $i = 1, 2, 3$; $p = r, k, l$; $m_{ip} = \rho_p^i a_{ip} b_{ip} h_{ip}$ whereas for:
$i = 1, 2$, $p = k => a_{ip} = L_i - \alpha_i - \beta_i$, $m_{ip} = \rho_p^i b_{ip} h_{ip}(L_i - \alpha_i - \beta_i)$;
$i = 3$, $p = k => h_{ip} = L_i - \alpha_i - \beta_i$, $m_{ip} = \rho_p^i a_{ip} b_{ip}(L_i - \alpha_i - \beta_i)$.
The tensor of inertia I_{sp}^i of the p-th element of the link i is equal to:

$$I_{sp}^i = \begin{bmatrix} \frac{m_{ip}}{12}(b_{ip}^2 + h_{ip}^2) & 0 & 0 \\ 0 & \frac{m_{ip}}{12}(a_{ip}^2 + h_{ip}^2) & 0 \\ 0 & 0 & \frac{m_{ip}}{12}(a_{ip}^2 + b_{ip}^2) \end{bmatrix} \tag{39}$$

The determination of the tensor $I_{p,s}^i$ consists in the determination of the tensor of inertia of the element p (in the manipulator under analysis $p = 1(r), 2(k), 3(l)$) with respect to the centre of gravity of the link i.

The vectors of locations of centres of gravity of the p-th element of the link i have the form:

$$\overline{r}_{cp}^i = [r_{cp}^x, r_{cp}^y, r_{cp}^z]^T \tag{40}$$

where: $i = 1, 2, 3$; $p = r, k, l$; in the local systems of reference of the respective elements of the link.
The vectors of positions of the local systems of reference of individual elements of

the link in the local system of reference of elements (link) that is reduced to the centre of gravity of the link take the form:

$$\overline{r}^i_{op} = [x^i_{op}, y^i_{op}, z^i_{op}]^T \tag{41}$$

in the case of the analysed form of the link i:

$$\overline{r}^i_{or} = [x^i_{or}, y^i_{or}, z^i_{or}]^T, \overline{r}^i_{ok} = [x^i_{ok}, y^i_{ok}, z^i_{ok}]^T, \overline{r}^i_{ol} = [x^i_{ol}, y^i_{ol}, z^i_{ol}]^T \tag{42}$$

The tensor of inertia $I^i_{p,s}$ is equal to:

$$I^i_{p,s} = \left[\begin{array}{l} \frac{m_{ip}}{12}(b^2_{ip} + h^2_{ip}) + m_{ip}[(r^y_{cp} + y^i_{op})^2 + (r^z_{cp} + z^i_{op})^2] \\ -m_{ip}[(r^x_{cp} + x^i_{op})(r^y_{cp} + y^i_{op})] \\ -m_{ip}[(r^x_{cp} + x^i_{op})(r^z_{cp} + z^i_{op})] \end{array} \right.$$

$$\begin{array}{l} -m_{ip}[(r^x_{cp} + x^i_{op})(r^y_{cp} + y^i_{op})] \\ \frac{m_{ip}}{12}(a^2_{ip} + h^2_{ip}) + m_{ip}[(r^x_{cp} + x^i_{op})^2 + (r^z_{cp} + z^i_{op})^2] \\ -m_{ip}[(r^y_{cp} + y^i_{op})(r^z_{cp} + z^i_{op})] \end{array} \tag{43}$$

$$\left. \begin{array}{l} -m_{ip}[(r^x_{cp} + x^i_{op})t(r^z_{cp} + z^i_{op})] \\ -m_{ip}[(r^y_{cp} + y^i_{op})(r^z_{cp} + z^i_{op})] \\ \frac{m_{ip}}{12}(a^2_{ip} + b^2_{ip}) + m_{ip}[(r^x_{cp} + x^i_{op})^2 + (r^y_{cp} + y^i_{op})^2] \end{array} \right]$$

The tensor of inertia of the i-th link is thus equal to:

$$I_{si} = \sum_p I^i_{p,s} \tag{44}$$

where: $p = r, k, l$.
Employing relationship (43), Eq. (44) assumes the form:

$$I_{si} = \begin{bmatrix} B^i_x & -B^i_{xy} & -B^i_{xz} \\ -B^i_{xy} & B^i_y & -B^i_{yz} \\ -B^i_{xz} & -B^i_{yz} & B^i_z \end{bmatrix} \tag{45}$$

where: $i = 1, 2, 3$; $p = r, k, l$; B^i_x, B^i_y, B^i_z - mass moments of inertia of the link i with respect to the axis x_i, y_i, z_i, respectively; $B^i_{xy}, B^i_{yz}, B^i_{xz}$ - centrifugal moments of inertia of the link i with respect to the planes of the system of co-ordinates, whose beginning lies in the centre of gravity of the link, whereas:

$$
\begin{aligned}
B_x^i &= \sum_p m_{ip}[\tfrac{1}{12}(b_{ip}^2 + h_{ip}^2) + (r_{cp}^y + y_{op}^i)^2 + (r_{cp}^z + z_{op}^i)^2] \\
B_y^i &= \sum_p m_{ip}[\tfrac{1}{12}(a_{ip}^2 + h_{ip}^2) + (r_{cp}^x + x_{op}^i)^2 + (r_{cp}^z + z_{op}^i)^2] \\
B_z^i &= \sum_p m_{ip}[\tfrac{1}{12}(a_{ip}^2 + b_{ip}^2) + (r_{cp}^x + x_{op}^i)^2 + (r_{cp}^y + y_{op}^i)^2] \\
B_{xy}^i &= \sum_p m_{ip}(r_{cp}^x + x_{op}^i)(r_{cp}^y + y_{op}^i) \\
B_{xz}^i &= \sum_p m_{ip}(r_{cp}^x + x_{op}^i)(r_{cp}^z + z_{op}^i) \\
B_{yz}^i &= \sum_p m_{ip}(r_{cp}^y + y_{op}^i)(r_{cp}^z + z_{op}^i)
\end{aligned}
\tag{46}
$$

Let us consider an exemplary link of the manipulator under analysis, Fig. 7.

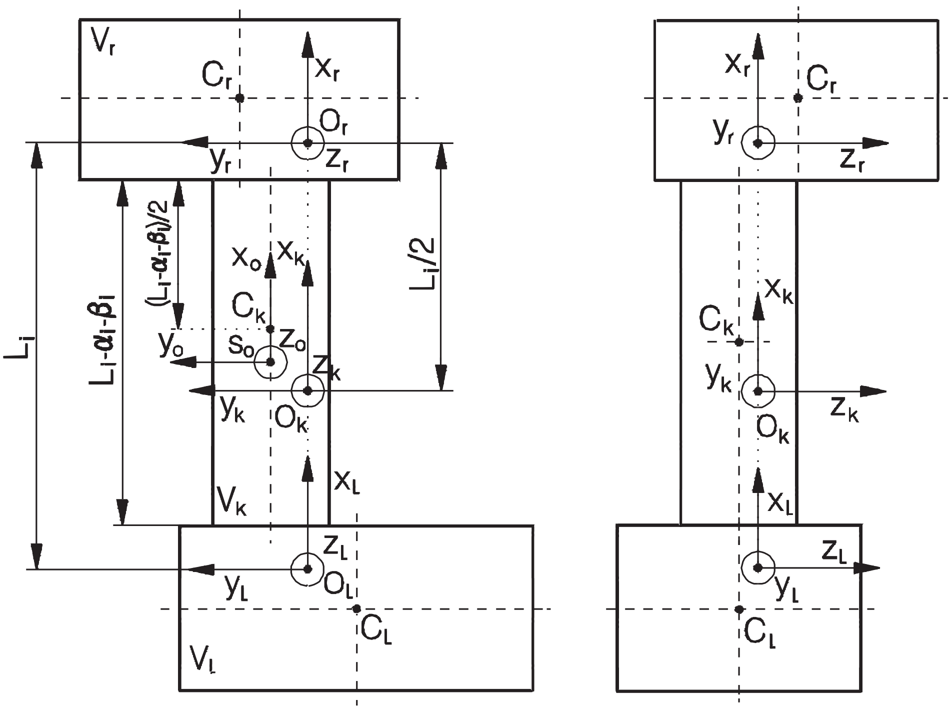

Figure 7 Design scheme of the i-th link of the manipulator under analysis. Systems of co-ordinates correspond to links 1 and 2

For the links of the manipulator under analysis, let us assume the following volumetric relationships of the corresponding elements of the link:

$$V_r = V_k = V_l \tag{47}$$

and of the components of the vectors of positions of centres of gravity of individual link elements in their local systems of reference, respectively:

$$\begin{aligned} r_{cr}^{x} + r_{ck}^{x} + r_{cl}^{x} = 0 \\ r_{cr}^{y} + r_{ck}^{y} + r_{cl}^{y} = 0 \\ r_{cr}^{z} + r_{ck}^{z} + r_{cl}^{z} = 0 \end{aligned} \tag{48}$$

As a result of the assumptions made (Eqs. (47), (48) and (6)-(12)), the position of the centre of gravity of the i-th link of the manipulator in the point whose coordinates are as follows:

$$\begin{aligned} i = 1,2 => (-\tfrac{L_i}{2}, 0, 0) \\ i = 3 => (0, 0, -\tfrac{L_i}{2}) \end{aligned} \tag{49}$$

has been determined in corresponding, local systems of reference of links.

6. Transformations of dimensions between local systems of reference

A transformation of limits of integration will be presented on the example of the description of the orientation of the vector that is described by the limits of integration defined in the systems $x_{ml}y_{ml}z_{ml}0_{ml}$ and $x_m y_m z_m 0_m$. Let us consider the case shown in Fig. 8. Then, matrix of rotation (17) assumes the form:

$$A_o = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \tag{50}$$

After the substitution of Eq. (50) into relationship (23), we obtain:

$$\begin{aligned} x_m = x_{ml}\cos(\alpha) + z_{ml}\sin(\alpha) \\ y_m = y_{ml} =< -\tfrac{w_p}{2}, \tfrac{w_p}{2} > \\ z_m = -x_{ml}\sin(\alpha) + z_{ml}\cos(\alpha) \end{aligned} \tag{51}$$

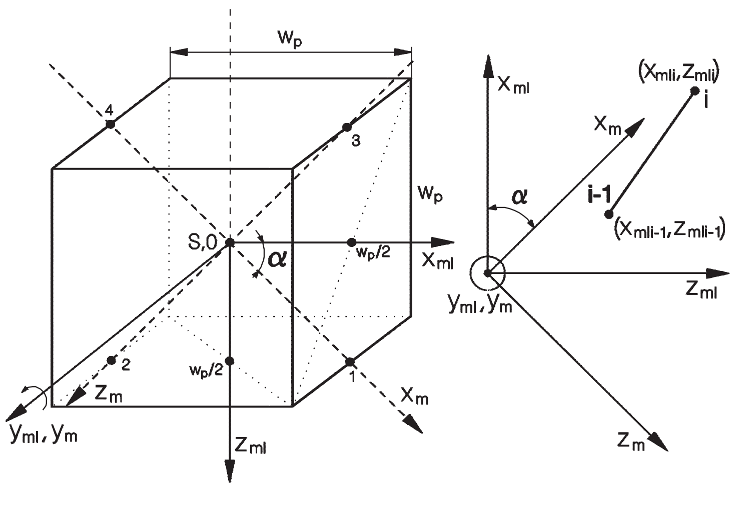

Figure 8 Transformations of dimensions between local systems of reference

As a result of the analysis of Fig. 8, the co-ordinates of selected points ($i = 1$, 2, 3, 4) of the solid body, in the local system of reference $x_{ml}y_{ml}z_{ml}0_{ml}$, are presented in Table 1.

Table 1

Considered point (i =1, 2, 3, 4)	x_{ml}	y_{ml}	z_{ml}
1	$w_p/2$	0	$w_p/2$
2	$-w_p/2$	0	$w_p/2$
3	$w_p/2$	0	$-w_p/2$
4	$-w_p/2$	0	$-w_p/2$

After the transformation of the local systems of reference, according to Eq. (51), the co-ordinates of the points under analysis assume the values included in Table 2, respectively.

Table 2

Point ($i = 1, 2, 3, 4$)	x_m	y_m	z_m
1	$w_p/2\sin(\alpha) + w_p/2\cos(\alpha)$	0	$-w_p/2\sin(\alpha) + w_p/2\cos(\alpha)$
2	$w_p/2\sin(\alpha) - w_p/2\cos(\alpha)$	0	$w_p/2\sin(\alpha) + w_p/2\cos(\alpha)$
3	$-w_p/2\sin(\alpha) + w_p/2\cos(\alpha)$	0	$-w_p/2\sin(\alpha) - w_p/2\cos(\alpha)$
4	$-w_p/2\sin(\alpha) - w_p/2\cos(\alpha)$	0	$w_p/2\sin(\alpha) - w_p/2\cos(\alpha)$

The equations of limits of integration of the cube under consideration, generating the equations of straight lines, with respect to the co-ordinate axes x and z, in the form of a system of equations, have the following form:

$$\left[\begin{array}{l} x_m = \pm\frac{w_p}{2\cos(\alpha)} + z_m tg(\alpha) \\ z_m = \pm\frac{w_p}{2\cos(\alpha)} - x_m tg(\alpha) \end{array}\right. \tag{52}$$

for $\alpha \neq (\pi/2, 3\pi/2)$. In the case when the angle $\alpha = \pi/4$, Fig. 8, the co-ordinates of the points under analysis assume values included in Table 3.

If we simplify equation (52), the corresponding equations of limits of

$$\left[\begin{array}{l} x_m = \pm\frac{\sqrt{2}}{2}w_p + z_m \\ z_m = \pm\frac{\sqrt{2}}{2}w_p - x_m \end{array}\right. \tag{53}$$

integration have the form:

Let us consider a homogeneous cube with a side length $w_p = a$ and let us assume the local systems of reference as in Fig. 9.

Table 3

Considered point ($i = 1, 2, 3, 4$)	x_m	y_m	z_m
1	$\sqrt{2}/2w_p$	0	0
2	0	0	$\sqrt{2}/2w_p$
3	0	0	$-\sqrt{2}/2w_p$
4	$-\sqrt{2}/2w_p$	0	0

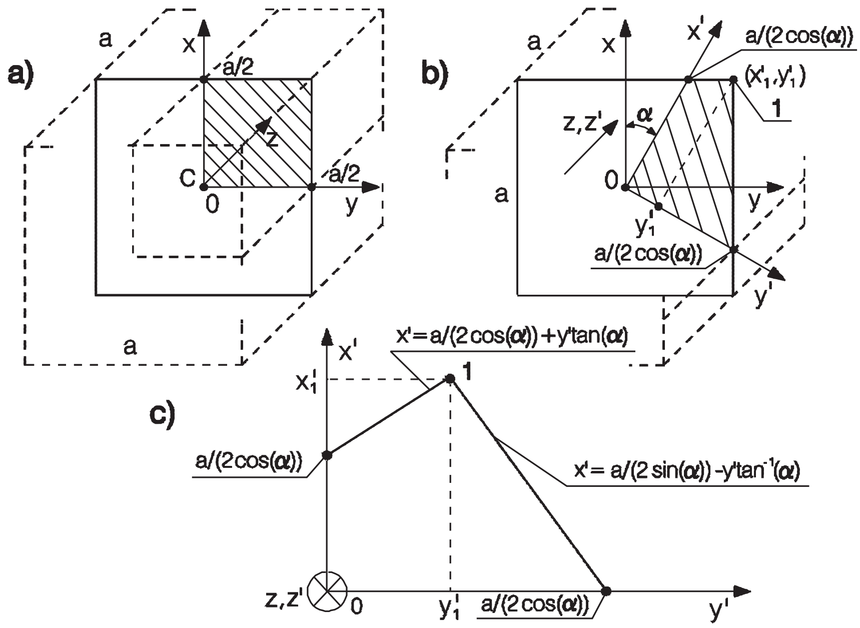

Figure 9 Scheme of transformations of geometrical dimensions in local systems of reference

Solving the system of equations that corresponds to Eq. (52) (that is to say, we have to remember about the notations of the axis of the assumed orientation of systems of reference), for instance through a substitution of Eq. (52b) into (52a)), we obtain the co-ordinates of point 1, Fig. 9, in the form:

$$(x', y') = \Big(\frac{a}{2}\left[\sin(\alpha) + \cos(\alpha)\right], \ \frac{a}{2}\left[\cos(\alpha) - \sin(\alpha)\right]\Big) \tag{54}$$

and other quantities shown in Figs. 9 and 10 as well.

For example, the moment of inertia of the specified part of the solid body, whose cross-section in the plane $x - y$ is hatched in the figures (for the cases from Figs.

9a, 9b, 9c and 10) with respect to the axis z is equal to:

$$B_{z'} = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} [\int_{0}^{\frac{a}{2}\cos(\alpha) - \frac{a}{2}\sin(\alpha)} \int_{0}^{\frac{a}{2\cos(\alpha)} + y'\tan(\alpha)} (x'^2 + y'^2) dx' dy' + \\ + \int_{\frac{a}{2}\cos(\alpha) - \frac{a}{2}\sin(\alpha)}^{\frac{a}{2\cos(\alpha)}} \int_{0}^{\frac{a}{2\sin(\alpha)} - y'\cot(\alpha)} (x'^2 + y'^2) dx' dy'] dz' = \rho \frac{a^5}{24} \tag{55}$$

and (Fig. 10):

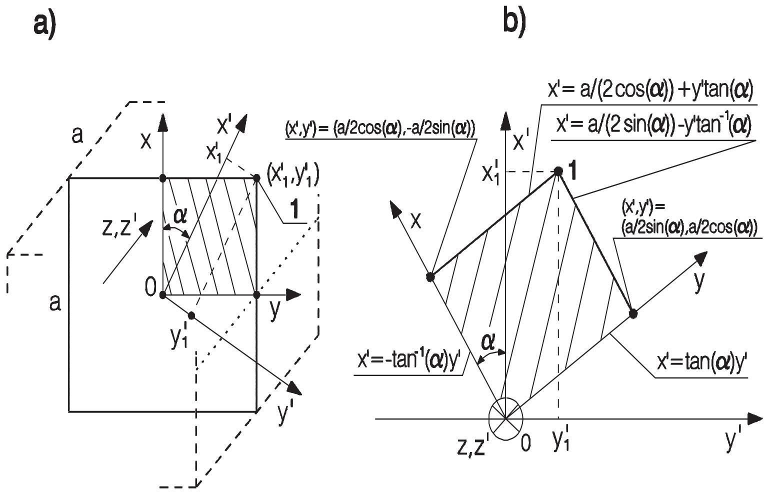

Figure 10 Scheme of transformations of geometrical dimensions in local systems

$$B_{z'} = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[\int_{-\frac{a}{2}\sin(\alpha)}^{0} \int_{-\tan^{-1}(\alpha)y'}^{\frac{a}{2\cos(\alpha)} + y'\tan(\alpha)} (x'^2 + y'^2) dx' dy' + \right. \\ \int_{0}^{\frac{a}{2}[\cos(\alpha) - \sin(\alpha)]} \int_{\tan(\alpha)y'}^{\frac{a}{2\cos(\alpha)} + y'\tan(\alpha)} (x'^2 + y'^2) dx' dy' + \\ \left. \int_{\frac{a}{2}[\cos(\alpha) - \sin(\alpha)]}^{\frac{a}{2}\cos(\alpha)} \int_{\tan(\alpha)y'}^{\frac{a}{2\sin(\alpha)} - y'\tan^{-1}(\alpha)} (x'^2 + y'^2) dx' dy' \right] dz' = \frac{\rho a^5}{24} \tag{56}$$

where: $\alpha \neq (\pi/2,\ 3\pi/2)$, ρg material density of the cube. As a result of the transformation of geometrical dimensions, carried out on the basis of Eqs. (23), (50) for the cases as in Figs. 9 and 10, we obtain:

$$B_{z'} = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{0}^{\frac{a}{2}} \int_{0}^{\frac{a}{2}} (x^2 + y^2) dx dy dz = \rho \frac{a^5}{24} \tag{57}$$

7. Conclusions

An analytical way of the determination of inertia tensor co-ordinates of manipulator links and manipulated objects on the basis of homogenous transformation matrices has been presented. The main idea of this method consists in a division of the solid body under analysis into elements and parts convenient for analysis, an attribution of local systems of reference to them in a way that makes calculations of element tensors of inertia easier, transformations of the local systems of reference, employing matrices of homogeneous transformations, to the system of reference fixed in the centre of gravity of the link. Thus, using matrices of homogeneous transformations that are commonly used in manipulator kinematics, we can simplify the algorithm for the determination of co-ordinates of tensors of inertia of links and objects manipulated.

Acknowledgments

The investigations were supported by the State Committee for Scientific Research (KBN), Poland, within Research Project 5T07A00525.

References

[1] **Armstrong B., Khatib O., Burdick J.**: The explicit dynamic model and inertia parameters of the PUMA 560 arm. *Proc. of the IEEE, Int. Conference on Robotics and Automation*, San Francisco, (1986), 510-518.

[2] **Craig J.J.**: Introduction to robotics, Mechanics and control, WNT, Warsaw (1993), Poland.

[3] **Leyko J.**: Dynamics of material systems, PWN, Warsaw, Poland.

[4] **Morecki A., Knapczyk J.**: Fundamentals of robotics, Theory and elements of manipulators and robots, WNT, Warsaw (1999) , Poland.

[5] **Paul R.P.**: Robot manipulator. Mathematics, Programming and Control, *MIT Press*, Cambridge M.A., (1981).

[6] **Sciavicco L., Siciliano B.**: Modeling and control of robot manipulators. New York, McGraw-Hill Comp., (1996).

[7] **Spong M.W., Vidyasagar M.**: Robot dynamics and control. New York, *J. Wiley*, (1989).

[8] **Szwedowicz J.**: Analysis of dynamics of the manipulator with flexible links. *Ph.D. Dissertation*, Gdansk Technical University, (1991), Poland.

[9] **Dawson D.M., Bridges M.M, Qu Z.**: Nonlinear control of robotic systems for environmental waste and restoration, Prentice Hall Inc., (1995).

[10] **Vukobratovi M.**: Dynamics of manipulation robots - theory and application, (1992).

[11] **Stiquito, Conrad J. M., Mills J. W.**: Advanced experiments with a simple and inexpensive robot, Los Alamitos, IEEE Comp. Society, (1998).