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Dynamic Stability of Rectangular Orthotropic Plates Subjected to Combined In–plane Pulse Loading in the Elasto–plastic Range

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Dynamic stability of rectangular simply supported plates has been investigated in the elasto–plastic range. The paper concerns orthotropic plates subjected to combined in–plane rectangular pulse loading of short duration corresponding usually to the period of natural fundamental flexural vibrations. Analytical–numerical solution of dynamic stability of plate being part of a structure has been presented. For orthotropic material Hill's criterion of plasticity is applied. The calculations have been performed taking into account the plastic flow theory for composite material. The static tensile stress–strain curve has been approximated by bilinear characteristic. On the diagrams the plate deflection as a function of dynamic load is shown. The large–deflection equations are solved by Galerkin's method. Nonlinear differential equations dependent on time have been computed applying Runge–Kutta fourth–order method.

Keywords: Composites, dynamic stability of plates, plasticity, failure criteria

1. Introduction

At present the design of thin–walled constructions made of composites is generally applied in the technology. Lately composites have been used commonly in army, automotive industry, aircraft industry as well as marine structures. The advantages of application of composites structures are huge because they include the high strength and stiffness properties and additionally are light. Despite the high strength the composites are characterised by very good fatigue life, sound insulation properties or electromagnetic transparency and can work in the wide scope of temperature (usually to 120^o C).

In the literature the character of load is defined by amplitude, duration and shape of pulse [6]. The load may be as follows: impact load (the waves phenomena and takeover of hit energy should be taken into consideration), dynamic load and quasi–static load. It has been assumed that the effect of dynamic buckling takes place at moderate great amplitudes of loading and duration equal or close to the period of natural fundamental vibrations of structures.

The term of dynamic stability loss contains itself a lot of physical phenomena and can be defined in many different ways. The constructions can bear the pulse loading significantly greater than the static buckling load but the dynamic buckling occurs under the condition that construction should include initial deflections (imperfections).

Seeking the critical dynamic load different criteria can be applied, determined by the appearance of some particular phenomena during dynamic buckling (for example obtaining the assumed proper deflection or shortening) or criteria allowing to find out failure load. Taking into account only the elastic range, in order to determine critical dynamic load Budiansky–Hutchinson's criterion [9, 10] or Volmir's criterion [19] can be used. According to Budiansky-Hutchinson's criterion dynamic stability loss occurs when some dynamic load causes the rapid increase of plate deflection. Volmir's criterion assumes the dynamic stability loss of plates (panels, plated structure) when the plate deflection reaches the value equal to the one or half plate thickness.

In the literature one can find many works devoted to the analysis of dynamic stability of structures. Zizicas [23] as the first presented in 1952 theoretical solution for simply supported plate. Volmir [19] showed response of rectangular plate for various shapes pulse loading. Authors Weller, Abramovich and Yaffee [1] applied Budiansky–Hutchinson's criterion for isotropic plates under sinusoidal pulse. The scientists Air–Gur and Simonetta in paper [2] analysed composite plate clamped on all its edges and proposed four criteria of dynamic stability loss. Petry and Fahlbush [16] obtained many results for isotropic plate considering different pulse duration and initial imperfection. Those authors proposed failure criterion which assumes that the construction is dynamic stable under pulse loading when the effective stress (found by Huber–Mises formula) in each point of construction is not greater than limit stress of material during dynamic buckling.

The aim of this paper is to investigate dynamic stability of composite plates considering elasto–plastic range of material. It was desirable to conduct the research on the behaviour of plate subjected to untypical combined load that earlier was considered only in static solution. Apart from that, in the work the maximal dynamic load, which the plate can carry, has been searched taking into account the material characteristics.

2. Problem formulation

In the figure 1 the thin–walled structure subjected to the vertical dynamic force is shown [21]. The vertical force dependent upon the time causes in the considered upper flange of the structure the linearly variable dynamic bending moment.

The value of the force $N_x^{(2)}(t)$ depends on the increase (decrease) of the bending moment along the structure and is described by the coefficient η (Eq. 2). The force $T_2(t)$ is caused by the torque and the shear force. At simple torsion of the thin - walled structure the absolute value of the force $T_2(t)$ is constant on the whole periphery of the plate. In study the part of the compressed flange of the structure is considered, restricted by the membranes and the webs, treated as a rectangular plate simply supported on all its edges. This assumption results from the fact that the thickness of webs and membranes is considerably smaller than the flange one.

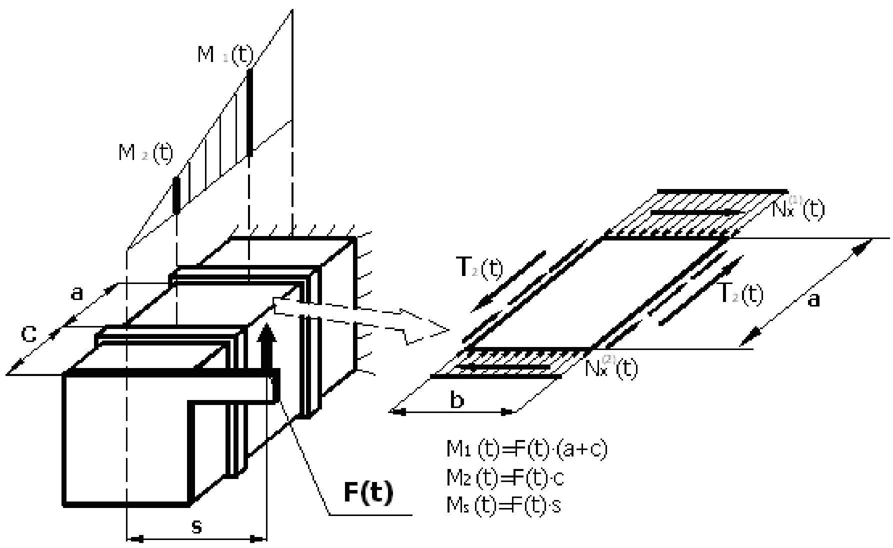


Figure 1 The structure loaded by pulse force F(t) causing bending and torsion

Let's consider a rectangular plate (flange of the construction shown in Fig. 1) of length $\boldsymbol{a}$, width $\boldsymbol{b}$ and constant thickness $\boldsymbol{h}$, simply supported along all its edges, loaded as shown in Fig. 2. The plate has the initial imperfection $\boldsymbol{w}_0$ in z – direction. Loading of a plate is described by two coefficients [21]:

$$\eta = \frac{N_x^{(2)}(t)}{N_x^{(1)}(t)} \qquad \zeta = \frac{T_2^{(}t)}{N_x^{(1)}(t)}. \tag{1}$$

In analysed one–layer composite plate it is assumed that the principal directions of orthotropy coincide with plate edges. Material properties of plates are described by: E_1, E_2 – Young's moduli in 1-th and 2-nd direction respectively, ν_{21}, ν_{12} – Poisson's ratio, G_{12} – shear modulus in 1-2 plane.

3. Solution of the problem

Using the classical thin plates theory the large-deflection equations have been delivered and solved by Galerkin's method [14]. The problem is conducted on the basis of the analytical-numerical method of solution. Some of the obtained results are compared with the finite element method calculations (ANSYS 9.0).

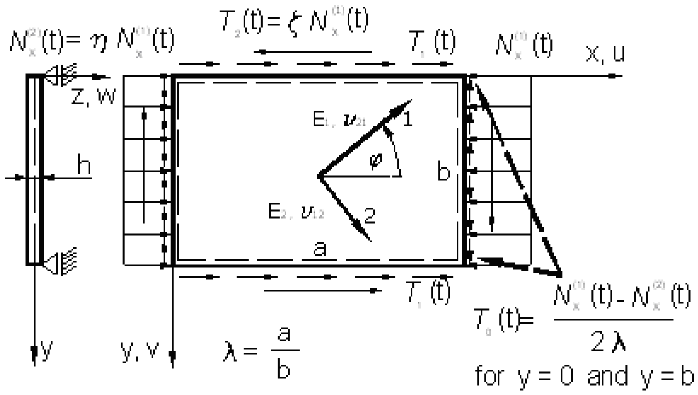
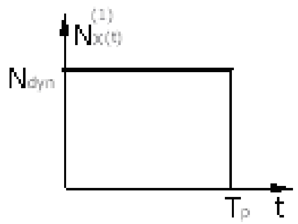


Figure 2 Loading and geometry of a composite plate



for $0 \le t \le T_p$ $\quad N_x^{(1)} = N_{dyn}^{(1)}$ for $t > T_p$ $\quad N_x^{(1)} = 0$

The nonlinear strain – displacement relations considering the initial imperfection are assumed as [14]:

$$
\begin{aligned}
\varepsilon_x &= \varepsilon_{xm} + \varepsilon_{xb} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 (w - w_0)}{\partial x^2} \\
\varepsilon_y &= \varepsilon_{ym} + \varepsilon_{yb} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 (w - w_0)}{\partial y^2} \\
\gamma_{xy} &= \gamma_{xym} + \gamma_{xyb} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} - 2z\frac{\partial^2 (w - w_0)}{\partial x \partial y}
\end{aligned}
\tag{2}
$$

where

u, v, w – displacement functions in x, y, z direction, respectively,

w –total deflection,

w_0 – initial imperfection,

$\varepsilon_{xm}, \varepsilon_{ym}, \gamma_{xym}$ – membrane strains,

$\varepsilon_{xb}, \varepsilon_{yb}, \gamma_{xyb}$ – bending strains.

In the analysis the plastic flow theory for orthotropic material is used [7, 8, 12, 13]. The stress and strain variations for composite plate in the elasto-plastic range considering Prandtl–Reuss equations have been derived in a form [12]:

$$\begin{aligned} \delta\sigma_x &= \bar{R}_{11}\delta\varepsilon_x + \bar{R}_{12}\delta\varepsilon_y + \bar{R}_{13}\delta\gamma_{xy} \\ \delta\sigma_y &= \bar{R}_{21}\delta\varepsilon_x + \bar{R}_{22}\delta\varepsilon_y + \bar{R}_{23}\delta\gamma_{xy} \\ \delta\tau_{xy} &= \bar{R}_{31}\delta\varepsilon_x + \bar{R}_{32}\delta\varepsilon_y + \bar{R}_{33}\delta\gamma_{xy} \end{aligned} \tag{3}$$

where $\bar{R}_{11}$, $\bar{R}_{12}$, $\bar{R}_{21}$, $\bar{R}_{13}$, $\bar{R}_{31}$, $\bar{R}_{22}$, $\bar{R}_{23}$, $\bar{R}_{32}$, $\bar{R}_{33}$ are variable coefficients depending on the loading, material property and effective stress (See appendix).

The membrane forces $\boldsymbol{N}_x, \boldsymbol{N}_y, \boldsymbol{N}_{xy}$ and bending and twisting moments $\boldsymbol{M}_x, \boldsymbol{M}_y, \boldsymbol{M}_{xy}$ are defined as [14]:

$$\begin{aligned} \{N_x\ , N_y,\, N_{xy}\} &= \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} \{\sigma_{xm}, \sigma_{ym},\, \tau_{xym}\}\, dz \\ \{M_x\ , M_y,\, M_{xy}\} &= \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} z\,\{\sigma_{xb}, \sigma_{yb},\, \tau_{xyb}\}\, dz \end{aligned} \tag{4}$$

The differential equations of the dynamic equilibrium of thin plates are [14]:

$$\begin{aligned} N_{x,x} + N_{xy,y} &= \rho h \frac{\partial^2 u}{\partial t^2} \\ N_{y,y} + N_{xy,x} &= \rho h \frac{\partial^2 v}{\partial t^2} \\ (N_x w_{,x} + N_{xy} w_{,y} - M_{x,x} - M_{xy,y})_{,x} & \\ +(N_y w_{,y} + N_{xy} w_{,x} - M_{y,y} - M_{xy,x})_{,y} &= \rho h \frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{5}$$

where ρ – material density of a plate.

Applying equations (3) and (4), introducing Airy's force functions

$$(N_x = \frac{\partial^2 \Phi}{\partial y^2} h, \quad N_y = \frac{\partial^2 \Phi}{\partial x^2} h, \quad N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} h)$$

after the neglecting the in-plane terms of inertia

$$\frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial^2 v}{\partial t^2} = 0$$

the three equations of the dynamic equilibrium (5) reduce to the one which follows as:

$$\begin{aligned}\rho h\frac{\partial^2 w}{\partial t^2} &= h\left(\frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)\\ &\quad-\int\limits_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{\partial^4(w-w_0)}{\partial x^2\partial y^2}\left(2\bar{R}_{12}+4\bar{R}_{33}\right)+\frac{\partial^4(w-w_0)}{\partial x^4}\bar{R}_{11}\right.\\ &\quad\left.+4\frac{\partial^4(w-w_0)}{\partial x^3\partial y}\bar{R}_{13}+4\frac{\partial^4(w-w_0)}{\partial x\partial y^3}\bar{R}_{23}+\frac{\partial^4(w-w_0)}{\partial y^4}\bar{R}_{22}\right]z^2dz\end{aligned}\tag{6}$$

Investigating the behavior of a plate in the elasto-plastic region relations between stresses and strains have to be known in entire range. In the computations the experimental tensile stress–strain curve is represented by the empirical equations of suitable forms. In this paper real static tensile curves have been approximated by bilinear characteristics for composite material, shown in Fig. 3. The parameters σ_{10}, $\sigma_{20}$are initial yield stresses in 1-th and 2-nd direction, respectively.

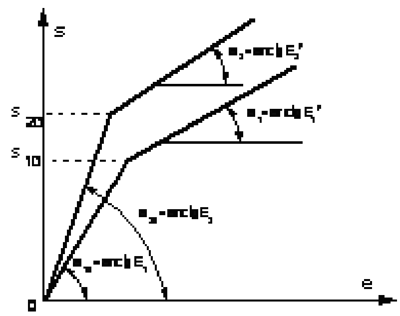


Figure 3 Approximated material characteristics for composite

The moment of the entering from the elastic region into elasto–plastic region is determined by yield criterion. In order to analyse an orthotropic material the anisotropic criteria have to be considered. In this work Hill's criterion for orthotropic materials has been applied.

For plane stress state Hill's criterion is written in a form [7, 12, 13, 15]:

$$\sigma_{eff}^2=\bar{a}_1\sigma_1^2+\bar{a}_2\sigma_2^2-\bar{a}_{12}\sigma_1\sigma_2+3\bar{a}_3\tau_{12}^2,\tag{7}$$

where, σ, τ are stress components, $\bar{a}_1 \div \bar{a}_3$ are parameters of anisotropy and σ_{eff} means effective stress.

For simply supported plate it is assumed that all edges remain straight and parallel during dynamic buckling.

The initial and boundary conditions:

- the initial conditions: for $t = 0$

$$w = w_0 \qquad \frac{\partial w}{\partial t} = 0$$

- boundary conditions:

$$\begin{aligned}
w(x,y)\big|_{x=0}^{x=a} &= 0 \\
M_x\big|_{x=0}^{x=a} &= 0 \\
N_x(x=0) &= \frac{1}{b}\int_0^b \Phi,_{yy}(\eta)dy = -N_x^{(1)}\eta \\
N_x(x=a) &= \frac{1}{b}\int_0^b \Phi,_{yy}(\eta)dy = -N_x^{(1)}(1-\eta) - N_x^{(1)}\eta \\
N_{xy}\big|_{x=0}^{x=a} &= \frac{1}{b}\int_0^b \Phi,_{xy}\,dx = 0
\end{aligned}$$

$$\begin{aligned}
w(x,y)\big|_{y=0}^{y=b} &= 0 \\
M_y\big|_{y=0}^{y=b} &= 0 \\
N_y\big|_{y=0}^{y=b} &= \frac{1}{a}\int_0^a \Phi,_{xx}(\eta)dx = 0 \\
N_{xy}(y=0) &= -\frac{1}{a}\int_0^a \Phi,_{xy}(\eta)dy = -\frac{1}{2}\frac{b}{a}(\eta-1)N_x^{(1)}(t) \\
N_{xy}(y=b) &= -\frac{1}{a}\int_0^a \Phi,_{xy}(\eta)dy = \frac{1}{2}\frac{b}{a}(\eta-1)N_x^{(1)}(t)
\end{aligned}$$

In order to satisfy boundary conditions the deflection function is taken as:

$$w(x,y,t) = w_{11} + w_{21} = A_{11}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + A_{21}(t)\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} \tag{8}$$

Assumed two parameters of deflection function (8) results from the possibility of calculation program but for considered material property and dimensions of plate this function is significantly.

Before the integration of the dynamic equilibrium equation (5), the plate has been divided into $l \times s \times t = 10 \times 10 \times 4$ equal elements pieces and for each element variable coefficients $\bar{R}_{11}^{l,s,t}$, $\bar{R}_{22}^{l,s,t}$, $\bar{R}_{33}^{l,s,t}$, $\bar{R}_{12}^{l,s,t}$, $\bar{R}_{13}^{l,s,t}$, $\bar{R}_{23}^{l,s,t}$ are attributed.

The differential equations system depending upon the time has been computed in Mathematica 5.0 code using Runge–Kutta fourth–order method. In Fig. 4 the flow chart of main program is presented.

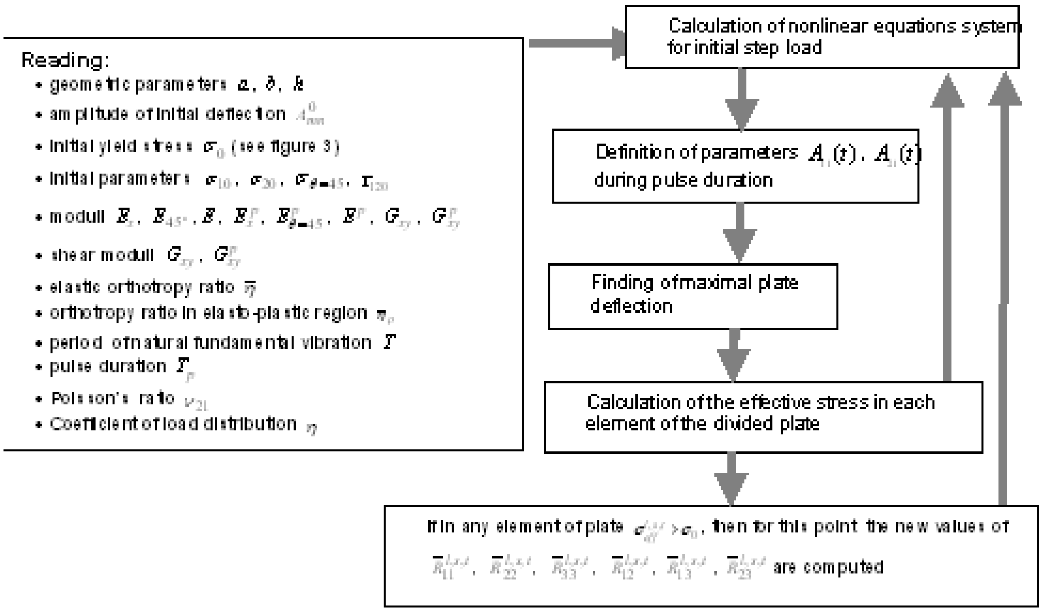


Figure 4 The flow chart of the calculation program

4. Results of calculations

The dynamic response for the orthotropic plate subjected to in–plane rectangular pulse loading is presented. The results are obtained for the case when the principal directions of orthotropy coincide with the plate edges ($\varphi = 0°$, see Fig. 2) taking into consideration various η parameters and $\zeta = 0$. The density of the material amounts 2000 kg/m^3. In the presented diagrams there are marked:

T_p – duration of pulse loading;
T – period of natural flexural vibrations of considered plate;
η – coefficient of plate loading;
N_{cr} – static buckling load of the considered plate;
N_{dyn} – load amplitude;
σ_{cr} – static buckling stress of the considered plate;
E_1 – Young's modulus in 1-th (x) direction;
E_2 – Young's modulus in 2-nd (y) direction;
G_{12} – shear modulus in the 1-2 plane (x-y plane);
$w_{0\,\max}$ – initial maximal deflection of plate;

$w_{\max}/h$ – quotient of maximal deflection of plate with respect to the thickness of considered plate;

σ_{10}, σ_{20}, $\sigma_{\theta=45}$ – initial yield stresses in the directions 1, 2 and rotated by $\theta = 45°$ to the x-direction, respectively;

σ_0 – initial yield stress in the reference direction;

τ_0 – initial yield stresses for the pure shear test;

E_1^p, E_2^p, $E_{\theta=45}^p$ – moduli of linear hardening in the directions 1, 2 and rotated by $\theta = 45°$ to the x-direction, respectively;

G_{12}^p – shear modulus of linear hardening in the 1-2 plane (x-y plane);

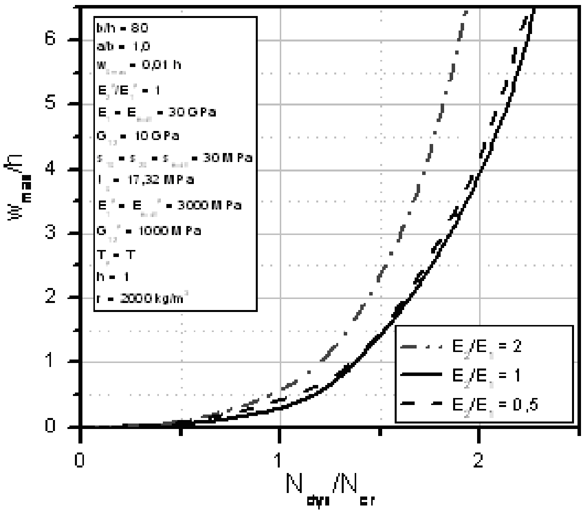


Figure 5 The response of the compressed plate for various ratio of the orthotropy in the elastic region

The influence of the material orthotropy in the elastic region on the response of plate is shown in the Fig. 5. In the elasto–plastic region the quotient of $E_2^p/E_1^p = 1$ is assumed. For $E_2^p/E_1^p = 2$ the sudden increase of deflection of the plate appears at $N_{dyn} \cong 1,2N_{cr}$ and for the two other considered cases at $N_{dyn} \cong 1,4N_{cr}$. It arises so because for such a material property and for $E_2^p/E_1^p = 2$ the static buckling load is the greatest and its value corresponds to 70 % of the initial yield stress in the reference direction. For $E_2^p/E_1^p = 1$ and $E_2^p/E_1^p = 0,5$ curves are almost the same. In Fig. 6 the maximal deflection of plate dependent on dynamic load for various coefficient η is presented. For all cases the courses of curves are different but for $\eta = 0$ and $\eta = 0,5$ the rapid growth of deflection follows at $N_{dyn} \geq 2,0N_{cr}$. Considering the pure compression ($\eta = 1$) in spite of the lowest critical buckling stress in this case, the load carrying capacity ends for $N_{dyn} \geq 1,8N_{cr}$

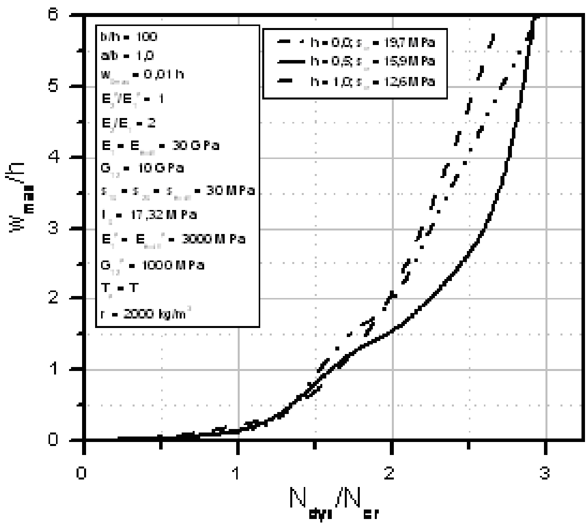


Figure 6 Comparison of the influence of the load coefficient on the maximal deflection of plate

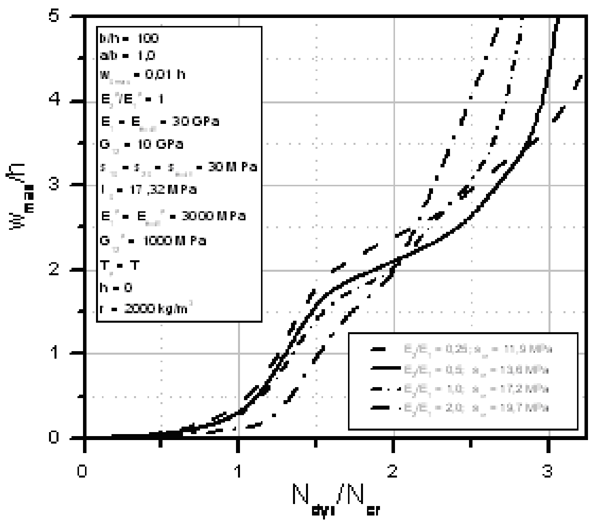


Figure 7 The response of the combined loaded plate for various ratio of the orthotropy in the elastic region

In Fig. 6 the maximal deflection of plate dependent on dynamic load for various coefficient η is presented. For all cases the courses of curves are different but for $\eta = 0$ and $\eta = 0,5$ the rapid growth of deflection follows at $N_{dyn} \geq 2,0N_{cr}$. Considering the pure compression ($\eta = 1$) in spite of the lowest critical buckling stress in this case, the load carrying capacity ends for $N_{dyn} \geq 1,8N_{cr}$

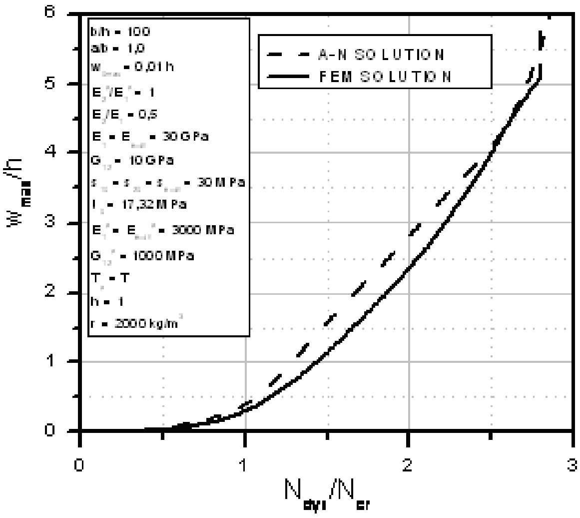


Figure 8 Comparison of the obtained results by analytical-numerical method solution with the result by FEM solution for compressed plate

Analysing the orthotropic plates subjected to the combined load ($\eta = 0$) for various quotients of E_2/E_1 (shown in Fig. 7) the fastest increase of the response of plate is observed for $E_2/E_1 = 2$ and $E_2/E_1 = 1$ at $N_{dyn} \geq 1,9N_{cr}$. It occurs similarly for pure compression (Fig. 5). For $E_2/E_1 < 1$ the curves go somewhat gently and for this material property the plate can carry the greater dynamic load referred to its critical buckling load. The comparison of the results from the analytical–numerical solution with the numerical solution obtained by finite elements method for pure compression is shown in Fig. 8. The received curved are similar each other.

5. Conclusions

The present paper provides the results of the behaviour of orthotropic plate under rectangular in–plane pulse loading. The main purpose of the work was the estimation of maximal dynamic load for various cases of material property taking into account elasto–plastic region and the flow theory of plasticity. It can be assumed that the load carrying capacity of plate reaches its boundary when for given dynamic load follows the sudden growth of deflection. It occurs in many considered cases if in material of the construction there exist many points in which initial yield limit is exceeded. It was observed that for each way of load the deflection of plate grows to infinity when the dynamic stress is close to initial yield stress of material.

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Appendix

$$
\begin{aligned}
\bar{R}_{11} &= \frac{E_1}{(1-\bar{\eta}\nu_{21}^2)}\left[1-\frac{(2\bar{a}_1-\nu_{21}\bar{\eta}\bar{a}_{12})^2}{9R_0}\bar{\ell}^2\right]\\
\bar{R}_{12}=\bar{R}_{21} &= \frac{E_1\bar{\eta}}{(1-\bar{\eta}\nu_{21}^2)}\left[\nu_{21}-\frac{(2\bar{a}_1-\nu_{21}\bar{\eta}\bar{a}_{12})(2\nu_{21}\bar{a}_1-\bar{a}_{12})}{9\bar{R}_0}\bar{\ell}^2\right]\\
\bar{R}_{13}=\bar{R}_{31} &= G_{12}\frac{(\nu_{21}\bar{\eta}\bar{a}_{12}-2\bar{a}_1)2\bar{a}_3}{3\bar{R}_0}\bar{\ell}^2\bar{\varphi}^2\\
\bar{R}_{22} &= \frac{E_1}{(1-\bar{\eta}\nu_{21}^2)}\left[\bar{\eta}-\bar{\eta}^2\frac{(2\nu_{21}\bar{a}_1-\bar{a}_{12})^2}{9R_0}\bar{\ell}^2\right]\\
\bar{R}_{23}=\bar{R}_{32} &= G_{12}\frac{(\bar{\eta}\bar{a}_{12}-2\nu_{21}\bar{\eta}\bar{a}_1)2\bar{a}_3}{3\bar{R}_0}\bar{\ell}^2\bar{\varphi}^2\\
\bar{R}_{33} &= 1-\frac{4\bar{a}_3^2\bar{\eta})}{\bar{R}_0}\frac{G_{12}(1-\nu_{21}^2\bar{\eta})}{E_1}\bar{\varphi}^2\\
R_0 &= \left[\frac{E^{p*}E^*}{E^*-E^{p*}}\right]\frac{4}{9}(1-\bar{\eta}\nu_{21}^2)(\bar{a}_1\bar{\ell}^2+3\bar{a}_3\bar{\varphi}^2)\\
&\quad +\bar{\ell}^2\frac{1}{9}(4\bar{a}_1^2-4\bar{\eta}\nu_{21}\bar{a}_1\bar{a}_{12}+\bar{\eta}\bar{a}_{12}^2)+G^*4\bar{a}_3^2\bar{\varphi}^2
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\ell} &= (1-\eta)\frac{x}{a}+\eta\\
\bar{\varphi} &= -\frac{1}{2}\frac{b}{a}(\eta-1)(1-\frac{2y}{b})\\
E^{p*} &= \frac{E^p}{E_1}\\
E^* &= \frac{E}{E_1}\\
G^* &= \frac{G_{12}(1-\nu_{21}^2\bar{\eta})}{E_1}
\end{aligned}
$$