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Magnetohydrodynamics Stability of a Compressible Fluid Layer Below a Vacuum Medium

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The instability analysis of a compressible fluid layer acted by capillary, pressure gradient, external gravity and electromagnetic forces has been developed on the basis of the linear perturbation technique.

A general eigenvalue relation is derived and discussed analytically and numerically for all values of amplitude disturbances. This model has many applications in Astronomy as the interface of an expanding supernova remnant and in Geology during the geological drillings in the crust of the earth.

Keywords: Compressible fluid layer, magnetohydrodynamics (MHD)

1. Introduction

The stability of superposed fluids in the absence of magnetic field was studied by Rayleigh–Taylor and Kelvin–Helmholtz [1]. The field of Magnetohydrodynamics (MHD) was initiated by Alfven [2]. The set of equations which describe MHD are a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. However the investigation of Hydrodynamics stability analysis of the interface of two contacted fluids has been started little bit earlier than the comprehensive work [3]. Here the study of the stability analysis of superposed fluids concerns allowance for surface tension at interfaces between fluids and the effect of a horizontal magnetic field transverse to the direct of gravity **g** on the character of the equilibrium of a compressible inviscid fluid of zero resistivity. The purpose of this study is elaborating the dynamical oscillations and instability of a compressible magnetic fluid layer subject io the external gravity, capillary and pressure gradient forces. A general dispersion relation is derived which is valid for all values of amplitude disturbances. From the derived dispersion relation we can determine the stable and unstable regions.

Several limiting cases in the literature could be recoverd from the present results upon assuming appropriate simplifications. Upon applicable suitable boundary conditions, a general dispersion relation is derived and discussed.

2. Formulation of the problem

We study the stability of fluid layer subject to the external gravity. Consider an initially flat layer of inviscid compressible fluid that occupies the half–space $z < 0$, initial flat interface at $z = 0$ and the upper half–space $z > 0$ is a vacuum as illustrated in Fig. 1. The analysis will investigate the stability of the equilibrium in response to the pressure gradient, gravity, capillary and magnetic forces.

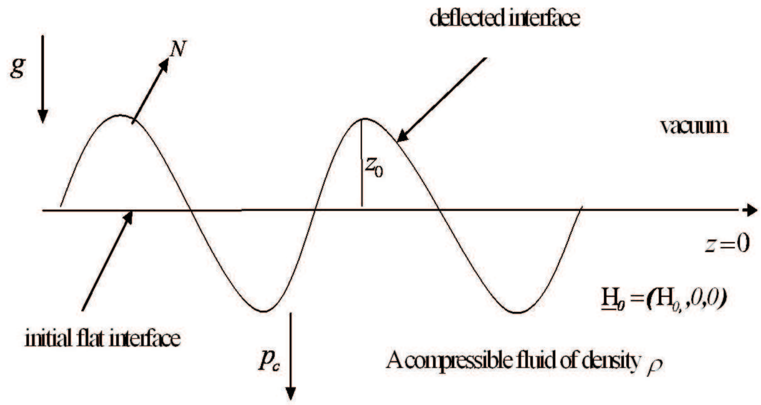

Figure 1 Sketch of a gravitating compressible fluid layer under the effect of the capillary pressure and a horizontal magnetic field

3. Basic Equations

The fundamental equations for such a study are the combination of the ordinary fluid dynamics, Maxwell electromagnetic equations and those of the perfect gas. Under the present circumstances these equations are as follows

Equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{\mu}{\rho}(\mathbf{H}\cdot\nabla)\mathbf{H} = -\nabla\Pi, \qquad \Pi = \frac{P}{\rho} + \mu\frac{(\mathbf{H}\cdot\mathbf{H})}{2\rho} + gz \tag{1}$$

Continuity equation of a compressible fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{2}$$

Equation of gas state

$$P = K\rho^{\gamma} \tag{3}$$

Equation of conservation of energy

$$\rho c_v\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)T + P\nabla \cdot \mathbf{u} = 0 \tag{4}$$

Equation of curvature pressure due to capillary force

$$P_c = S(\nabla \cdot \mathbf{N}) \tag{5}$$

$$\mathbf{N} = \frac{\nabla F}{|\nabla F|}, \qquad F = z - z_0(x, y) \tag{6}$$

Equation of conservation of flux

$$\nabla \cdot \mathbf{H} = 0 \tag{7}$$

The case when the electrical conductivity of the medium may be considered as infinite is of particular interest. The resistivity is then zero and the equation for $\mathbf{H}$ becomes [1]

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \mathbf{H}) \tag{8}$$

It's assumed that in the vacuum region , there's no current flow:

$$\nabla \cdot \mathbf{H}^v = 0 \tag{9}$$

$$\nabla \times \mathbf{H}^v = 0 \tag{10}$$

Here ρ, u and P are the fluid mass density, velocity vector and kinetic pressure, g is the gravity acceleration, S the surface tension coefficient, $\mathbf{N}$ the unit outward normal vector to the fluid surface, $F = z - z_0(x, y)$ is the equation of the fluid boundary surface and z_0 is the elevation of the deflected interface $\mathbf{H} = (H_0, 0, 0)$ is the magnetic field intensity and μ is the magnetic field permeability. The forces acting on the model are the forces due to gradient pressure, capillary, gravity and the electromagnetic force (Lorentz force) $\mu(\nabla \times \mathbf{H}) \times \mathbf{H}$.

From Eq. (5) and (6), the curvature pressure due to capillary force is

$$P_c = -S\left(\frac{\partial^2 z_0}{\partial x^2} + \frac{\partial^2 z_0}{\partial y^2}\right) \tag{11}$$

From the continuity of the normal component of the total stress tensor across the interface at $z = 0$, the unperturbed kinetic pressure of the fluid in the equilibrium state

$$P_0 = K^2 S z_0 - \frac{1}{2}\mu H_0^2 \tag{12}$$

which is a simple linear combination of the pressure due to capillary force at $z = 0$ and the electromagnetic force in the fluid.

4. Perturbation Analysis

From small departures from the equilibrium state, the different variable quantities could be expressed as its unperturbed part plus a fluctuation part [4].

$$Q(x,\, y,\, z,\, t) = Q_0(z) + \varepsilon Q_1(x,\, y,\, z) \tag{13}$$

where 0 indicates the steady state value and 1 the perturbation. $Q(x,\, y,\, z,\, t)$ stands for each of u, Π, P, ρ, P_c, T, N and ε is the amplitude of the perturbation at any instant of time. The time–space dependence is given by

$$Q_1(x,y,z,t) = \hat{Q}_1(z)e^{\sigma t - i(k_x x + k_y y)} \tag{14}$$

where $\hat{Q}_1(z)$ is an amplitude function dependent on z, σ is the growth rate and $k^2 = k_x^2 + k_y^2$ is the square of the longitudinal wave number. Based on the expensions (13) and (14), the relevant perturbation equations are given by:

$$\sigma u_1 - (\mu l \rho_0)(\mathbf{H}_0 \cdot \nabla)\mathbf{H}_1 = -\nabla \Pi_1 \tag{15}$$

where

$$\Pi_1 = \frac{P_1}{\rho_0} + g z_0 + \frac{\mu}{\rho_0}(\mathbf{H}_0 \cdot \mathbf{H}_1) \tag{16}$$

$$P_1 = a^2 \rho_1, \qquad a = \sqrt{\frac{\gamma P_0}{\rho_0}} \tag{17}$$

$$\rho_0 a^2 (\nabla \cdot \mathbf{u}_1) = -\sigma P_1 \tag{18}$$

where a is the sound speed in the gas defined by $a = (\gamma P_0/\rho_0)^{\frac{1}{2}}$ and $\gamma = \frac{c_p}{c_v}$ is the ratio of specific heats of the fluid.

$$\sigma \mathbf{H}_1 = \mathrm{curl}(\mathbf{u_1} \times \mathbf{H_0}) \tag{19}$$

For the vacuum region we have

$$\nabla \cdot \mathbf{H}_1^V = 0 \tag{20}$$
$$\nabla \times \mathbf{H}_1^V = 0 \tag{21}$$

By the aid of the expansion (14), Eqs (15) and (18) are combined to give the magnetic field intensity and the velocity vector in the form

$$\mathbf{H}_1 = \frac{-ik_x H_0 \mathbf{u}_1}{\sigma} + \frac{H_0 P_1 \mathbf{e}_x}{\rho_0 a^2} \tag{22}$$
$$\mathbf{u}_1 = \left(\frac{-\sigma}{\sigma^2 + \Omega_A^2}\right)\left[\nabla \Pi_1 + \left(\frac{i\Omega_A^2 P_1}{\rho_0 k_x a^2}\right)\mathbf{e}_x\right] \tag{23}$$

where $\mathbf{e_x}$ is a unit vector in the x direction and Ω_A is the Alfven wave frequency defined in terms of H_0 as:

$$\Omega_A = k_x H_0 (\mu/\rho_0)^{\frac{1}{2}} \tag{24}$$

Combining the x–components of the vector quantities given by Eqs (22) and (23)and utilizing equation (16), we get

$$H_{1x} = \frac{k_x^2 H_0 \Pi_1}{\sigma^2 + \Omega_A^2} - \frac{\Omega_A^2 P_1 H_0}{\rho_0 a^2(\sigma^2 + \Omega_A^2)} + \frac{H_0 P_1}{\rho_0 a^2} \tag{25}$$

Using Eqs (15) and (16) we get

$$u_{1x} = \frac{ik_x}{\sigma}\left(\frac{P_1}{\rho_0} + gz_0\right) \tag{26}$$

Using Eqs (25) and (26) in (16) we get

$$\Pi_1 = \frac{P_1}{\rho_0}\chi + \left(\frac{\sigma^2 + \Omega_A^2}{\sigma^2}\right) gz_0 \tag{27}$$

with

$$\chi = 1 + \frac{\mu H_0^2}{\rho_0 \sigma^2 a^2}(\sigma^2 + k_x^2 a^2) \tag{28}$$

Using Eqs (18) and (23) we get

$$\nabla^2 \Pi_1 = \frac{P_1}{\rho_0}\frac{\sigma^2}{a^2} \tag{29}$$

Using Eq. (27) in (29)and by the aid of the Eq. (14) we get

$$\frac{d^2\hat{\Pi}_1}{dz^2} - \left(k^2 + \frac{\sigma^2}{a^2\chi}\right)\hat{\Pi}_1 + \left(\frac{\sigma^2 + \Omega_A^2}{a^2\chi}\right) g\hat{z}_0 = 0 \tag{30}$$

Upon applying appropriate boundary condition on the interface [5], applying the continuity of the normal velocity and using the linearized form of z–component of velocity that $u_{1z} = \frac{\partial z_0}{\partial t}$ where $z_0(x, y, t)$ is the interface when the surface position changes with time, so we can deduce that

$$\hat{z}_0 = -\frac{1}{\sigma^2 + \Omega_A^2}\frac{d\hat{\Pi}_1}{dz} \tag{31}$$

Using Eq. (31) in Eq. (30) we get the second order differential equation as

$$\frac{d^2\hat{\Pi}_1}{dz^2} - C\frac{d\hat{\Pi}_1}{dz} - E\hat{\Pi}_1 = 0 \tag{32}$$

where

$$C = \frac{g}{a^2\chi} \tag{33}$$

$$E = k^2 + \frac{\sigma^2}{a^2\chi} \tag{34}$$

Under certain circumstances, the solution of the Eq. (32) is

$$\hat{\Pi}_1 = \frac{2(\sigma^2 + \Omega_A^2)}{C + \sqrt{C^2 + 4E}}\hat{z}_0 e^{\frac{1}{2}(C + \sqrt{C^2+4E})z} \tag{35}$$

and consequently

$$\Pi_1 = \hat{\Pi}_1(z) e^{\sigma t - i(k_x x + k_y y)} \tag{36}$$

According to Eqs (11), (13) and (14), the perturbed curvature pressure due to capillary force is given by

$$P_{1c} = k^2 S \hat{z}_0 e^{\sigma t - i(k_x x + k_y y)} \tag{37}$$

5. Dispersion Relation

To obtain the required eigen value relation we have to apply some dynamic condition which states that the normal component of the total stress tensor due to the kinetic, magnetic pressures and the curvature pressure due to capillary force must be continuous across the deformed interface and taking into account that the upper region is a vacuum.

This condition gives the required dispersion relation as

$$\frac{2\sigma^2}{C + \sqrt{C^2 + 4E}} = g - \frac{k^2 S}{\rho_0} - \frac{2\Omega_A^2}{C + \sqrt{C^2 + 4E}} \tag{38}$$

6. Discussions and Limiting cases

Eq. (38) is the required hydromagnetic stability criterion for a compressible semi-infinite fluid layer which is ambient with a vacuum medium from above and is acted by elctromagnetic and pressure gradient (due to the surface tension) forces.

We can deduce from Eq. (38) that the compressibility has a stabilizing influence for all wavelengths [5]. As $a \to \infty$ and consequently $C = 0$, $E = k$ and the eigen value relation (38) reduces to

$$\sigma^2 = gk \left(1 - \frac{k^2 S}{\rho_0 g} - \frac{\mu k_x^2 H_0^2}{\rho_0 g k}\right) \tag{39}$$

which is similar to the dispersion relation of incompressible fluid layer, that derived in [6]. Define the Alven velocity as

$$V_A = \sqrt{\frac{\mu H_0^2}{\rho_0}} \tag{40}$$

Rewrite Eq. (39) using the relation (40), we get

$$\sigma^2 = gk \left(1 - \frac{k^2 S}{\rho_0 g} - \frac{V_A^2 k_x^2}{gk}\right) \tag{41}$$

In the absence of the horizontal magnetic field, the relation (39) becomes

$$\sigma^2 = gk \left(1 - \frac{k^2 S}{\rho_0 g}\right) \tag{42}$$

In this case the system is unstable for all wave numbers in the range $0 < k < k_c$, where

$$k_c = \left(\frac{\rho_0 g}{S}\right)^{\frac{1}{2}} \tag{43}$$

and it's stable for all disturbances with $k \geq k_c$, where the equalities correspond to the marginal stability.

Also in the absence of the surface tension the relation (39) becomes

$$\sigma^2 = gk(1 - \frac{\mu k_x^2 H_0^2}{\rho_0 gk}) \tag{44}$$

which indicates that the term contains k_x^2 has an effect which is the same as a surface tension in Eq. (42). In the absence of both surface tension and the magnetic field, Eq. (39) reduces to the well–known relation

$$\sigma^2 = gk \tag{45}$$

which the amplitude of the disturbance grows most rapidly and it's known by Rayleigh–Taylor instability [1].

The (k, σ) – relationships derived in the manner we have described are illustrated in Figs 2–3, which are drawn using MATLAB program. According to Eq. (42) and by measuring σ and k in the units $\sqrt{gk}\,\text{sec}^{-1}$, $\sqrt{\rho_0 g/S}$ cm^{-1}, it will be observed in Fig. 2 by measuring that the system under the effect of gravity and surface tension is unstable in the domain $0 < k < 1$ and stable in the domain $1 \leq k < \infty$

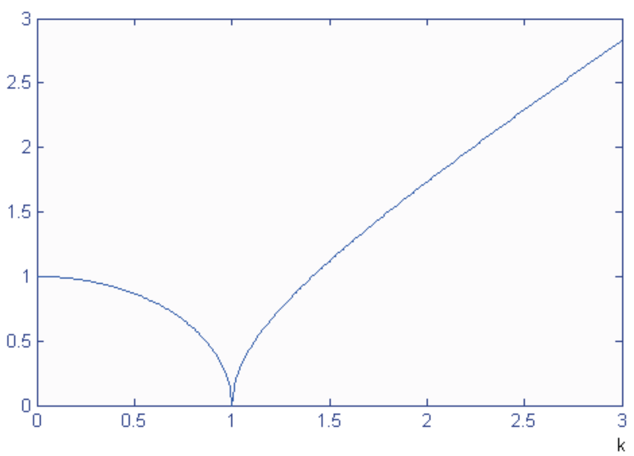

Figure 2 The growth rate σ versus the wave number k

According to Eq. (41) and by measuring σ, k and V_A in the units $\sqrt{gk}\,\text{sec}^{-1}$, $\sqrt{\rho_0 g/S}$ cm^{-1} and $\sqrt{gk}/k_x$ cm sec^{-1} respectively, we can study the stability of the system under the effect of gravity, surface tension and the magnetic field as shown in Fig. 3, in which the curves labelled 1, 2, 3, 4, 5 and 6 are for values of $V_A = 0.1$, 0.3, 0.5, 0.7, 1.0 and 1.2, respectively. When $V_A = 0.1$, 0.3, 0.5, 0.7 the unstable domains are $0 < k < 0.995$, $0 < k < 0.954$, $0 < k < 0.866$, $0 < k < 0.714$ and the stable domains are $0.995 \leq k < \infty$, $0.954 \leq k < \infty$, $0.866 \leq k < \infty$,

$0.714 \leq k < \infty$, where the equalities correspond to the marginal stability, but when $V_A = 1.0$ and $V_A = 1.2$ the system is stable for all wavelengths.

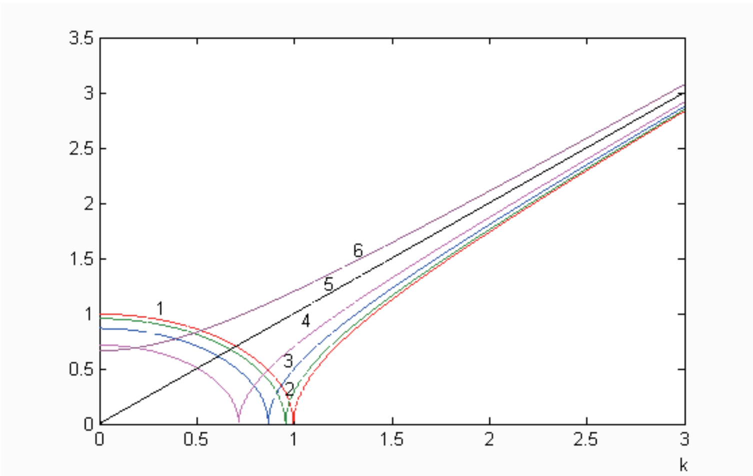

Figure 3 The growth rate σ versus the wave number k for different values of V_A

7. Conclusion

For a compressible fluid layer below a vacuum acted by capillary, pressure gradient, external gravity and electromagnetic forces. The compressibility has a stabilizing influence for all wavelengths and surface tension has strong stabilizing effect for all sufficiently short wavelengths but the system remains unstable for all sufficiently long wave lengths. The capillary instability of the compressible fluid may be completely suppressed above a certain value of the basic magnetic field.

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