Mechanics and Mechanical Engineering
Vol. 11, No 1 (2007) 37–48

Optimization of Thin-Walled Beams Subjected to Bending in Respect of Local Stability and Strenght

Tadeusz Gałkiewicz, Marian Królak and Tomasz Kubiak
Technical University of Łódź, Department of Strength of Materials and Structures
Stefanowskiego 1/15, 90-924 Łódź, Poland

Received (1 March 2007)
Revised (15 March 2007)
Accepted (8 May 2007)

A method for optimization (with some limitations) of thin-walled beams with a trapezoidal cross-section is presented in the paper. Beams are subjected to bending. They undergo local buckling in the elastic range. The dimensions of beam cross-section walls are optimized. Two cases was consider. In the first one the smallest weight of the beam (the smallest area cross-section) for given critical buckling moment was calculated. In the second case the maximum critical moment for given area of the cross-section was calculated. Next, the ultimate bending moment in the postcritical state is found for the optimized beams.

Keywords: thin-walled beams, buckling, optimization, postbucling state, ultimate load limit

1. Introduction

Thin-walled beams that are light due to the fact their material strength properties are better exploited are often used as load-carrying elements in mechanical (e.g. crane beams) and civil engineering (e.g. bridges) structures. Compressed thin walls of such beams, especially walls of larger widths, are subjected to buckling (local stability loss) at low values of compressive stresses.

The present study is aimed at such a selection of dimensions of the beam cross-section as to obtain either the smallest area of the beam cross-section (the smallest weight of the beam) under the given critical bending moment or the largest critical bending moment (the highest load-carrying capacity of the beam) for the given area of the beam cross-section, as well as the ratio of the thickness to the width t_1/b_1 of the compressed flange..

As thin-walled beams, especially those with closed profiles and flat walls, can operate safely after elastic local buckling, a reserve of the load-carrying capability residing in postcritical elastic states should be employed in the calculations of limit loads [1, 2].

In the present paper, an analytical optimization method of cross-section dimensions of thin-walled single-cell beams with an isosceles-trapezoidal shape (in the special case, with a rectangular shape), subjected to pure plane bending in the plane going through the beam axis and the symmetry axis of the cross-section, will be presented [5]. The considerations will be devoted to beams in which the local buckling of walls occurs within the range of elastic strains ($\sigma_{cr} \leq \sigma_{prop}$) and the load-carrying capacity of beams is in the postcritical (postbuckling) state.

2. Formulation of the Problem. Basic Assumptions

The cross-section geometry of the beams under analysis is defined by the dimensions shown in Fig. 1.

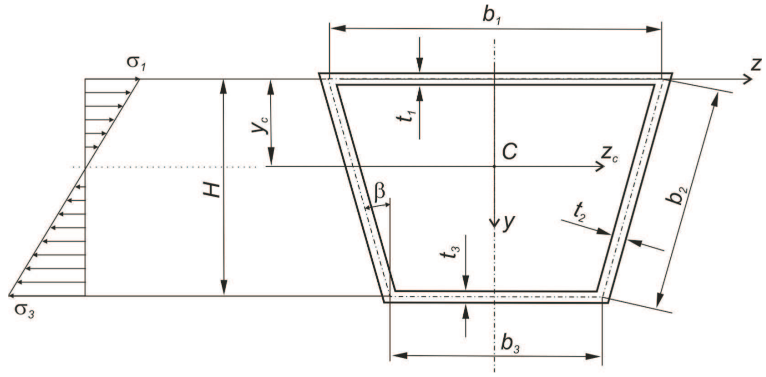

Figure 1 Dimensions of the cross-section of the beams under analysis

We will analyze beams made of an isotropic material characterized by the Young modulus E and the Poisson's ratio ν. Individual walls (flanges and webs) of the beam are assumed to have a constant but different thickness and to be ideally flat (lack of geometrical imperfections). Local buckling of the walls can occur only within the elastic strain range, and the limits of proportionality of the material σ_{prop} in tension and compression are equal as regards the absolute value. It has been assumed that the beam is not prone to local stability loss. Beams are subjected to pure plane bending in the plane going through the beam axis and the symmetry axis of the cross-section. The sense of the bending moment M_b is assumed to be such that the upper flange of the beam (dimensions $b_1 \times t_1$) is under compression, whereas the lower flange is under tension. A distribution of bending stresses (in the prebuckling state) along the beam height is shown in Fig. 1.

In the case of an ideally elastic-plastic material model, the limit of proportionality of the material is identified with the yield point ($\sigma_{prop} = \sigma^y$).

3. Solution to the Problem

In the problem to be solved, the following quantities will be considered as given, namely:

E– Young modulus of the beam material,

ν – Poisson's ratio,
σ_{prop} – limit of proportionality,
β – angle of deflection of side walls (webs) from the vertical plane,
k_1 – stability coefficient of the compressed (upper) flange,
σ_{1cr} – critical stress of local buckling of the upper flange,
σ_{2cr} – critical stress of local buckling of webs,
α – parameter dependent of the position of the center of gravity of the beam cross-section.

Instead of stresses σ_{1cr}, the ratio t_1/b_1 (ratio of the thickness to the width of the flange under compression) can be adopted and then the critical stress can be calculated from the following formula:

$$\sigma_{1cr} = k_1 \frac{\pi^2 E}{12\left(1-\nu^2\right)} \left(\frac{t_1}{b_1}\right)^2 \tag{1}$$

The characteristics of the beam cross-section (Fig. 1), such as:
– cross-section area A,
– position of the center of cross-section area y_c,
– inertia moment I_{zc} with respect to the bending neutral axis z_c,
– sectional modulus Z,
– parameter α,
can be expressed by the following formulae:

$$A = \sum_{i=1}^{4} A_i = \sum_{i=1}^{4} b_i t_i$$

$$\begin{aligned} y_c &= \frac{A_2 + A_3}{A} \cdot H \\ I_{zc} &= \left(\frac{H}{\alpha}\right)^2 \left[A\left(\alpha-1\right) - A_2 \frac{\alpha^2}{3}\right] \\ Z &= \frac{I_{zc}}{y_{\max}} = \frac{H}{\alpha}\left[A - \frac{\alpha^2}{3\left(\alpha-1\right)} A_2\right] \\ \alpha &= \frac{H}{y_c} = 1 + \left|\frac{\sigma_3}{\sigma_1}\right| = \frac{A}{A_2 + A_3} \end{aligned} \tag{2}$$

The quantities occurring in formulae (2) such as b_i, t_i (for i= 1,2,3,4), y_c, H, σ_1 and σ_2 are shown in Fig. 1. The parameter α depends on the position of the center of area of the beam cross-section or on the value of membrane stresses in beam flanges (for a linear distribution of stresses along the beam height H). Further on, it will be assumed that $\alpha \geq 2$, that is to say, $y_c = \frac{1}{2}H$ and $y_{\max} = H - y_c = \frac{H}{\alpha}\left(\alpha - 1\right)$.

The bending moment of the beam (in the precritical state) is described with the formula:

$$M_b = Z \cdot \sigma_3 = \frac{H}{\alpha}\left[\left(\alpha-1\right)A - \frac{\alpha^2}{3} A_2\right] \sigma_1 \tag{3}$$

or

$$M_b = \sigma_1 \cos\beta \left[\frac{\alpha - 1}{\alpha} A \cdot b_2 - \frac{\alpha}{3} t_2 b_2^2 \right] \tag{4}$$

In webs of the analyzed beams, there exist normal stresses that are linearly variable along beam widths, compressive stresses σ_1 in the junction with the upper flange, and tensile stresses σ_3 in the junction with the lower flange (Fig. 1).

Critical stresses of local buckling of the long rectangular plate loaded in such a manner can be described by the formula:

$$\sigma_{2cr} = k_2 \frac{\pi^2 E}{12\left(1 - \nu^2\right)} \left(\frac{t_2}{b_2}\right)^2, \tag{5}$$

where σ_{2cr} corresponds to the highest (as regards the absolute value) compressive stress in the web. The stress σ_{2cr} described by formula (5) has been assumed to be positive.

Minimum values of the stability coefficient k_2 can be calculated from the following approximated formulae:

$$k_2 = 6.15\alpha^2 - 2.35\alpha + 4 \quad for \quad 0 \le \alpha \le 4 \tag{6}$$

or

$$k_2 = 78\alpha - 149.5\sqrt{\alpha} + 79.3 \quad for \quad 1 \le \alpha \le 4 \tag{7}$$

Formula (6) results from the parabolic distribution of $k_2 = k_2(\alpha)$ presented in the form of a diagram in [3], whereas formula (7) has been taken from [4]. In further calculations we will employ formula (7) as it yields slightly lower values of k_2 in the α range of interest to us.

The optimization of the beams under consideration in respect of local stability will be conducted on the assumption that the compressed flange and webs are subjected to buckling simultaneous, i.e.:

$$\sigma_{1cr} = \sigma_{2cr} \tag{8}$$

It follows from this assumption (see formulae (1) and (5)) that:

$$t_2 = \frac{t_1}{b_1} \sqrt{\frac{k_1}{k_2}} b_2 \tag{9}$$

To simplify the further formulae, we will introduce two dimensionless quantities, namely:

$$\begin{aligned} B_1 &= \tfrac{b_1}{t_1} = \pi \sqrt{\tfrac{k_1 E}{12(1-\nu^2)\sigma_{1cr}}} \\ B_2 &= \tfrac{b_1}{t_1} \sqrt{\tfrac{k_2}{k_1}} = \pi \sqrt{\tfrac{k_2 E}{12(1-\nu^2)\sigma_{1cr}}} \end{aligned} \tag{10}$$

Formula (4), after employing relationships (9) and (10), takes the following form:

$$M_{cr} = \sigma_{1cr} \cos\beta \left(\frac{\alpha - 1}{\alpha} A \cdot b_2 - \frac{\alpha}{3} \frac{b_2^3}{B_2} \right) \tag{11}$$

3.1. *Case of A=const*

In this subsection, formulae for the optimal dimensions of the cross-section of flanges and webs and the maximum value of the critical bending moment for the given total area of the beam cross-section (A = const) are derived. When the derivative of the critical bending moment (11) with respect to the web width b_2 equals zero:

$$\frac{\mathrm{d}M_{cr}}{\mathrm{d}b_2} = \sigma_{1cr}\cos\beta\left(\frac{\alpha-1}{\alpha}A - \frac{\alpha}{B_2}b_2^2\right) = 0$$

then:

$$b_2^{opt} = \frac{1}{\alpha}\sqrt{(\alpha-1)\,B_2A}. \tag{12}$$

It follows from (9) and (12) that:

$$t_2^{opt} = \frac{1}{\alpha}\sqrt{(\alpha-1)\,\frac{A}{B_2}} \tag{13}$$

and thus :

$$A_2^{opt} = b_2^{opt}t_2^{opt} = \frac{\alpha-1}{\alpha^2}A. \tag{14}$$

When the optimal web width (12) is substituted into formula (11), a formula for the maximum value of the critical bending moment is obtained:

$$M_{cr}^{\max} = \frac{2}{3}\frac{\alpha-1}{\alpha^2}\sqrt{(\alpha-1)\,B_2}A^{\frac{3}{2}}\sigma_{1cr}\cos\beta, \tag{15}$$

where:

$$\alpha = \frac{A}{A_2^{opt} + A_3^{opt}}. \tag{16}$$

From (16) and (14) we obtain:

$$A_3^{opt} = \frac{A}{\alpha} - A_2^{opt} = \frac{A}{\alpha^2}. \tag{17}$$

Formulae for the remaining optimal dimensions of the beam cross-section are as follows:

$$\begin{gathered} A_1^{opt} = A - 2A_{2opt} - A_3^{opt} = \left(\tfrac{\alpha-1}{\alpha}\right)^2 \cdot A \\ b_1^{opt} = \tfrac{\alpha-1}{\alpha}\sqrt{B_1\cdot A}, \qquad b_3^{opt} = b_1^{opt} - 2b_2^{opt}\cdot\sin\beta \\ t_1^{opt} = \tfrac{\alpha-1}{\alpha}\sqrt{\tfrac{A}{B_1}}, \qquad t_3^{opt} = \tfrac{A_3^{opt}}{b_3} \end{gathered} \tag{18}$$

Taking into consideration the assumption that buckling of the compressed flange and webs is simultaneous and a slightly safer design of the beam with respect to the wall local stability loss, it is postulated to assume the stability coefficient k_1 for the uniformly compressed upper flange to be equal to the coefficient for the compressed, simply supported at all edges, rectangular plate, that is to say, k_1= 4.

For sufficiently thick beam walls (and strictly speaking, for certain values of t_1/b_1 and t_2/b_2), the beam load-carrying capacity is affected by the material strength and not by the local stability loss. In this case, the maximum value of the bending moment (4) has been determined on the assumption that A= const for the specific (assumed) thickness t_2 of the web.

From the condition that $\frac{\mathrm{d}M_b}{\mathrm{d}b_2} = 0$, the optimal width of webs has been obtained:

$$b_2^{opt} = \frac{3\,(\alpha-1)}{2\alpha^2}\frac{A}{t_2},$$

for which the final formula for the maximum bending moment of the beam (with respect to the material strength) has the following form:

$$M_b^{\max} = \frac{3\,(\alpha-1)^2}{2\alpha^2}\frac{A}{t_2}. \tag{19}$$

The variability of the function $M_{cr}(t_2)$ and $M_b^{max}(t_2)$ described with formulae (11) and (19) is plotted in Fig 2. The optimal thickness t_2^{opt} of webs, for which the critical moment of local buckling reaches its maximum, and the limit thickness t_2^{ult} of webs, above which the load-carrying capacity results from the material strength (beam walls are not subject to buckling), are shown in Fig. 2.

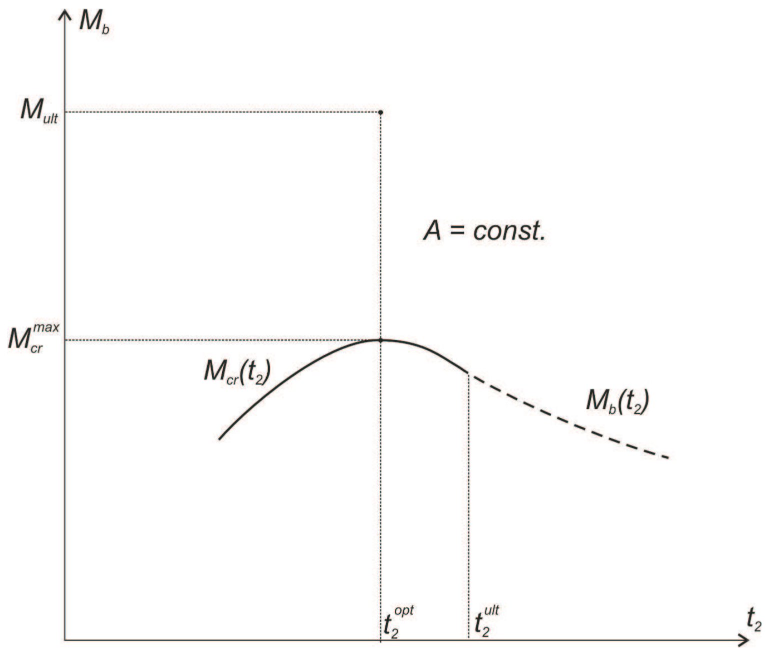

Figure 2 Bending moment as a function of the beam web width t_2

A value of t_2^{ult} can be calculated from the formula:

$$t_2^{ult} = \sqrt{\frac{3\,(\alpha-1)\,A}{2\alpha^2 B_2}}. \tag{20}$$

In Fig. 2, the limit moment M_{ult} corresponding to the optimal thickness t_2^{opt} of webs (and to the optimal dimensions of the beam cross-section with respect to the local stability loss), after reaching of which a beam failure phase begins, is marked with a point. The value of M_{ult} can be calculated with professional FEM software.

3.2. *Case of M $_b$=const*

Here, the optimal values – with respect to the local stability or strength – of thickness and width of beam walls, for which (for the given bending moment M_b) the beam cross-section area will reach its minimum value, will be determined. The formula for the beam cross-section area has been obtained from relationship (11):

$$A = \frac{\alpha}{\alpha - 1}\left(\frac{M_{cr}}{b_2 \sigma_{1cr} \cos\beta} + \frac{\alpha b_2^2}{3B_2}\right). \tag{21}$$

From the condition that $\frac{\mathrm{d}A}{\mathrm{d}b_2} = 0$, we obtain:

$$b_2^{opt} = \sqrt[3]{\frac{3B_2 M_{cr}}{2\alpha\sigma_{1cr}\cos\beta}}, \tag{22}$$

which allows us to derive a formula for the minimum area of the beam cross-section:

$$A^{\min} = \left[\frac{3M_{cr}\alpha^2}{2\sqrt{(\alpha-1)^3}\, B_2 \sigma_{1cr}\cos\beta}\right]^{\frac{2}{3}} \tag{23}$$

The remaining optimal dimensions of the beam cross-section can be calculated from formulae (13) and (18), respectively.

If the thickness of webs t_2 is larger that the limit thickness t_2^{ult}, described by relationship (20), then not the local stability but strength decides about the load-carrying capacity of the beam. Thus, the beam cross-section area determined from relationship (4) is defined by the following formula:

$$A = \frac{\alpha}{\alpha - 1}\left(\frac{M_b}{\sigma_1 b_2 \cos\beta} + \frac{\alpha}{3} t_2 b_2\right) \tag{24}$$

From the condition that $\frac{\mathrm{d}A}{\mathrm{d}b_2} = 0$ (for t_2= const), we obtain:

$$b_2 = \sqrt{\frac{3M_b}{\alpha\sigma_1 t_2 \cos\beta}} \quad for \quad b_2 \geq b_2^{ult} \tag{25}$$

After the substitution of (25) into (24), the formula for $A = A(t_2)$ takes the following form:

$$A = 2\frac{\alpha}{\alpha - 1}\sqrt{\frac{\alpha M_b t_2}{3\sigma_1 \cos\beta}} \quad for \quad t_2 \geq t_2^{ult} \tag{26}$$

The variability of the function $A = A(t_2)$, described by formulae (21) and (26) for $M_b = M_{cr}$= const, is plotted in Fig. 3. The notations on the horizontal axis are the same as in Fig. 2.

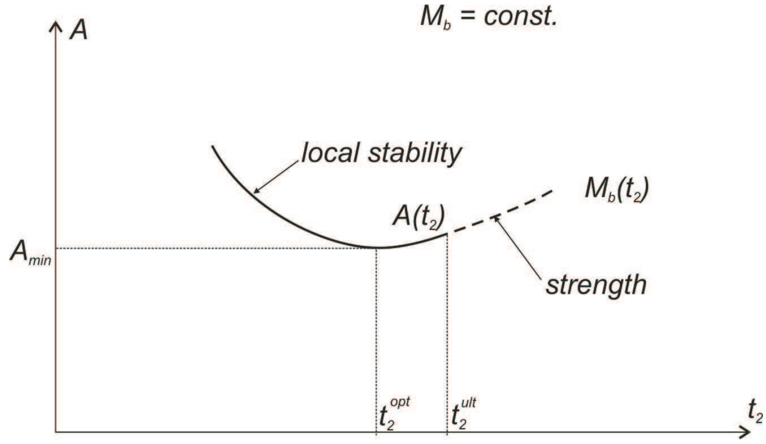

Figure 3 Change in the beam cross-section area A as a function of the web thickness t_2

4. Parametric Analysis

After solving the problem, an influence of some parameters on maximum values of the critical moments of local buckling M_{cr}^{max} has been analyzed and limit values of the bending moments M_{ult} (that cause a failure of the beam) have been calculated by means of an analysis of the postcritical state with FEM codes.

4.1. *Influence of the parameter α on the values of* M_{cr}^{max}

For a rectangular beam ($\beta = 0$) made of steel characterized by the following properties: E= $2{\cdot}10^5$ MPa, $\nu = 0.3$, $\sigma_{prop} = \sigma^{\,y}$= 226 MPa, whose cross-section is equal to A= 0.04 m^2 and the ratio of the thickness to the width of the compressed flange $t_1/b_1 = 0.01$ ($\sigma_{1cr} = 72.3$ MPa), a diagram of M_{cr}^{max} as a function of the parameter $\alpha = 1 + \left|\frac{\sigma_3}{\sigma_{1cr}}\right| = \frac{H}{y_c} = \frac{A}{A_2^{opt} + A_3^{opt}}$,
which varies within the range $2 \le \alpha \le 3$, has been plotted. Results of the calculations are listed in Table 1 and plotted in Fig. 4, respectively.

Table 1 Optimal values of beam parameters as a function of the parameter α

α [–]	A_1^{opt} [m^2]	A_2^{opt} [m^2]	A_3^{opt} [m^2]	σ_3 [MPa]	M_{cr}^{max} [MN·m]
2.00	0.0100	0.0100	0.0100	72.30	1.507
2.25	0.0123	0.00988	0.0079	90.38	1.770
2.50	0.0144	0.00960	0.0064	108.45	1.988
2.75	0.0162	0.00926	0.0053	126.53	2.172
3.00	0.0178	0.00889	0.0044	144.60	2.327

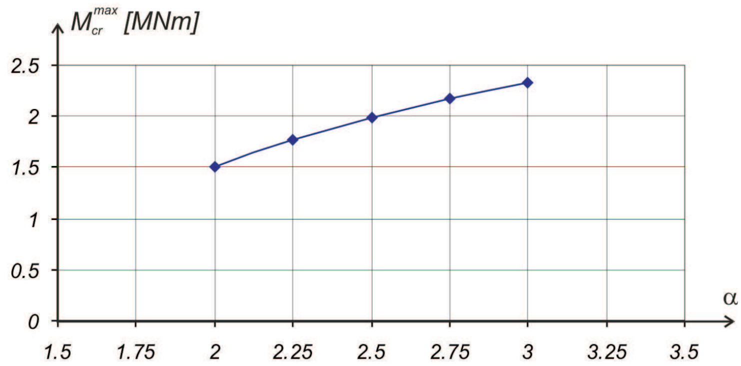

Figure 4 M_{cr}^{max} as a function of the parameter α

Attention should be paid to the values of stresses σ_3 in the lower flange (under tension) which rapidly grow with an increase in the parameter α and can exert an influence on the limit moment M_{ult}.

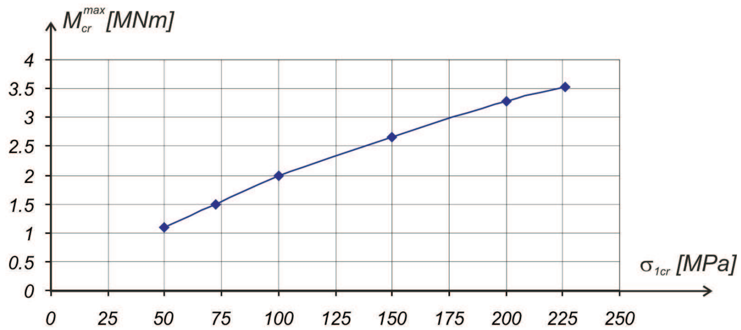

Figure 5 M_{cr}^{max} as a function of the critical stress $\sigma_1 cr$

4.2. *Influence of the critical stresses σ_{1cr} on the values of M_{cr}^{max}*

For the beam with the properties as those mentioned in subsection 4.1 and the parameter $\alpha = 2$, optimal dimensions of the beam wall cross-section have been determined and the maximum values of the critical moment M_{cr}^{max} have been calculated. Figure 5 shows a plot of M_{cr}^{max} as a function of the critical stresses σ_{1cr} for the flange under compression.

The obtained value of M_{cr}^{max}= 3.54 [MNm] for $\sigma_{1cr}= \sigma_{prop}= \sigma^{\,y}$= 226 MPa is the maximum value of the critical moment for the thin-walled rectangular beam (β = 0) with two axes of symmetry of the cross-section α = 2) with the cross-section area A= 0.04 m^2. The optimal critical moment obtained is the limit moment at the same time, as buckling occurs at the stresses equal to the yield point of the beam material. All the numerical data and the results of calculations for the beam under analysis, including the optimal dimensions of the wall cross-section, are presented in Table 2.

Table 2 Data and calculation results

Quantity	Cross-section of the beam			
	rectangular		trapezoidal	
E[MPa]	$2{\cdot}10^5$	$2{\cdot}10^5$	$2{\cdot}10^5$	$2{\cdot}10^5$
ν	0.3	0.3	0.3	0.3
$\sigma^{\ y}$[MPa]	226	226	226	226
σ_{1cr}[MPa]	226	72.3	72.3	72.3
σ_3[MPa]	226	72.3	108.5	144.6
α	2.0	2.0	2.5	3.0
β [deg]	0	0	10	15
k_1	4	4	4	4
k_2	23.875	23.875	37.92	54.36
A[m^2]	0.04	0.04	0.04	0.04
A_1[m^2]	0.01	0.01	0.0144	0.0178
A_2[m^2]	0.01	0.01	0.0096	0.0089
A_3[m^2]	0.01	0.01	0.0064	0.0044
b_1[m]	0.752	1.00	1.20	1.333
b_2[m]	1.176	1.563	1.719	1.8105
b_3[m]	0.752	1.00	0.603	0.3961
t_1[m]	0.0133	0.010	0.0120	0.0133
t_2[m]	0.0085	0.0064	0.0056	0.0049
t_3[m]	0.0133	0.010	0.0106	0.0112
M_{cr}^{opt}[MNm]	3.54	1.50	1.96	2.25
M_{ult}[MNm]	3.54	2.70	3.06	3.36

In the last line of the table, the values of limit bending moments calculated (for the optimized beams) with the FEM ANSYS package are given.

4.3. *Influence of the angle β and the parameter α on the values of M_{cr}^{max} and M_{ult}*

Beside the results for the beam analyzed in subsection 4.2, in Table 2 there are calculation data and results of three other thin-walled beams, including one with a rectangular cross-section ($\beta = 0$, $\alpha = 2$) and two beams with a trapezoidal cross-section ($\beta = 10$ deg, $\alpha = 2.5$ and $\beta = 15$ deg, $\alpha = 3.0$).

All the beams under consideration are made of the same material, and their upper flange (under compression) is subjected to local buckling at the stresses σ_{1cr}= 72.3 MPa (i.e. t_1/t_2= 0.01). Bending moments as a function of the web thickness are plotted in Fig. 6. The behavior of these three beams in the post-critical state, obtained from the calculations made with a FEM ANSYS package, is shown in Fig. 7. The results of calculation is presented in Table 3, where L is the length of the one segment of bended girder. To ensure the same condition in analytical (presented above) and finite element method the length of one segment of the girder L have been asumed to obtain the lowest value of critical bending moment M_{cr}.

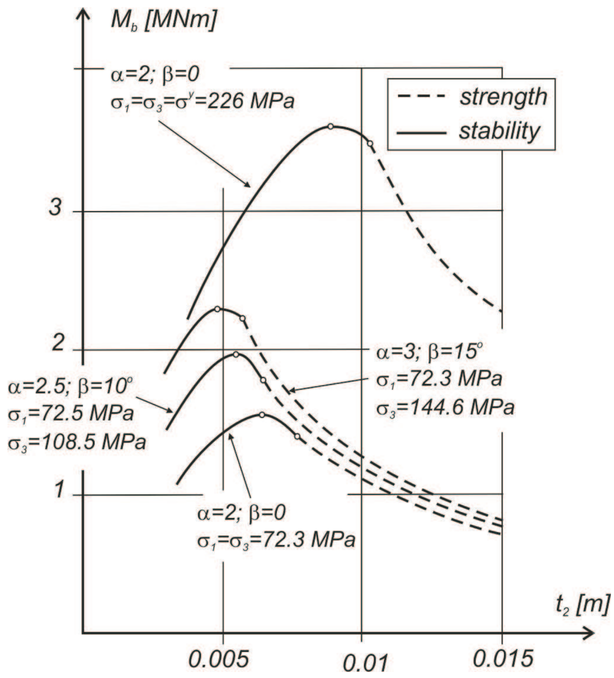

Figure 6 M_b versus t_2

Table 3 Critical and ultimate moment obtained using FEM

	L [m]	M_{cr} [MNm]	M_{ult} [MNm]
rectangle $\beta = 0$, $\alpha = 2$	1.05	1.587	2.697
trapezoid $\beta = 10$, $\alpha = 2.5$	1.22	1.840	3.064
trapezoid $\beta = 15$, $\alpha = 3.0$	1.36	2.074	3.358

5. Conclusions

The solution to the problem presented allows one to design easily thin-walled beams in bending, characterized by optimal dimensions of the wall (flanges and webs) cross-section. Beams subjected to local bending of walls under compression within

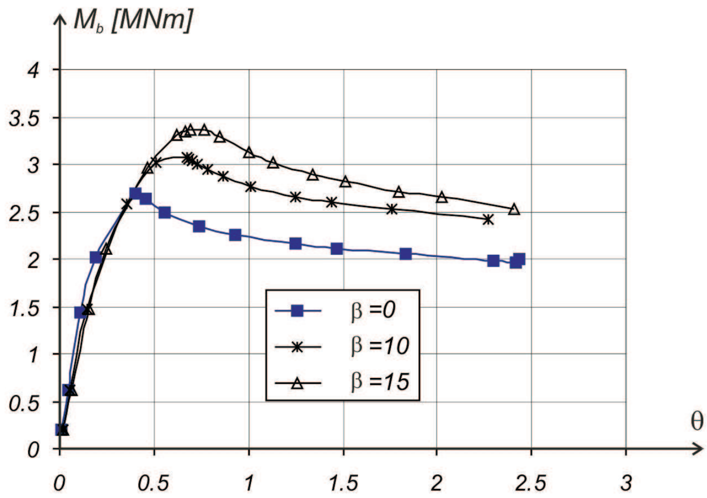

Figure 7 Bending moments in the optimized beams in the precritical and postcritical states as a function of the rotation angle θ of the beam cross-section

the elastic range and reaching the limit load-carrying capacity in the postcritical or critical state can be thus designed. In some cases, the limit moment can be achieved through the material plasticization in the region under tension (in the lower flange), especially in beams with the coefficient $\alpha > 2$, when $|\sigma_3| > |\sigma_1|$. Thin-walled beams with triangular cross-sections need a certain modification of the formulae obtained in this study and therefore should be analyzed separately.

References

[1] **Królak M.** (Ed.), Postcritical states and the limit load-carrying capacity of thin-walled girders with flat walls, *PWN, Warsaw- Łódź,* 1990 (*in Polish*)

[2] **Kołakowski Z., Kowal-Michalska K.**, Selected problems of instabilities in composite structures, *Technical University of Łódź, Łódź*, 1999

[3] **Wolmir A.S.**, Stability of deformable structures, *Nauka*, Moscow, 1967 (*in Russian*)

[4] **Gałkiewicz T.**, Approximate stability analysis of rib walls of the box girder subjected to bending with compression or tension, *Proceedings of Symposium on Stability of Structures*, Łódź, 1979 (*in Polish*)

[5] **Magnucki K., Oswald M.**, Stability and optimization of three-layer structures, *Institute for Sustainable Technologies*, Radom, 2002 (*in Polish*)