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Mass Moments of Interia and Static Moments of a Rigid Body

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This article defines mass moments of inertia and static moments of a rigid body for an "umbrella" system, i.e. a system comprising of three elements: any plane in space, a straight line perpendicular to this plane and their point of intersection (pole). From this system we derive the relationships between mass moments of inertia and static moments for models with two parallel planes and a line perpendicular to them or models with two parallel lines and a plane perpendicular to them. A special case of these relationships is the Steiner's theorem.

Keywords: Rigid body, mass moments of inertia, static moments, Steiner's theorem, "umbrella" system

1. Introduction

The notion, definitions and methods of calculation of mass moments of inertia of a rigid body belong to the fundamental problems of mechanics. Textbooks usually describe this problem in a Cartesian coordinate system and for such a system introduce a notion of the moment of inertia relative to successive planes and the axes of the system and the point (pole) being the centre of the coordinate system. In addition, all relationships occurring between these quantities are defined in the same coordinate system.

In the paper a different model of the system – composed of three elements: a pole, and a straight line and plane perpendicular to each other – has been introduced. The layout of the elements there reminds an umbrella.

For this model, mass moments of inertia and static moments of a rigid body have been defined and fundamental relationships between them have been determined.

2. Description of the model

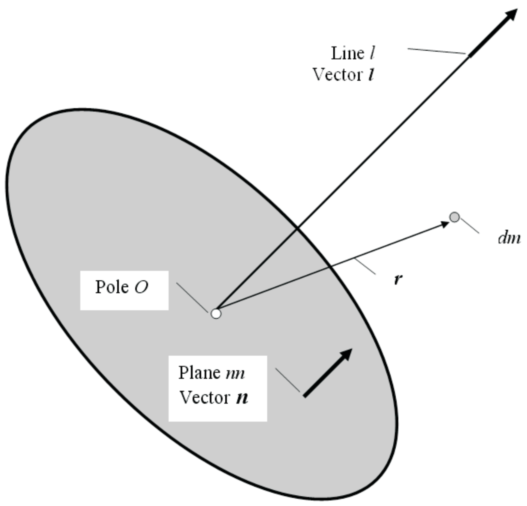

Figure 1 Pole - straight line - plane system (the umbrella)

Let us assume any point (pole) O in space and draw through this point any plane nn and a straight line l, which is perpendicular to this plane. The system of these three elements: the pole–straight line–plane in space is shown in Fig. 1. The position of the plane nn in space is determined by a unit vector $\mathbf{n}$ perpendicular to this plane. The position of the straight line l in space is determined by the unit vector $\mathbf{l}$, lying on this straight line. The position of the point (pole) in space can be given using its coordinates. Thereby, the pole–straight line–plane system is univocally determined in space.

In view of the perpendicularity of the plane nn and the straight line l. we state that the unit vectors $\mathbf{n}$ and $\mathbf{l}$ are perpendicular and have their absolute values equal.

$$|\mathbf{n}| = |\mathbf{l}| = l \qquad \text{and} \quad \mathbf{n} \parallel \mathbf{l} \tag{1}$$

Moments of inertia of any rigid body are products of the second degree of the parameters of a body of a density ρ filling an area V. Thus, these are scalar quantities.

In these products there occurs a radius vector $\mathbf{r}$, determining the position of an elementary mass d_m

$$d_m = \rho\, d_v$$

where: ρ is the volumetric mass density, d_v is the elementary volume.

We introduce a notion of the moment of inertia relative to the pole O, the straight line l and the plane nn in the form of volume integrals from the following squares of scalar products and vector products [1]

$$I_o = \iiint\limits_v \mathbf{r}^2 \rho\, d_v \qquad I_l = \iiint\limits_v (\mathbf{l} \times \mathbf{r})^2 \rho\, d_v \qquad I_{nn} = \iiint\limits_v (\mathbf{n} * \mathbf{r})^2 \rho\, d_v \tag{2}$$

The static moments of a body relative to the same system (the pole O – the straight line l – the plane nn) are moments of the first degree, hence they are vectors and are equal to, respectively

$$\mathbf{m}_o = \iiint\limits_v \mathbf{r} \rho\, d_v \qquad \mathbf{m}_l = \iiint\limits_v (\mathbf{l} \times \mathbf{r}) \rho\, d_v \qquad \mathbf{m}_{nn} = \mathbf{n} \iiint\limits_v (\mathbf{n} * \mathbf{r}) \rho\, d_v \tag{3}$$

3. Relationships between the moments of inertia in relation to the elements of the model.

Let us consider the square of a vector product $(\mathbf{l} \times \mathbf{r})^2$ being a sub–integral function in an expression for the moment of inertia I_1 relative to the straight line l. Using Lagrange's identity and (1), we can express this function by means of scalar products $(\mathbf{l} * \mathbf{r})$ and $(\mathbf{n} * \mathbf{r})$.

$$(\mathbf{l} \times \mathbf{r})^2 = \mathbf{l}^2 \mathbf{r}^2 - (\mathbf{l} * \mathbf{r})^2 = \mathbf{r}^2 - (\mathbf{n} * \mathbf{r})^2 \tag{4}$$

Following multiplication of both sides of expression (4) by ρd_v and integration, we get

$$\iiint\limits_v (\mathbf{l} \times \mathbf{r})^2 \rho\, d_v = \iiint\limits_v \mathbf{r}^2 \rho\, d_v - \iiint\limits_v (\mathbf{l} * \mathbf{n})^2 \rho\, d_v \tag{5}$$

$$I_1 = I_o - I_{nn} \qquad \text{thus} \quad I_o = I_1 + I_{nn} \tag{6}$$

Property (6) can be formulated as follows:

The moment of inertia of a rigid body relative to any pole is a sum of the moments of inertia of this body relative to two, mutually perpendicular elements, a straight line and a plane drawn through this pole. We now introduce the following increments of the moments of inertia:

$$\begin{array}{lll} - & \text{between straight lines 1 and 2 of as} & \Delta I_{1-2} = I_2 - I_1 \\ - & \text{between the poles } O_1 \text{ and } O_2 \text{ as} & \Delta I_{o1-2} = I_{o2} - I_{o1} \\ - & \text{between the planes 11 and 22 as} & \Delta I_{11-22} = I_{22} - I_{11} \end{array} \tag{7}$$

3.1. *Conclusion 1*

Let us assume that (Fig.2)there is a common plane $11 = 22$ ($I_{11} = I_{22}$) containing poles O_1 and O_2

two straight lines l_1 and l_2, passing through the poles O_1 and O_2, respectively, are parallel to each other and perpendicular to this plane ($\mathbf{l}_1 \parallel \mathbf{l}_2$ and $\mathbf{r}_{12} \perp \mathbf{l}_1$).

Then, according to (6) we can write a relationship between the moments of inertia I_{o1}, I_{o2} relative to the poles O_1 and O_2 and the moments of inertia I_1, I_2 relative to the straight lines l_1 and l_2

$$I_{11} = I_{o1} - I_1 \quad \text{and} \quad I_{22} = I_{o2} - I_2$$

Since $I_{11} = I_{22}$, then

$$I_{o2} - I_{o1} = I_2 - I_1 \tag{8}$$

Having used increments (7), we have:

$$\Delta I_{o1-2} = \Delta I_{1-2} \quad \text{for} \quad \mathbf{l}_1 \times \mathbf{l}_2 = 0 \quad \text{and} \quad \mathbf{r}_{12} * \mathbf{l}_1 = 0 \tag{9}$$

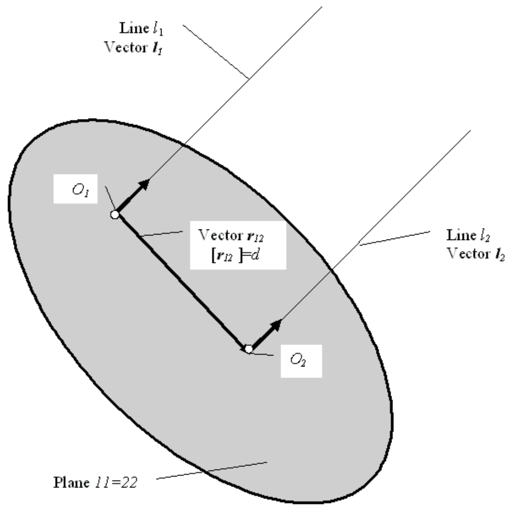

Figure 2 A plane - two straight lines model

An increment of the moment of inertia of a rigid body between any two poles is the same as An increment of the moment of inertia between two parallel straight lines passing through these two poles, and perpendicular to the common plane drawn through these poles.

3.2. *Conclusion 2*

Let us assume that (Fig. 3)

- there exists a straight line l of a moment of inertia I_1 containing the poles O_1 and O_2
- two planes 11 and 22 passing through the poles O_1 and O_2, respectively are parallel and perpendicular to this plane ($\mathbf{n}_1 \parallel \mathbf{n}_2$ and $\mathbf{r}_{12} \parallel \mathbf{l}$)

Then, according to (6), we can write a relationship between the moments of inertia I_{o1}, I_{o2} relative to the poles O_1 and O_2 and the moments of inertia I_{11} and I_{22} relative to both planes 11 and 22.

$$I_l = I_{o1} - I_{11} \quad \text{and} \quad I_l = I_{o2} - I_{22}$$

Thus,

$$I_{o2} - I_{o1} = I_{22} - I_{11}$$

Having used increments (7), we have:

$$\Delta I_{o1-2} = \Delta I_{11-22} \quad \text{for} \quad \mathbf{n}_1 \times \mathbf{n}_2 = 0, \quad \mathbf{r}_{12} \times \mathbf{l} = 0 \quad \text{and} \quad \mathbf{r}_{12} \in l \tag{10}$$

An increment of the moment of inertia between any two poles is the same as An increment of the moment of inertia between two parallel planes passing through these poles and perpendicular to the straight line joining these poles.

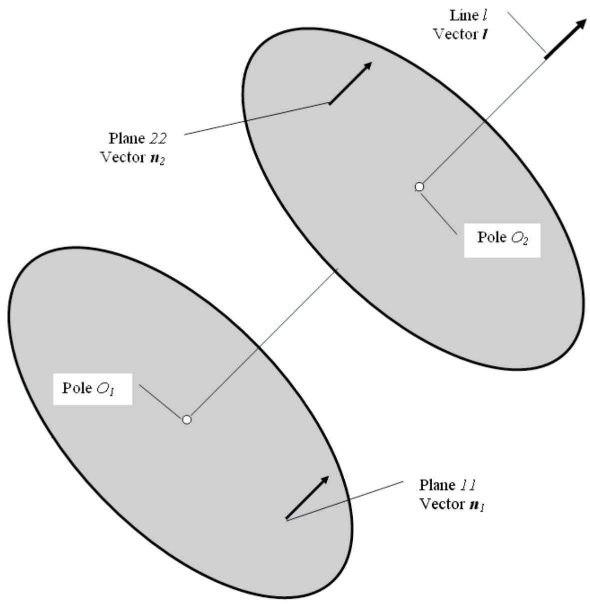

Figure 3 A two planes - straight line model

3.3. *Conclusion 3*

Let us assume that (Fig. 4)

- there are two parallel straight lines l_1 and l_2 containing the poles O_1 and O_2, ($\mathbf{l}_1 \parallel \mathbf{l}_2$)
- two planes 11 and 22 passing through the poles O_1 and O_2, respectively are parallel to each other and perpendicular to both straight lines ($\mathbf{n}_1 \parallel \mathbf{n}_2$ and $\mathbf{n}_1 \parallel \mathbf{l}_1$).

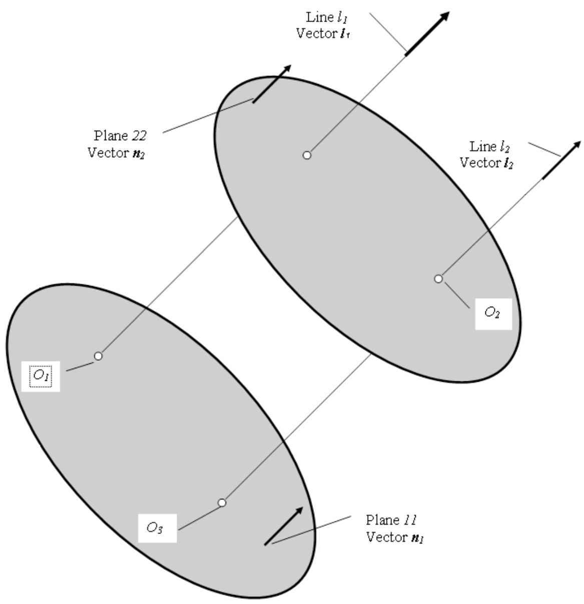

Figure 4 A two planes - two straight lines model

Let us introduce a third pole O_3, lying on the straight line l_2 and, simultaneously, lying in plane 11, the same in which the pole O_1 is lying. It can easily be seen that the poles O_1 and O_2 satisfy the assumptions of conclusion 2, while the poles O_2 and O_3 taken together satisfy the assumptions of conclusion 1.

For the poles O_1 and O_3, making use of (9), we can write

$$I_{o3} - I_{o1} = I_2 - I_1$$

While for the poles O_2 and O_3, making use of (10), we can write

$$I_{o2} - I_{o3} = I_{22} - I_{11}$$

Adding the last two equations, we get

$$I_{o2} - I_{o1} = I_2 - I_1 + I_{22} - I_{11}$$

Having used increments (7), we have:

$$\Delta I_{o1-2} = \Delta I_{1-2} + \Delta I_{11-22} \tag{11}$$

An increment of the moment of inertia of a rigid body between any two poles is a sum of two increments:

- *an increment of the moment of inertia between two parallel straight lines passing through these poles and perpendicular to the common plane drawn through these poles, and*

- *an increment of the moment of inertia between two parallel planes passing through these poles and perpendicular to a straight line joining these poles.*

4. Increment of the static moment vector between two poles

We take into consideration two poles O_1 and O_2 distant from each other by a segment d. Their mutual position is determined

by distance vectors $\mathbf{r}_{12}$ and $\mathbf{r}_{21}$, $\mathbf{O}_1\mathbf{O}_2 = \mathbf{r}_{12}$ and $\mathbf{O}_2\mathbf{O}_1 = \mathbf{r}_{21}$, where

$$r_{12} = r_{21} = d \quad \text{and} \quad \mathbf{r}_{12} = -\mathbf{r}_{21} \tag{12}$$

Let vector–radii $\mathbf{r}_1$ and $\mathbf{r}_2$ determine the position of the elementary mass $d_m = \rho\, d_v$, as shown in Fig. 5.

Thus we have

$$\mathbf{r}_1 = \mathbf{r}_{12} + \mathbf{r}_2 \tag{13}$$

We multiply both sides of equation (13) by$d_m = \rho\, d_v$ and integrate it

$$\iiint\limits_v \mathbf{r}_1 \rho\, d_v = \mathbf{r}_{12} \iiint\limits_v \rho\, d_v + \iiint\limits_v \mathbf{r}_2 \rho\, d_v \tag{14}$$

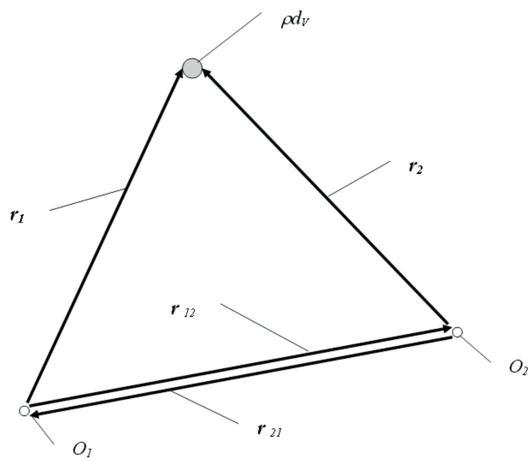

Figure 5 Position of the elementary mass in space

It therefore follows that

$$\begin{aligned} \mathbf{m}_{o1} &= m\mathbf{r}_{12} + \mathbf{m}_{o2} \\ \mathbf{m}_{o2} - \mathbf{m}_{o1} &= -m\mathbf{r}_{12} \end{aligned}$$

An increment of the static moment vector between the points O_1 and O_2 is equal to

$$\Delta \mathbf{m}_{o1-2} = \mathbf{m}_{o2} - \mathbf{m}_{o1} \tag{15}$$

Then, having taken into account (12)

$$\Delta \mathbf{m}_{o1-2} = m\mathbf{r}_{21} \tag{16}$$

An increment of the static moment vector of a rigid body between any two points is a vector parallel to the axis drawn through these points, proportional to the mass of the body and to the distance vector of these points.

Squaring equation (13) we obtain

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{12} + \mathbf{r}_2 \\ \mathbf{r}_1^2 &= \mathbf{r}_{12}^2 + \mathbf{r}_2^2 + 2\mathbf{r}_{12} * \quad \mathbf{r}_2 \end{aligned}$$

Then multiplying the last equation by $d_m = \rho\, d_v$ and integrating, we get

$$\iiint\limits_v \mathbf{r}_1^2 \rho\, d_v = \iiint\limits_v \mathbf{r}_{12}^2 \rho\, d_v + \iiint\limits_v \mathbf{r}_2^2 \rho\, d_v + 2 \iiint\limits_v (\mathbf{r}_{12} * \mathbf{r}_2)\rho\, d_v \tag{17}$$

As it can be proved that

$$\iiint\limits_v (\mathbf{r}_{12} * \mathbf{r}_2)\rho\, d_v = \mathbf{r}_{12} * \iiint\limits_v \mathbf{r}_2 \rho\, d_v = \mathbf{r}_{12} * \mathbf{m}_{o2} \tag{18}$$

besides

$$\iiint\limits_v \mathbf{r}_{12}^2 \rho\, d_v = \mathbf{r}_{12}^2 \iiint\limits_v \rho\, d_v = d^2 m = m\, d^2$$

equation (17) assumes the form

$$I_{o1} = m\, d^2 + I_{o2} + 2\, \mathbf{r}_{12} * \mathbf{m}_{o2} \tag{19}$$

We apply the same procedure to the relationship $\mathbf{r}_2 = \mathbf{r}_{21} + \mathbf{r}_1$as to (13) and having squared and integrated it, we have

$$I_{o2} = m\, d^2 + I_{o1} + 2\, \mathbf{r}_{21} * \mathbf{m}_{o1} \tag{20}$$

Then, we can write it as

$$\begin{aligned} I_{o2} - I_{o1} &= m\, d^2 + 2\, \mathbf{r}_{21} * \mathbf{m}_{o1} \\ \Delta I_{o1-2} &= m\, d^2 + 2\, \mathbf{r}_{21} * \mathbf{m}_{o1} \end{aligned} \tag{21}$$

Adding equations (19) and (20), we obtain

$$\mathbf{r}_{12} * \mathbf{m}_{o2} + \mathbf{r}_{21} * \mathbf{m}_{o1} + m\, d^2 = 0$$

We use relationship (12) $\mathbf{r}_{12} = -\mathbf{r}_{21}$ and transform it into the form

$$\mathbf{r}_{12} * \mathbf{m}_{o1} + \mathbf{r}_{21} * \mathbf{m}_{o2} = m\,d^2 \tag{22}$$

For any two poles of a rigid body distant from each other by a segment d a sum of scalar products of the static moment vectors going out of these poles and the distance vectors from the second pole amounts to $m\,d^2$.

We make use of (12) and (13) again and transform expression (22) into

$$\begin{aligned} m\,d^2 &= \mathbf{r}_{12} * (\mathbf{m}_{o1} - \mathbf{m}_{o2}) = \mathbf{r}_{21} * (\mathbf{m}_{o2} - \mathbf{m}_{o1}) \\ m\,d^2 &= \mathbf{r}_{12} * \Delta\mathbf{m}_{o2-1} = \mathbf{r}_{21} * \Delta\mathbf{m}_{o1-2} \end{aligned} \tag{23}$$

For any two points of a rigid body distant from each other by a segment d a scalar product of the vector of distances of these points multiplied by the vector of increment of the static moment between these points is equal to a product of the body mass and the square of the distances between these points.

5. Increments of the moments of inertia between two elements of a system.

Let us return to expression (21)

$$\Delta I_{o1-2} = m\,d^2 + 2\,\mathbf{r}_{21} * \mathbf{m}_{o1}$$

Let us put expression (23) of the form given below into it

$$m\,d^2 = \mathbf{r}_{21} * (\mathbf{m}_{o2} - \mathbf{m}_{o1})$$

and reduce similar terms. Having taken into consideration (7) and (12) we have

$$\Delta I_{o1-2} = \mathbf{r}_{21} * (\mathbf{m}_{o1} + \mathbf{m}_{o2}) = -\mathbf{r}_{12} * (\mathbf{m}_{o1} + \mathbf{m}_{o2}) \tag{24}$$

Let us introduce a designation of the vector of a sum of the static moments of a rigid body for any two points

$$\sum \mathbf{m}_{o1-2} = \mathbf{m}_{o1} + \mathbf{m}_{o2} \tag{25}$$

We can now the increment of the moment of inertia (24) write in the form

$$\Delta I_{o1-2} = \mathbf{r}_{21} * \sum \mathbf{m}_{o1-2} \tag{26}$$

An increment of the moment of inertia of a rigid body between any two poles is equal to a scalar product of the vector of distances of these points and the vector of a sum of the static moments at these points.

If the assumptions of conclusion 1 (Fig. 2) are satisfied, i.e. when the poles O_1 and O_2 are the points of intersection of the two parallel straight lines l_1 and l_2 with a straight lines perpendicular to it, then according to (19) we have

$$\Delta I_{o1-2} = \Delta I_{1-2}$$

which means that

$$\Delta I_{1-2} = \mathbf{r}_{21} * \sum \mathbf{m}_{o1-2} \tag{27}$$

An increment of the moment of inertia of a rigid body between two parallel straight lines is equal to a scalar product of the vector of distances of the two nearest points of these straight lines and the vector of a sum of the static moments at these points.

If one of these points (e.g. pole 1) is the centre of mass of a body, the static moment at this point is equal to zero ($\mathbf{m}_{o1} = 0$), then – having taken into consideration (23) – from equation (27) we obtain Steiner's theorem

$$I_2 = I_1 + md^2$$

where:

- I_2 is the moment of inertia of a body in relation to a straight line parallel to the central axis,
- I_1 is the moment of inertia of a body with respect to a straight line, being the central line,
- d is the distance of both parallel straight lines.

If the assumptions of conclusion 2 are met (Fig. 3), i.e. when the poles O_1 and O_2 are points of intersection of two parallel planes 11 and 22 with a straight line perpendicular to them, then – according to (10) – we have

$$\Delta I_{o1-2} = \Delta I_{11-22}$$

which means that

$$\Delta I_{11-22} = \mathbf{r}_{21} * \sum \mathbf{m}_{o1-2} \tag{28}$$

An increment of the moment of inertia of a rigid body between two parallel planes is equal to a scalar product of the vector of distances of the traces of these lines with a straight line perpendicular to it and the vector of a sum of the static moments at these points.

If one of these points (e.g. pole 1) is the centre of mass of a body, we notice again that the static moment at this point is equal to zero ($\mathbf{m}_{o1} = 0$), then – having taken into consideration (23) – we obtain the following relationship from equation (28)

$$I_{22} = I_{11} + m\,d^2 \tag{29}$$

where:

- I_{22} is the moment of inertia of a body in relation to any plane parallel to the plane containing the centre of mass of the body
- I_{11} is the moment of inertia of a body in relation to the plane containing the centre of mass of the body
- d is the distance between the parallel planes

The moment of inertia of a rigid body in relation to any plane is equal to the moment of inertia in relation to a plane parallel to it and passing through the centre of mass of the body increased by a product of the mass of the body and a square of the distances between the planes.

In this case, when the pole O_1 is the centre of mass of a body – after substituting ($\mathbf{m}_{o1} = 0$) and taking into account relationship (23) – it results from equation (27) that

$$I_{o2} = I_{o1} + m\,d^2 \tag{30}$$

where:

- I_{o2} is the moment of inertia of a body in relation to any pole
- I_{o1} is the moment of inertia of a body in relation to the pole–centre of mass
- d is the distance between both poles.

The moment of inertia of a rigid body in relation to any pole is equal to the moment of inertia in relation to the pole which is the centre of mass of the body increased by a product of the mass of the body and the square of the distances between both poles.

6. Summary

The formulae derived before (26–28) can be written as one and formulated as one general statement

An increment of the moment of inertia of a rigid body between two parallel planes or straight lines is the same as an increment of the moment of inertia between two poles which are traces of these planes (straight lines) with a common normal, whereas an increment of the moment of inertia of a rigid body between any two poles is equal to a scalar product of the vector of the distances between these points and the vector of a sum of the static moments at these points.

7. Final Conclusions

In the paper it has been proved that in a rigid body there are relationships between products of the second degree, moments of inertia in relation to any pole–straight line–plane system and products of the first degree, vectors of the static moments calculated in relation to the same system. These relationships are independent of the position of the pole–straight line–plane system in relation to a Cartesian or any other coordinate system assumed in this body since these relationships are scalar products of the vectors and the latter are independent of a system of reference.

In the paper relationships between increments of the mass moments of inertia of a body and scalar products of the vectors of the mutual position of the system elements and the vectors of the static moments of a rigid body.

In addition, well-known Steiner's theorem, concerning the moments of inertia between two parallel axes, one of which is the central axis, results from these relationships as a special case.

Nomenclature

I_o, I_{nn}, I_l	mass moments of inertia relative to the pole O, the plane nn and the straight line l
$\mathbf{m}_o$, $\mathbf{m}_{nn}$, $\mathbf{m}_l$	static moments of a body to the same system
m	mass of a rigid body