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Multi–Products of Unit Vectors and Vectors. Basic Notions

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This article describes basic types of products of multiple unit vectors and vectors and gives simple examples of these. A case of the product of three vectors is used to illustrate the relationships between different types of multi–products, as well as between their vector and scalar products.

Keywords: Multi–product of vectors, first, second and third kind of product even and odd product sum of products

1. Introduction

Whenever a product of vectors is referred to in the literature, most frequently a product of two vectors is meant: a scalar one $\mathbf{a}\,\mathbf{b}$, or a vector one $\mathbf{a} \times \mathbf{b}$. For three vectors a mixed product $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\mathbf{c}$ and a double vector product $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are also defined. In addition, for four vectors a triple vector product in the form $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ has been defined.

Thus, a question arises: what will result from the multiplication of several, three or more vectors and what will the result of such an operation be if we follow the rules of scalar multiplication of two vectors $\mathbf{a}\,\mathbf{b}$?

The following two situations can be the starting point for considerations on the multiplication of vectors:

- when the vectors lie in an n–dimensional space, determined by the system of axes and then they are described in this system of axes by coordinates. We will deal with this case later.

- when, at least, at the beginning we disregard the system of coordinates, while we relate the axes occurring in the problem only to the vectors.

2. Multiplication of unit vectors of axes. Types of products

If the axes occurring in a problem are related only to vectors, then with any vector $\mathbf{v}$, lying in the space on any line of operation, an axis v, coinciding with this vector is related. The axis v is determined by a positive unit vector, a unit vector of this axis $\mathbf{e_v}$ lying on it. Since the axis unit vector is defined as

$$\mathbf{e_v} = \mathbf{v}/v$$

hence the vector $\mathbf{v}$ can be written as

$$\mathbf{v} = v\,\mathbf{e}_v \tag{1}$$

Let us imagine another axis t lying in the same space, with the unit vector of this axis $\mathbf{e}_t$ lying on it. Let us assume that both axes, v and t, form between them a known angle φ_{vt}.

Let us now see an obvious relationship between the unit vector $\mathbf{e}_v$, lying on the axis v, and the unit vector $\mathbf{e}_t$ lying on the other axis t. Here we make use of a well-know definition of a scalar product of two vectors, or two unit vectors, in the form:

$$\mathbf{e}_v\,\mathbf{e}_t = \cos\varphi_{vt}$$

We multiply both sides of equation (1) by the unit vector $\mathbf{e}_t$ and we get:

$$\mathbf{v}\,\mathbf{e}_t = v\,\mathbf{e}_v\mathbf{e}_t = v\cos\varphi_{vt} \tag{2}$$

Let us now try to multiply again both sides of equation (2) by the unit vector of the axis $\mathbf{e}_t$.

$$\mathbf{v}\,\mathbf{e}_t\mathbf{e}_t = v\,\mathbf{e}_v\mathbf{e}_t\mathbf{e}_t = v(\mathbf{e}_v\mathbf{e}_t)\mathbf{e}_t = v\cos\varphi_{vt}\mathbf{e}_t = v_\mathbf{t} \tag{3}$$

Then on the right side of equation (3) we obtain a vector $\mathbf{v}_t$, which is a projection of the vector $\mathbf{v}$ on the axis t. Equation (3) is the result of double multiplication of equation (1) by the unit vector $\mathbf{e}_t$. As a result, after these operations we have obtained a new vector, lying on a different direction (line of operation), on the axis t.

A doubt arises whether such reasoning is correct. Let us notice that we have multiplied equation (1) by $\mathbf{e}_t$ twice, thus, in fact, we have multiplied equation (1) by one. It is known that the square of the same unit vector is equal to one ($\mathbf{e}_t\mathbf{e}_t = 1$), it is also known that if we multiply both sides of any equation times one, it remains unchanged.

Hence, we can question the correctness of the reasoning while making operations in transformations (2–3) and claim that equation (3) should assume the form,

$$\mathbf{v}\,\mathbf{e}_t\mathbf{e}_t = v\,\mathbf{e}_v(\mathbf{e}_t\mathbf{e}_t) = v\,\mathbf{e}_v = \mathbf{v} \tag{4}$$

In reality, this problem amounts to the question what the product of the unit vector of one axis times the square of the unit vector of another axis is; in other words, it boils down to the question about the value of the expression $\mathbf{e}_v\mathbf{e}_t\mathbf{e}_t$. As we have demonstrated above, the result of such multiplication depends on the order in which the unit vectors are multiplied. Performing this operation, we have two options of choosing the order of multiplication.

- We can first multiply the homogeneous unit vectors, which means that

$$\mathbf{e}_v\,\mathbf{e}_t\mathbf{e}_t = \mathbf{e}_v(\mathbf{e}_t\mathbf{e}_t) = \mathbf{e}_v \quad \text{since} \quad \mathbf{e}_t\mathbf{e}_t = 1 \tag{5}$$

This would be the *first kind of product of three* unit vectors, which can be denoted as $\mathbf{f}^3$. This kind of product will be referred to by a person who thinks that equation (1) should be left unchanged, i.e. should be transformed into equation (4).

- Or we can first multiply heterogeneous unit vectors, i.e. assume that

$$\mathbf{e}_v\mathbf{e}_t\mathbf{e}_t = (\mathbf{e}_v\mathbf{e}_t)\mathbf{e}_t = \cos\varphi_{vt}\mathbf{e}_t \tag{6}$$

This is the *second kind of product of three* unit vectors, which will be denoted as $\mathbf{s}^3$.

Such a kind of product occurs in the example given above during transformation of equation (1) into equation (3).

3. Scalar related to an axis. Product of scalar times vector

Such reasoning makes us realise that if any scalar c is presented as a product of a number c times the square of the unit vector of any axis t, i.e. as an expression

$$c = c1 = c\,\mathbf{e}_t\mathbf{e}_t \tag{7}$$

then the scalar c thus defined can be called a *scalar related to an axis t*.

A scalar related to an axis always has the same value, independent of the axis chosen, hence it satisfied the condition given in the definition of a scalar.

If we multiply a scalar related to an axis c by the vector $\mathbf{v} = v\,\mathbf{e}_v$, we will obtain the following definition of a product of a vector and scalar related to an axis.

$$c\,\mathbf{v} = c\,v\,\mathbf{e}_v = c\,v\,\mathbf{e}_v\mathbf{e}_t\mathbf{e}_t = \begin{cases} c\,v\,\mathbf{e}_v & \text{if} \quad \mathbf{f}^3 \\ c\,v\,\cos\varphi_{vt}\mathbf{e}_t & \text{if} \quad \mathbf{s}^3 \end{cases} \tag{8}$$

According to this formula, the product of a scalar c times the vector $\mathbf{v}$ is a new vector:

- which lies on the same axis v of a length equal to the product of cv – if we assume the first kind of odd product of three unit vectors $\mathbf{f}^3$,
- which may lie on any axis of a length equal to the product of $c\,v$ and the scalar product of unit vectors of both axes v and t – if we assume the second kind of product $\mathbf{s}^3$. It can then be said that in this case we obtain a vector of a projection of the vector $\mathbf{v}$ on any axis t.

It should be noted that if we calculate the product of $c\,v$ for the axis t coinciding with the axis v, then the value of the products for both cases $\mathbf{f}^3$ and $\mathbf{s}^3$ is the same and amounts to $c\,v\,\mathbf{e}_v$.

Let us also notice that if the axes t and v are perpendicular, the product of the second kind (if $\mathbf{s}^3$) is equal to zero.

4. Product of a vector equation times a unit vector. Even and odd products

Let us now deal with the question whether we can multiply both sides of any vector equation by a unit vector or any power of this unit vector. Let us assume the simplest equation of the sum of two vectors

$$\mathbf{a} = \mathbf{b} + \mathbf{c}$$

in the form

$$a\,\mathbf{e}_a = b\,\mathbf{e}_b + c\,\mathbf{e}_c \tag{9}$$

and multiply both sides of it by $\mathbf{e}_t$ or $\mathbf{e}_t^3$, or times the unit vector $\mathbf{e}_t$ raised to a higher but odd power. In each of these cases we have to do with a product of an even number of unit vectors, $\mathbf{f}^2$ and $\mathbf{s}^2$ or $\mathbf{f}^4$ and $\mathbf{s}^4$, etc.

Such a kind of product can be called an *even product* of several unit vectors. A scalar or a scalar equation is the result of an even product.

And so, for example, after the multiplication of both sides of equation (9) times $\mathbf{e}_t^3$ we get

$$a\,\mathbf{e}_a\mathbf{e}_t^3 = b\,\mathbf{e}_b\mathbf{e}_t^3 + c\,\mathbf{e}_c\mathbf{e}_t^3 \tag{10}$$

No matter which kind of product we assume, the first or the second, thus $\mathbf{f}^4 = \mathbf{s}^4$, we will get the same scalar equation of the form

$$a\cos\varphi_{at} = b\cos\varphi_{bt} + c\cos\varphi_{ct} \tag{11}$$

This equation is true and confirms the well–known theorem of Chasles that the coordinate of the resultant vector on any axis is equal to the algebraic sum of the coordinates of the component vectors on the same axis.

Let us now multiply both sides of equation (9) by $\mathbf{e}_t^2$ or $\mathbf{e}_t^4$, or times the unit vector $\mathbf{e}_t$ raised to a higher but even power. Then we have to do with a product of an odd number of unit vectors, $\mathbf{f}^3$ and $\mathbf{s}^3$ or $\mathbf{f}^5$ and $\mathbf{s}^5$, etc.

Such a kind of product can be called an *odd product* of several. A vector or vector equation is the result of an odd product.

And so, for example, after the multiplication of equation (9) times $\mathbf{e}_t^4$, we get

$$a\,\mathbf{e}_a\mathbf{e}_t^4 = b\,\mathbf{e}_b\mathbf{e}_t^4 + c\,\mathbf{e}_c\mathbf{e}_t^4 \tag{12}$$

If we calculate the *first kind* of product , i.e. $\mathbf{f}^5$, equation (12) will return to its initial form (9).

If we now calculate the *second kind* of product , i.e. $\mathbf{s}^5$, we will get an equation of the form

$$a\cos\varphi_{at}\mathbf{e}_t = b\cos\varphi_{bt}\mathbf{e}_t + c\cos\varphi_{ct}\mathbf{e}_t \tag{13}$$

which is a projection of equation (9) onto the axis t. This equation is also true.

The reasoning cited above leads to the conclusion that both sides of any equation of the vector sum can be multiplied by any unit vector of an axis or vector raised to any power and, as a result, we will obtain a vector or scalar equation which is always true. The kind of the equation obtained depends on two different features of the product obtained:

- evenness or oddness
- the priority chosen for the multiplication of unit vectors of an axis.

5. Product of three vectors

5.1. *Classical notation*

Let us now deal with a product of three vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$, lying on any three axes in the space, described as

$$\mathbf{a} = a\,\mathbf{e}_a, \qquad \mathbf{b} = b\,\mathbf{e}_b, \qquad \mathbf{c} = c\,\mathbf{e}_c$$

The product of the vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ also depends on the order of their multiplication. By multiplying these vectors in different orders, we obtain three different resultant vectors, of different lengths and lying on different directions – $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$.

$$\mathbf{a}\,\mathbf{b}\,\mathbf{c} = a\,b\,c\,\mathbf{e}_a\mathbf{e}_b\mathbf{e}_c = \begin{cases} a\,b\,c\cos\varphi_{bc}\mathbf{e}_a = (\mathbf{b}\,\mathbf{c})\mathbf{a} \\ a\,b\,c\cos\varphi_{ac}\mathbf{e}_b = (\mathbf{c}\,\mathbf{a})\mathbf{b} \\ a\,b\,c\cos\varphi_{ab}\mathbf{e}_c = (\mathbf{a}\,\mathbf{b})\mathbf{c} \end{cases} \tag{14}$$

where

$$(\mathbf{bc})\mathbf{a} \neq (\mathbf{ca})\mathbf{b} \neq (\mathbf{ab})\mathbf{c}$$

The vector sum of the right sides of products (14) is a constant value for the product of the three vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$, regardless of the order of their multiplication. This vector in form (15) will be denoted as $\mathbf{p}^3$ and called a vector of the *function of the sum of products* of three vectors.

$$\mathbf{p^3} = (\mathbf{b}\,\mathbf{c})\mathbf{a} + (\mathbf{c}\,\mathbf{a})\mathbf{b} + (\mathbf{a}\,\mathbf{b})\mathbf{c} \tag{15}$$

The product of the same vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ in an n–dimensional Cartesian space is different. To start with, let us assume that the space is three–dimensional, described by an orthogonal system of axes $O\,x\,y\,z$, and that the vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ determined by the coordinates given below are lying in this space

$$\mathbf{a}(a_x, a_y, a_z), \quad \mathbf{b}(b_x, b_y, b_z), \quad \mathbf{c}(c_x, c_y, c_z)$$

Then

$$\begin{aligned} \mathbf{a} &= a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \\ \mathbf{b} &= b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z \\ \mathbf{c} &= c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z \end{aligned} \tag{16}$$

Thus, the product of three vectors can be written as

$$\mathbf{abc} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z)(b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z)(c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z) \tag{17}$$

After multiplication of the right side of (17) and taking into consideration that $\mathbf{e}_x^3 = \mathbf{e}_x$ and that

$$\mathbf{e}_x^2\mathbf{e}_y = \mathbf{e}_y \qquad \text{if} \qquad \mathbf{f}^3$$

$$\mathbf{e}_x^2\mathbf{e}_y = 0 \qquad \text{if} \qquad \mathbf{s}^3$$

we isolate in it the *first* ($\mathbf{f}^3$) and the *second* ($\mathbf{s}^3$) *kind of product* in the form of the following vectors

$$\begin{aligned}
\mathbf{f}^3 &= [a_xb_xc_x + (b_yc_y + b_zc_z)a_x + (a_yc_y + a_zc_z)b_x + (a_yb_y + a_zb_z)c_x]\mathbf{e}_x \\
&+ [a_yb_yc_y + (b_xc_x + b_zc_z)a_y + (a_xc_x + a_zc_z)b_y + (a_xb_x + a_zb_z)c_y]\mathbf{e}_y \quad (18)\\
&+ [a_zb_zc_z + (b_xc_x + b_yc_y)a_z + (a_xc_x + a_yc_y)b_z + (a_xb_x + a_yb_y)c_z]\mathbf{e}_z \\
\mathbf{s}^3 &= a_xb_xc_x\mathbf{e}_x + a_yb_yc_y\mathbf{e}_y + a_zb_zc_z\mathbf{e}_z \qquad (19)
\end{aligned}$$

The vector $\mathbf{s}^3$ has coordinates equal to products of homogeneous components and is the result of multiplication of the vectors on the system axes. Thus, it can be called an *axis product* of three vectors.

After multiplying both sides of equation (19) by 2 and adding equation (18) and (19), on the right side of the sum obtained we get an expression in which scalar products of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ appear

$$\begin{aligned}
\mathbf{f}^3 + 2\mathbf{s}^3 &= (a_xb_x + a_yb_y + a_zb_z)(c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z) \\
&+ (a_xc_x + a_yc_y + a_zc_z)(b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) \\
&+ (b_xc_x + b_yc_y + b_zc_z)(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) \\
\mathbf{f}^3 + 2\mathbf{s}^3 &= (\mathbf{a}\,\mathbf{b})\mathbf{c} + (\mathbf{a}\,\mathbf{c})\mathbf{b} + (\mathbf{b}\,\mathbf{c})\mathbf{a} \qquad (20)
\end{aligned}$$

thus, after taking into account (15), we get

$$\mathbf{f}^3 + 2\mathbf{s}^3 = \mathbf{p}^3 \qquad (21)$$

If we analyse equations (18) and (19), we will see that the coordinates of the vector $\mathbf{s}^3$ are contained within the coordinates of the vector $\mathbf{f}^3$. Thus, we can subtract both sides of equation (19) from both sides of equation (18). We will then get a new vector which is the difference of the vectors of the product of the first and the second kind.

Let us call this vector a *product of the third kind* of three vectors and denote it as $\mathbf{t}^3$

$$\mathbf{t}^3 = \mathbf{f}^3 - \mathbf{s}^3 \qquad (22)$$

$$\begin{aligned}
\mathbf{t}^3 &= [(b_yc_y + b_zc_z)a_x + (a_yc_y + a_zc_z)b_x + (a_yb_y + a_zb_z)c_x]\mathbf{e}_x \\
&+ [(b_xc_x + b_zc_z)a_y + (a_xc_x + a_zc_z)b_y + (a_xb_x + a_zb_z)c_y]\mathbf{e}_y \qquad (23)\\
&+ [(b_xc_x + b_yc_y)a_z + (a_xc_x + a_yc_y)b_z + (a_xb_x + a_yb_y)c_z]\mathbf{e}_z
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{t}^3 &= (A_{yz}a_x + B_{yz}b_x + C_{yz}c_x)\mathbf{e}_x \\
&+ (A_{xz}a_y + B_{xz}b_y + C_{xz}c_y)\mathbf{e}_y \qquad (24)\\
&+ (A_{xy}a_z + B_{xy}b_z + C_{xy}c_z)\mathbf{e}_z
\end{aligned}$$

where

$$\begin{array}{lll} A_{xy} = b_x c_x + b_y c_y & B_{xy} = a_x c_x + a_y c_y & C_{xy} = a_x b_x + a_y b_y \\ A_{yz} = b_y c_y + b_z c_z & B_{yz} = a_y c_y + a_z c_z & C_{yz} = a_y b_y + a_z b_z \\ A_{xz} = b_x c_x + b_z c_z & B_{xz} = a_x c_x + a_z c_z & C_{xz} = a_x b_x + a_z b_z \end{array} \tag{25}$$

The vector $\mathbf{t}^3$ can also be presented as a sum of vectors

$$\begin{aligned} \mathbf{t}^3 &= (A_{yz}\mathbf{a}_x + A_{xz}\mathbf{a}_y + A_{xy}\mathbf{a}_z) + (B_{yz}\mathbf{b}x + B_{xz}\mathbf{b}_y + B_{xy}\mathbf{b}_z) \\ &+ (C_{yz}\mathbf{c}_x + C_{xz}\mathbf{c}_y + C_{xy}\mathbf{c}_z) \end{aligned} \tag{26}$$

The coefficients A, B, C are equal to the scalar products of vector projections ($\mathbf{b}$, $\mathbf{c}$), $(\mathbf{a}, \mathbf{c})$, $(\mathbf{a}, \mathbf{b})$ onto the planes $y\,z$, $x\,z$, $x\,y$ of the system, respectively.

Thus, the product $\mathbf{t}^3$ depends on the value of the scalar products calculated in the planes of the coordinate system and therefore, we can call it a *plane product.*

Since

$$\mathbf{f}^3 = \mathbf{s}^3 + \mathbf{t}^3$$

equation (20) can now be written as

$$3\,\mathbf{s}^3 + \mathbf{t}^3 = (\mathbf{a}\,\mathbf{b})\mathbf{c} + (\mathbf{a}\,\mathbf{c})\mathbf{b} + (\mathbf{b}\,\mathbf{c})\mathbf{a} \tag{27}$$

so, after taking into account (15) we can write

$$3\,\mathbf{s}^3 + \mathbf{t}^3 = \mathbf{p}^3 \tag{28}$$

From the theory of vector products we know that

$$\begin{aligned} (\mathbf{a}\times\mathbf{c}) \times \mathbf{b} &= (\mathbf{a}\,\mathbf{b})\mathbf{c} - (\mathbf{b}\,\mathbf{c})\mathbf{a} \\ (\mathbf{a}\times\mathbf{b}) \times \mathbf{c} &= (\mathbf{a}\,\mathbf{c})\mathbf{b} - (\mathbf{b}\,\mathbf{c})\mathbf{a} \end{aligned}$$

Thus, equations (20), (21), (27) and (28) can be presented as the following sum of vector and scalar products

$$\mathbf{p}^3 = \mathbf{f}^3 + 2\,\mathbf{s}^3 = 3\,\mathbf{s}^3 + \mathbf{t}^3 = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} + 3\,(\mathbf{b}\,\mathbf{c})\mathbf{a} \tag{29}$$

5.2. *Matrix notation*

It is often easier to use matrix notation. We can then write vectors $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ as matrices $\mathbf{a}^T[a_x a_y a_z]$, $\mathbf{b}^T[b_x b_y b_z]$, $\mathbf{c}^T[c_x c_y c_z]$.

We introduce the tensor of the product of vectors $\mathbf{a}$ $\mathbf{b}$ in the form of matrix $\mathbf{P}_{ab}$ (and similarly $\mathbf{P}_{bc}$ and $\mathbf{P}_{ca}$).

$$P_{ab} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix} \tag{30}$$

In this case the product of tensor $\mathbf{P}_{ab}$ and the matrix corresponding to vector $\mathbf{c}$ can be written as:

$$\mathbf{P}_{ab}\,\mathbf{c} = (\mathbf{b}\,\mathbf{c})\mathbf{a} \tag{31}$$

Substituting all three tensors into equation (15), the sum of the products $\mathbf{p}^3$ becomes the sum of the product of matrices:

$$\mathbf{p}^3 = \mathbf{P}_{ab}\,\mathbf{c} + \mathbf{P}_{bc}\,\mathbf{a} + \mathbf{P}_{ca}\,\mathbf{b} \tag{32}$$

Therefore the right-hand sides of equations (19) and (20) can be replaced by the right-hand side of equation (32).

6. Final remarks

Making use of simple examples of products of several vectors described in a three-dimensional space, we have introduced basic notions ordering the description of a general case of *multi-products of vectors*, such as

- an even and odd product,
- a product of the first kind and a product of the second kind (the axis one) and of the third kind (the plane one)
- a function of the sum of products.

Taking three vectors as an example, relationships (20–21) and (27–29) occurring between these quantities have been demonstrated.

In addition, a notion of a scalar related to an axis has been introduced.

It has also been proved that it is possible to multiply vector equations times any power of the unit vector of any axis, the correctness of the equation being preserved.

This offers possibilities to solve a general case of an even product and an odd product for any number of vectors in an n–dimensional space.

Nomenclature

f^v	first kind of product of even (v) vectors
$\mathbf{f}^d$	first kind of product of odd (d) vectors
s^v	second kind of product of even vectors
$\mathbf{s}^d$	second kind of product of odd vectors
t^v	third kind of product of even vectors
$\mathbf{t}^d$	third kind of product of odd vectors
p^v	function of the sum of even vectors
$\mathbf{p}^d$	function of the sum of odd vectors
$\mathbf{P}_{ab}$	tensor of the product of vectors $\mathbf{a}\,\mathbf{b}$