Mechanics and Mechanical Engineering
Vol. 13, No. 1 (2009) 25–44

Generalized Thermo–Viscoelasticity under Three Theories

Mohamed I. A. Othman
Faculty of Education, Department of Mathematics,
Salalah-211, P.O. Box 2801, Sultanate of Oman

Received (10 October 2008)
Revised (15 March 2009)
Accepted (8 May 2009)

The model of the equations of the two–imensional generalized thermo–viscoelasticity based on Lord–Shulman (L–S), Green and Lindsay (G–L) and Classical dynamical–coupled (CD) theories are studied. The normal mode analysis is used to obtain the exact expressions for the temperature distribution, displacement components, and the thermal stresses. The resulting formulation is applied to two different concrete problems. The first concerns to the case of a heat punch across the surface of a semi–infinite thermo–viscoelastic half–space subject to appropriate boundary conditions. The second deals with a plate with thermo–isolated surfaces subjected to a time–dependent compression. A comparison is carried out between temperature, displacement and stress as calculated for each problem from view of the different theories of generalized thermo–viscoelasticity. Numerical results are given and illustrated graphically. Comparisons are made with the results predicted by three theories. The analysis presented in this paper is more general than any previous investigation.

Keywords: Generalized thermo–elasticity theories, temperature, displacement, stress

1. Introduction

Since the work of Maxwell, Boltzmann, Voigt, Kelvin and others, the linear visco–elasticity remains an important area of research. Gross [1], Staverman and Schwarzl [2], Alfery and Gurnee [3] and Ferry [4] investigated the mechanical–model representation of linear viscoelastic behavior results. Solution of boundary value problems for linear viscoelastic materials including temperature variations in both quasistatic and dynamic problems made great strides in the last decades, in the work of Biot [5], [6], Morland and Lee [7], Tanner [8] and Huilgol and Phan–Thien [9]. Bland [10] linked the solution of linear– viscoelasticity problems to corresponding linear elastic solutions. A notable works in this field was the work of Gurtin and Sternberg [11], Sternberg [12] and Ilioushin [13] offered an approximation method for the linear thermal viscoelastic problems. One can refer to the book of Ilioushin and Pobedria [14] for a formulation of the mathematical theory of thermal viscoelasticity and the solutions of some boundary value problems, as well as, to the work of Pobedria [15] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book of Koltunov [16].

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves.

Lord and Shulmann [17] introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier law. This new law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. The heat equation of this theory is of the wave-type, ensuring finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations remain the same as those for the coupled and the uncoupled theories. Dhaliwal and Sherief [18] extended this theory to general anisotropic media in the presence of heat sources. Ezzat et. al [19] surveyed a thermovisco–elasticity problem in two–dimensional by state space approach with one relaxation time. Othman [20] studied the problem of two-dimensional generalized electromagneto–thermovisco–elasticity based on the (L–S) theory for a thermally and electrically conducting half–space solid whose surface is subjected to a thermal shock.

Müller [21] was first introduced the theory of generalized thermoelasticity with two relaxation times. A more explicit version was then introduced by Green and Laws [22], Green and Lindsay [23] and independently by uhubi [24]. In this theory the temperature rates are considered among the constitutive variables. This theory also predicts finite speeds of propagation as in the (L–S) theory. It differs from the latter in that Fourier's law of heat conduction is not violated if the body under consideration has a center of symmetry. Ezzat and Othman [25] have studied some problems in electromagneto–thermoelastic waves with two relaxation times.

In the present paper, a comparison is carried out between displacement components temperature distribution and thermal stresses as calculated from the generalized thermo–viscoelasticity (L–S) and (G–L) theories for the problem under consideration. The (CD) theory is recovered as a special case. The results obtained in this study put in evidence the effects of the thermal relaxation times involved in the theories.

2. Formulation of the problem

We assume that there are no external forces or heat sources acting on a viscoelastic solid region. The solid is assumed to obey the equations of generalized thermo–visco–elasticity with thermal relaxation times, which consists of:

The equation of motion

$$\sigma_{ij,j} = \rho\, \ddot{u}_i \tag{1}$$

The generalized heat conduction equation

$$k T_{,ii} = \rho\, C_E\Big(\frac{\partial}{\partial t} \; + \; \tau \, \frac{\partial^2}{\partial t^2} \Big) T \, + \; \gamma \, T_o \, (\, \dot{u}_{i,i} + \; \tau \, \delta \, \ddot{u}_{i,i}) \tag{2}$$

The constitutive equation (Pobedria [15] and Fung [26])

$$S_{ij} = \int_0^t R(\, t - \tau\,) \, \frac{\partial \, e_{ij}(\, \bar{x}, \tau\,)}{\partial \, \tau} \; d\, \tau \; = \; \hat{R}\,(e_{ij}) \tag{3}$$

with the assumptions

$$\begin{array}{rcccl} \sigma_{ij}(\bar{x},t) & = & \dfrac{\partial \, \sigma_{ij}(\bar{x},t)}{\partial \, t} & = & 0 \\ \varepsilon_{ij}(\bar{x},t) & = & \dfrac{\partial \, \varepsilon_{ij}(\bar{x},t)}{\partial \, t} & = & 0 \\ & & -\infty < t < 0 & & \end{array} \tag{4}$$

where

$$\begin{array}{rcl} S_{ij} & = & \sigma_{ij} - \dfrac{\sigma_{kk}}{3} \, \delta_{ij} \\ e_{ij} & = & \varepsilon_{ij} - \dfrac{e}{3} \, \delta_{ij} \\ e & = & \varepsilon_{kk} \\ \sigma_{ij} & = & \sigma_{ji} \\ \bar{x} & \equiv & (x, y, z) \end{array}$$

and $R(t)$ is the relaxation function which can be taken (Koltunov [16]) in the form:

$$R(t) = 2\, \mu\, [\, 1 - A \int_0^t e^{-\, \beta \, t} \, t^{\alpha^* - 1} \; d\, t\,] \tag{5}$$

where, $(0 < \; \alpha^* < 1, \;\; A > 0, \;\; \beta > 0)$.

Assuming that the relaxation effects of the volume properties of the material are ignored, one can write for the generalized theory of thermo–viscoelasticity with thermal relaxation times

$$\sigma \;\; = K\, [\, e - 3 \;\, \alpha_T (\, T - T_o + \; \nu \, \dot{T}\,)\,] \tag{6}$$

where, $\sigma = \sigma_{ii}/3$.

Substituting from Eq. (6) into Eq. (3) we obtain

$$\sigma_{ij} \;\; = \hat{R}\,(\, \varepsilon_{ij} - \; \frac{e}{3} \; \delta_{ij}) \; + \; K\, e \, \delta_{ij} - \; \gamma\, (\, T - T_o + \; \nu \, \dot{T}\,)\, \delta_{ij} \tag{7}$$

From Eqs (1) and (7), it follows that

$$\rho\, \ddot{u}_i \;\; = \hat{R}\,(\, \frac{1}{2} \nabla^2 u_i + \frac{1}{6} e_{,i}) \; + \; K\, e_{,\, i} - \gamma\, (\, T - T_o + \nu \, \dot{T})_{,\, i} \tag{8}$$

In the above equations a dot denotes differentiation with respect to time, while a comma denotes material derivatives. The summation notation is used. We shall consider only the simplest case of the two–dimensional problem. We assume that all causes producing the wave propagation are independent of the variable z and that waves are propagated only in the xy–plane. Thus all quantities were appearing in Eqs (1)–(8) are independent of the z variable. Then the displacement vector has components $(\,u\,(x,\,y,\,t),\,v\,(x,\,y,\,t),\,0\,)$ (plane strain problem).

Moreover, the use of the relaxation times $\tau\,,\nu$ and a parameter δ makes the aforementioned fundamental equations possible for the three different theories:

1. Classical Dynamical Coupled theory (1956): $\tau\ = 0,\, \nu = 0\,,\, \delta = 0.$

2. Lord–Shulman's theory (1967): $\nu = 0,\, \tau > 0\,,\, \delta = 1.$

3. Green–Lindsay's theory (1972): $\nu \geq 0,\, \tau > 0\,,\, \delta = 0.$

Let us introduce the following non–dimensional variables

$$\begin{array}{rl} x' = c_o\eta_o x & y' = c_o\eta_o y \\ u' = c_o\eta_o u & v' = c_o\eta_o v \\ t' = c_o^2\eta_o t & \tau' = c_o^2\eta_o\tau \\ \nu' = c_o^2\eta_o\nu & \theta = \dfrac{\gamma(T-T_o)}{\rho c_o^2} \\ R' = \dfrac{2}{3K}R & {\sigma'}_{ij} = \dfrac{\sigma_{ij}}{K} \end{array}$$

In terms of these non–dimensional variables, Eqs (2), (7) and (8), taking the following form (dropping the dashes for convenience).

$$\frac{\partial^2 u}{\partial t^2} = \hat{R}(\phi) + \frac{\partial e}{\partial x} - (\frac{\partial\theta}{\partial x} + \nu\frac{\partial^2\theta}{\partial x\partial t}) \tag{9}$$

$$\frac{\partial^2 v}{\partial t^2} = \hat{R}(\psi) + \frac{\partial e}{\partial y} - (\frac{\partial\theta}{\partial y} + \nu\frac{\partial^2\theta}{\partial y\partial t}) \tag{10}$$

$$\nabla^2\theta = (\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2})\theta + \varepsilon\delta_o(\frac{\partial e}{\partial t} + \tau\delta\frac{\partial^2 e}{\partial t^2}) \tag{11}$$

$$\sigma_{xx} = \hat{R}(\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}) + e - (\theta + \nu\frac{\partial\theta}{\partial t}) \tag{12}$$

$$\sigma_{yy} = \hat{R}(\frac{\partial v}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}) + e - (\theta + \nu\frac{\partial\theta}{\partial t}) \tag{13}$$

$$\sigma_{zz} = -\frac{1}{2}\hat{R}(e) + e - (\theta + \nu\frac{\partial\theta}{\partial t}) \tag{14}$$

$$\sigma_{xy} = \frac{3}{4}\hat{R}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \tag{15}$$

where

$$\phi = \frac{\partial^2 u}{\partial x^2} + \frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{1}{4}\frac{\partial^2 v}{\partial x \partial y} \tag{16}$$

$$\psi = \frac{\partial^2 v}{\partial y^2} + \frac{3}{4}\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial x \partial y} \tag{17}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{18}$$

3. Normal mode analysis

Equations (9)–(18) are simplified by decomposing the solution in terms of modes so that

$$[u, v, \theta, \phi, \psi, e, \sigma_{ij}]\,(x, y, t) = [u^*, v^*, \theta^*, \phi^*, \psi^*, e^*, \sigma^*_{ij}](y) \exp{(\omega\, t + iax)} \tag{19}$$

It can be proved that:

$$\hat{R}(f(x,\, y,\, t)) = \int_0^t R(t-\tau)\frac{\partial f(x,\, y,\, t)}{\partial \tau} d\tau = \omega \bar{R}(\omega) f^*(y) \exp(\omega t + iax) \tag{20}$$

for any function $f(x,\, y,\, t)$ of class $C^{(1)}$, which satisfies the conditions:

$$f(x,\, y,\, t) = \frac{\partial f(x,\, y,\, t)}{\partial t} = 0, \quad (-\infty < t < 0) \tag{21}$$

where,

$$\bar{R}(\omega) = \int_0^\infty e^{-\omega t} R(t) dt \tag{22}$$

and ω is the (complex) time constant and a is the wave number in the x–direction.

This makes it possible to get from Eqs (9) and (10)

$$(D^2 - a^2)\Phi^*(y) + \omega \bar{R}(D^2 - a^2)e^*(y) = \omega^2 e^*(y) \tag{23}$$

where

$$\Phi^* = e^* - (1 + \nu\omega)\theta^* \tag{24}$$

Equation (11) simplifies to

$$\Phi^*(y) = \frac{1}{\varepsilon_1 \omega (1 + \tau\delta\omega)} \left[D^2 - a^2 - \omega_1 - \varepsilon_1 \omega \omega_2 (1 + \tau\delta\omega)\right] \theta^*(y) \tag{25}$$

where, $D = \frac{d}{dy}$.

Eliminating $\Phi^*(y)$ between Eq. (23) and (25), and using (24) we get:

$$(D^4 - a_1 D^2 + a_2)\theta^*(y) = 0 \tag{26}$$

where,

$$a_1 = \omega_1 + 2a^2 + \alpha\omega^2 + \alpha\varepsilon_1 \omega \omega_2 (1 + \tau\delta\omega) \tag{27}$$

$$a_2 = (a^2 + \alpha\omega^2)(a^2 + \omega_1) + \alpha\omega\varepsilon_1\omega_2 a^2 (1 + \tau\delta\omega) \tag{28}$$

Eq. (26) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)\theta^*(y) = 0 \tag{29}$$

where,

$$k_{1,2}^2 = (a^2 + \omega_3) \pm \omega_4 \tag{30}$$

$$\begin{aligned} \omega_1 &= \omega(1+\tau\omega) \\ \omega_2 &= (1+\nu\omega) \\ \omega_3 &= \frac{1}{2}[\omega_1 + \alpha\,\omega^2 + \alpha\,\varepsilon_1\omega\,\omega_2(1+\tau\,\delta\,\omega\,)] \\ \omega_4 &= \sqrt{a_1^2 - 4\,a_2} \\ \alpha &= \frac{1}{\omega\bar{R}+1} \end{aligned} \tag{31}$$

The solution of Eq. (29) is taken as:

$$\theta^*(y) = A_1\cosh(k_1y) + A_2\cosh(k_2y) + A_3\sinh(k_1y) + A_4\sinh(k_2y) \tag{32}$$

where A_1, A_2, A_3 and A_4 are some parameters depending on a and ω.

Substituting Eq. (32) into Eq. (25), we obtain:

$$\begin{aligned} \Phi^*(y) = &\left[\frac{k_1^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2(1+\tau\delta\omega)}{\varepsilon_1\omega(1+\tau\delta\omega)}\right][A_1\cosh(k_1y) + A_3\sinh(k_1y)] \\ &+ \left[\frac{k_2^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2(1+\tau\delta\omega)}{\varepsilon_1\omega(1+\tau\delta\omega)}\right][A_2\cosh(k_2y) + A_4\sinh(k_2y)] \end{aligned} \tag{33}$$

Substituting from Eqs (32) and (33) into Eq. (24) one obtain

$$\begin{aligned} e^*(y) &= \left(\frac{k_1^2 - a^2 - \omega_1}{\varepsilon_1\omega(1+\tau\delta\omega)}\right)[A_1\cosh(k_1y) + A_3\sinh(k_1y)] \\ &\quad + \left(\frac{k_2^2 - a^2 - \omega_1}{\varepsilon_1\omega(1+\tau\delta\omega)}\right)[A_2\cosh(k_2y) + A_4\sinh(k_2y)] \end{aligned} \tag{34}$$

Introducing the function

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{35}$$

we obtain from Eqs (9) and (10) after some manipulations:

$$(D^2 - a^2 - \alpha_o\omega^2)\Omega^* = 0 \tag{36}$$

then,

$$\Omega^*(y) = B_1\sinh(my) + B_2\cosh(my) \tag{37}$$

where B_1 and B_2 are some parameters depending on a and ω,

$$m^2 = a^2 + \alpha_o\omega^2, \quad \alpha_o = \frac{4}{3\omega\bar{R}} \tag{38}$$

Since,

$$\Omega^* = iav^* - Du^*, \quad e^* = i\,a\,u^* + D\,v^* \tag{39}$$

From Eqs (34), (37) and (39), we obtain:

$$\begin{aligned} u^*(y) \;=\; & \frac{ia}{\omega\varepsilon_1(1+\tau\delta\omega)}\left\{\left(\frac{k_1^2-a^2-\omega_1}{(k_1^2-a^2)}\right)[A_1\cosh(k_1y)+A_3\sinh(k_1y)]\right. \\ & \left.+\left(\frac{k_2^2-a^2-\omega_1}{(k_2^2-a^2)}\right)[A_2\cosh(k_2y)+A_4\sinh(k_2y)]\right\} \\ & -\frac{m}{\alpha_o\omega^2}\,[B_1\cosh(my)+B_2\sinh(my)] \end{aligned} \tag{40}$$

and

$$\begin{aligned} v^*(y) \;=\; & \frac{1}{\omega\varepsilon_1(1+\tau\delta\omega)}\left\{\frac{k_1(k_1^2-a^2-\omega_1)}{(k_1^2-a^2)}[A_1\sinh(k_1y)+A_3\cosh(k_1y)]\right. \\ & \left.+\frac{k_2(k_2^2-a^2-\omega_1)}{(k_2^2-a^2)}[A_2\sinh(k_2y)+A_4\cosh(k_2y)]\right\} \\ & +\frac{ia}{\alpha_o\omega^2}\,[B_1\sinh(my)+B_2\cosh(my)] \end{aligned} \tag{41}$$

Eqs (12)–(15) have the following form in the normal mode:

$$\sigma^*_{xx} = \omega\bar{R}(iau^* - \frac{1}{2}Dv^*) + e^* - \omega_2\theta^* \tag{42}$$

$$\sigma^*_{yy} = \omega\bar{R}(Dv^* - \frac{1}{2}iau^*) + e^* - \omega_2\theta^* \tag{43}$$

$$\sigma^*_{xy} = \frac{3}{4}\omega\bar{R}(Du^* + iav^*) \tag{44}$$

$$\sigma^*_{zz} = (1 - \frac{1}{2}\omega\bar{R})e^* - \omega_2\theta^* \tag{45}$$

Substituting from Eqs (32), (34), (40) and (41) into Eqs (42)–(45) we get

$$\begin{aligned} \sigma^*_{xx}(y) = & \frac{1}{\alpha\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}\{C_1[A_1\cosh(k_1y)+A_3\sinh(k_1y)] \\ & +C_2\,[A_2\cosh(k_2y)+A_4\sinh(k_2y)]\} - \frac{2iam}{\alpha_o^2\omega^2}[B_1\cosh(my)+B_2\sinh(my)] \end{aligned} \tag{46}$$

$$\begin{aligned} \sigma^*_{yy}(y) = & \frac{1}{\alpha\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}\{C_3\,[A_1\cosh(k_1y)+A_3\sinh(k_1y)] \\ & +C_4\,[A_2\cosh(k_2y)+A_4\sinh(k_2y)]\} - \frac{2iam}{\alpha_o^2\omega^2}[B_1\cosh(my)+B_2\sinh(my)] \end{aligned} \tag{47}$$

$$\sigma^*_{zz}(y) = \frac{1}{\alpha\alpha_o\omega^2(1+\tau\delta\omega)}\{C_5[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)]$$
$$+C_6\,[A_2\cosh(k_2 y){+}A_4\sinh(k_2 y)]\} \tag{48}$$

$$\sigma^*_{xy}(y) = \frac{2ia}{\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}\{C_7\,[A_1\sinh(k_1 y) + A_3\cosh(k_1 y)]$$
$$+C_8\,[A_2\sinh(k_2 y){+}A_4\cosh(k_2 y)]\} - \frac{(m^2+a^2)}{\alpha_o^2\omega^2}\,[B_1\sinh(my) + B_2\cosh(my)] \tag{49}$$

$$C_1 = \frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}\left[\alpha_o(k_1^2 - a^2) - 2\alpha k_1^2\right] - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{50}$$
$$C_2 = \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}\left[\alpha_o(k_2^2 - a^2) - 2\alpha k_2^2\right] - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{51}$$
$$C_3 = \frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}\left[\alpha_o(k_1^2 - a^2) + 2\alpha a^2\right] - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{52}$$
$$C_4 = \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}\left[\alpha_o(k_2^2 - a^2) + 2\alpha a^2\right] - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{53}$$
$$C_5 = (\alpha_o - 2\alpha)(k_1^2 - a^2 - \omega_1) - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{54}$$
$$C_6 = (\alpha_o - 2\alpha)(k_2^2 - a^2 - \omega_1) - \alpha\alpha_o\omega_2\omega\varepsilon_1(1+\tau\delta\omega) \tag{55}$$

$$C_7 = k_1\frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)} \tag{56}$$
$$C_8 = k_2\frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)} \tag{57}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. Assuming that all the relations (temperature ... etc.) are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Application

Problem I: A time–dependent heat punch across the surface of semi–infinite thermo–viscoelastic half–space (Ezzat and Othman [25]).

We shall consider a homogeneous isotropic thermo–viscoelastic solid occupying the region $G = \{\,(x,\,y,\,z)|\; x,\,z \in R,\; y \le 0\}$.

In the physical problem, we shall suppress the positive exponential, which are unbounded at infinitely. Thus we should replace each $\sinh(k_i y)$ by $[\frac{1}{2}\exp(k_i y)]$, $\cosh(k_i y)$ by $[\frac{1}{2}\exp(k_i y)]$, where $i = 1, 2$, $\sinh(my)$ by $[\frac{1}{2}\exp(my)]$ and $\cosh(my)$ by $[\frac{1}{2}\exp(my)]$.

Then, Eqs (32), (33), (40), (41) and (46)–(49) can be written as:

$$\theta^*(y) = A_1^*\exp(k_1 y) + A_2^*\exp(k_2 y) \tag{58}$$

$$\Phi^*(y) = \frac{1}{\varepsilon_1\omega(1+\tau\delta\omega)}[\alpha_1 A_1^*\exp(k_1 y) + \alpha_2 A_2^*\exp(k_2 y)] \tag{59}$$

where,

$$\alpha_1 = k_1^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2(1+\tau\delta\omega) \tag{60}$$

$$\alpha_2 = k_2^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2(1+\tau\delta\omega) \tag{61}$$

$$\begin{aligned} u^*(y) = \frac{ia}{\omega\varepsilon_1(1+\tau\delta\omega)} &\left\{ \frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)} A_1^*\exp(k_1 y) \right. \\ &\left. + \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)} A_2^*\exp(k_2 y) \right\} - \frac{m}{\alpha_o\omega^2} B_1^*\exp(-my) \end{aligned} \tag{62}$$

$$\begin{aligned} v^*(y) = \frac{-1}{\omega\varepsilon_1(1+\tau\delta\omega)} &\left\{ \frac{k_1(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)} A_1^*\exp(k_1 y) \right. \\ &\left. + \frac{k_2(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)} A_2^*\exp(k_2 y) \right\} - \frac{ia}{\alpha_o\omega^2} B_1^*\exp(my) \end{aligned} \tag{63}$$

$$\begin{aligned} \sigma_{xx}^*(y) &= \frac{1}{\alpha\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}[C_1 A_1^*\exp(k_1 y) + C_2 A_2^*\exp(k_2 y)] \\ &\quad - \frac{2iam}{\alpha_o^2\omega^2} B_1^*\exp(my) \end{aligned} \tag{64}$$

$$\begin{aligned} \sigma_{yy}^*(y) &= \frac{1}{\alpha\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}[C_3 A_1^*\exp(k_1 y) + C_4 A_2^*\exp(k_2 y)] \\ &\quad - \frac{2iam}{\alpha_o^2\omega^2} B_1^*\exp(my) \end{aligned} \tag{65}$$

$$\sigma_{zz}^*(y) = \frac{1}{\alpha\alpha_o\omega^2}[C_5 A_1^*\exp(k_1 y) + C_6 A_2^*\exp(k_2 y)] \tag{66}$$

$$\begin{aligned} \sigma_{xy}^*(y) &= \frac{-2ia}{\alpha_o\omega\varepsilon_1(1+\tau\delta\omega)}[C_7 A_1^*\exp(k_1 y) + C_8 A_2^*\exp(k_2 y)] \\ &\quad + \frac{(m^2 + a^2)}{\alpha_o^2\omega^2} B_1^*\exp(my) \end{aligned} \tag{67}$$

where $C_1 - C_8$ are given by Eqs (50)–(57).

The boundary conditions on the surface $y = 0$ are taken to be:

$$\theta\,(x, 0,\, t) = n(x,\, t) \tag{68}$$

$$\sigma_x y(x, 0,\, t) = 0 \tag{69}$$

$$\sigma_{yy}(x,\, 0,\, t) = P(x,\, t) \tag{70}$$

where n and p are given function of x and t, by using the normal mode analysis we get

$$\theta^*(a,\, 0,\, \omega) = n^* \tag{71}$$

$$\sigma^*_{yy}(a,\, 0,\, \omega) = P^* \tag{72}$$

$$\sigma^*_{xy}(a,\, 0,\, \omega) = 0 \tag{73}$$

Eqs (58), (65) and (67) together with Eqs (71), (72) and (73) respectively give:

$$A_1^* + A_2^* = n^* \tag{74}$$

$$\alpha_o\omega[C_3A_1^* + C_4A_2^*] - 2iam\varepsilon_1\alpha B_1^*(1+\tau\delta\omega) = \alpha\alpha_o^2\omega^2\varepsilon_1(1+\tau\delta\omega)P^* \tag{75}$$

$$2ia\alpha_o\omega\,[C_7A_1^* + C_8A_2^*] - (m^2 + a^2)\varepsilon_1(1+\tau\delta\omega)B_1^* = 0 \tag{76}$$

Solving Eqs (74)–(76) for the unknowns A_1^*, A_2^* and B_1^* one obtains:

$$A_1^* = -\frac{E_1}{E_3} \tag{77}$$

$$A_2^* = n^* + \frac{E_1}{E_3} \tag{78}$$

$$B_1^* = \frac{2iaE_2\alpha_o\omega}{\varepsilon_1(m^2 + a^2)(1+\tau\delta\omega)} \tag{79}$$

where,

$$\begin{aligned} E_1 &= a^2\,[n^*(C_4 + 4\alpha mC_8) - \alpha\varepsilon_1\alpha_o\omega P^*(1+\tau\delta\omega)] \\ &+m^2\,[n^*C_4 - \alpha\varepsilon_1\alpha_o\omega P^*(1+\tau\delta\omega)] \end{aligned} \tag{80}$$

$$E_2 = C_8n^* - (C_7 - C_8)\frac{E_1}{E_3} \tag{81}$$

$$E_3 = a^2\,[4m\alpha(C_7 - C_8) + C_3 - C_4] + m^2(C_3 - C_4) \tag{82}$$

Problem II: A plate with thermo–isolated surfaces $y = \pm L$, subjected to time dependent compression (Ezzat et al. [19]).

We shall consider a homogeneous isotropic thermo-viscoelastic infinite thick flate plate of a finite thickness 2L occupying the region G^* given by

$$G^* = \{\ (x,\, y,\, z)\,|\ \ x,\, z \in R,\ \ -L \le y \le L\,\}$$

with the middle surface of the plate coinciding with the plate $y = 0$.

The boundary conditions of the problem are taken as:

$$\frac{\partial\theta}{\partial y} = 0, \qquad \text{on} \qquad y = \pm L \tag{83}$$

$$\sigma_{yy} = -P_o(x,\, y,\, t), \qquad \text{on} \qquad y = \pm L \tag{84}$$

$$\sigma_{xy} = 0, \qquad \text{on} \qquad y = \pm L \tag{85}$$

by using the normal mode we obtain

$$\frac{\partial \theta^*}{\partial y} = 0, \qquad \text{on} \qquad y = \pm L \tag{86}$$

$$\sigma^*_{yy} = -P^*_o, \qquad \text{on} \qquad y = \pm L \tag{87}$$

$$\sigma^*_{xy} = 0, \qquad \text{on} \qquad y = \pm L \tag{88}$$

Eqs (32), (47) and (49) together with Eqs (86), (87) and (88) respectively give:

$$k_1\bar{A}_1 \sinh(k_1L) + k_2\bar{A}_2 \sinh(k_2L) = 0 \tag{89}$$

$$\begin{aligned}&\alpha_o\omega\left\{C_3\bar{A}_1\cosh(k_1L) + C_4\bar{A}_2\cosh(k_2L)\right\} - 2iam\alpha\varepsilon_1\bar{B}_1\cosh(mL)\\ &= -\alpha\varepsilon_1\alpha_o^2\omega^2P^*_o(1+\tau\delta\omega)\end{aligned} \tag{90}$$

$$\begin{aligned}&2ia\alpha_o\omega\left\{C_7\bar{A}_1\sinh(k_1L) + C_8\bar{A}_2\sinh(k_2L)\right\}\\ &-\varepsilon_1(m^2+a^2)(1+\tau\delta\omega)\bar{B}_1\sinh(mL) = 0\end{aligned} \tag{91}$$

where, $\bar{A}_1$ $\bar{A}_2$ are parameters depending on a and ω.

Equations (89), (90) and (91) can be solved for the three unknowns $\bar{A}_1$, $\bar{A}_2$ and $\bar{B}_1$

$$\bar{A}_1 = \frac{\alpha\alpha_o\varepsilon_1\omega(m^2+a^2)k_2P^*_o(1+\tau\delta\omega)^2\sinh(mL)\sinh(k_2L)}{\bar{\Delta}} \tag{92}$$

$$\bar{A}_2 = -\frac{\alpha\alpha_o\varepsilon_1\omega(m^2+a^2)k_1P^*_o(1+\tau\delta\omega)^2\sinh(mL)\sinh(k_1L)}{\bar{\Delta}} \tag{93}$$

$$\bar{B}_1 = \frac{2ia\alpha\alpha_o^2\omega^2(C_7k_2 - C_8k_1)P^*_o(1+\tau\delta\omega)\sinh(k_1L)\sinh(k_2L)}{\bar{\Delta}} \tag{94}$$

where,

$$\bar{\Delta} \quad = \quad k_1C_{11}\sinh(k_1L) - k_2C_{12}\sinh(k_2L) \tag{95}$$

$$\begin{aligned}C_{11} \quad = \quad &4a^2m\alpha C_8\sinh(k_2L)\cosh(mL)\\ &+(m^2+a^2)C_4(1+\tau\delta\omega)\cosh(k_2L)\sinh(mL)\end{aligned} \tag{96}$$

$$\begin{aligned}C_{12} \quad = \quad &4a^2m\alpha C_7\sinh(k_1L)\cosh(mL)\\ &+(m^2+a^2)C_3(1+\tau\delta\omega)\cosh(k_1L)\sinh(mL)\end{aligned} \tag{97}$$

5. Numerical results

As a numerical example, we have considered polymethyl methacrylate, which has wide applications in industry and medicine. Since we have $\omega = \omega_o + i\zeta$ where i imaginary unit, $e^{\omega t} = e^{\omega_o t}(cos\zeta t + isin\zeta t)$ and for small values of time, we can take $\omega = \omega_o$ (real). Taking $\alpha^* = 0.5$ in Eq. (5) and using Eq. (22) we get:

$$\bar{R}(\omega) = \frac{4\mu}{3K}\left[\frac{1}{\omega_o} - \frac{A\sqrt{\pi}}{\omega_o\sqrt{\omega_o + \beta}}\right] \tag{98}$$

The numerical constants are taken as:
$4\mu/3K = 0.8$, $A = 0.106$, $\varepsilon_1 = 0.045$, $\beta = 0.005$, $T_o = 773$ K, $\tau = 0.02$, $\nu = 0.05$, $\omega_o = 2$, $\alpha = 0.59037$.
The real part of the function θ, the displacement components u and stress component σ_{xx}, were evaluated for problem (I) on the plane $L = -4$, and for problem (II) on the plane ($y = 1$) where $L = 2$ and on the middle plane ($y = 0$) for the two different values of time namely $t = 0.001$ and $t = 0.2$. These results are shown in Figs 1–9. The graph shows the sixth curves predicted by the different theories of thermo–viscoelasticity. In these figures the solid lines represent the solution corresponding to using the (CD) theory of heat conduction ($\tau = \nu = 0$ and $\delta = 0$), the dashed lines represent the solution for the (G-L) theory ($\tau = 0.02$, $\nu = 0.05$ and $\delta = 0$) and the dotted dashed lines represent the solution for the (L–S) theory ($\tau = 0.02$, $\nu = 0$ and $\delta = 1$). The phenomenon of finite speeds of propagation is manifested in all these figures.

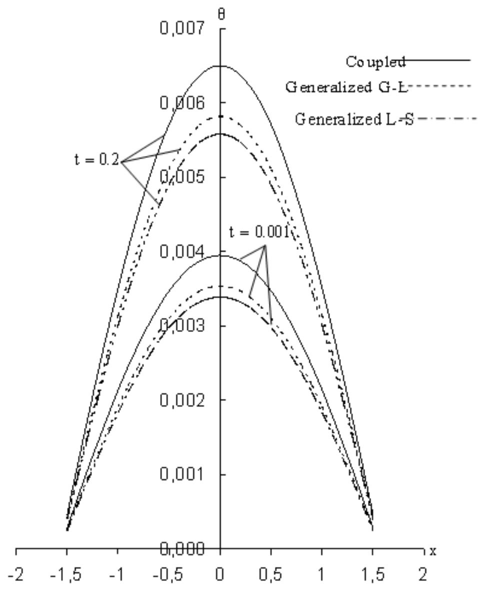

Figure 1 Temperature distribution θ for the problem I

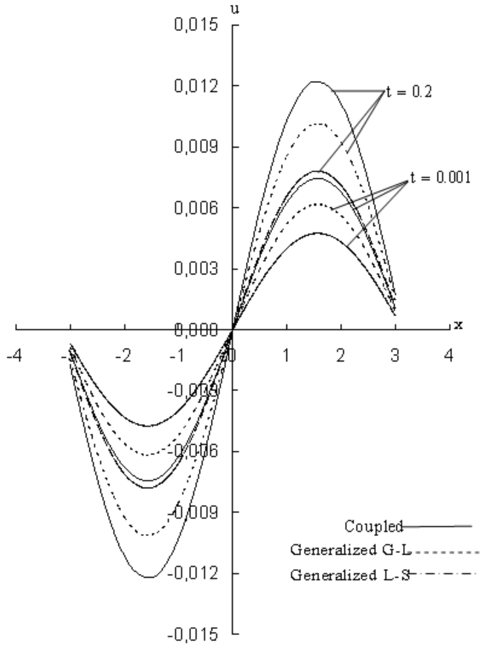

Figure 2 Horizontal displacement distribution u for the problem I

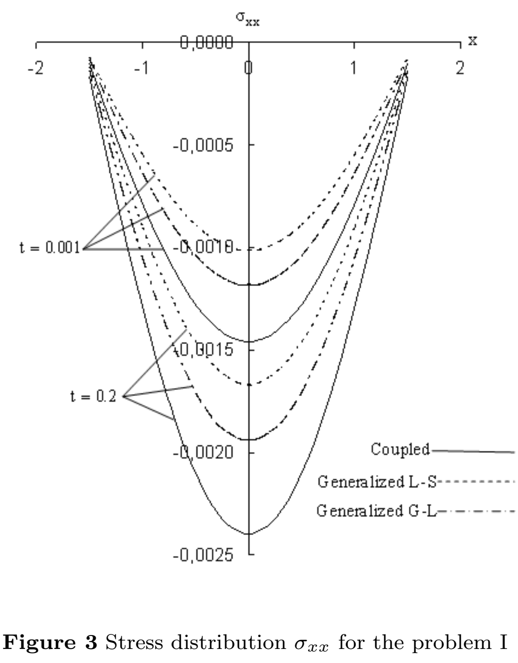

Figure 3 Stress distribution σ_{xx} for the problem I

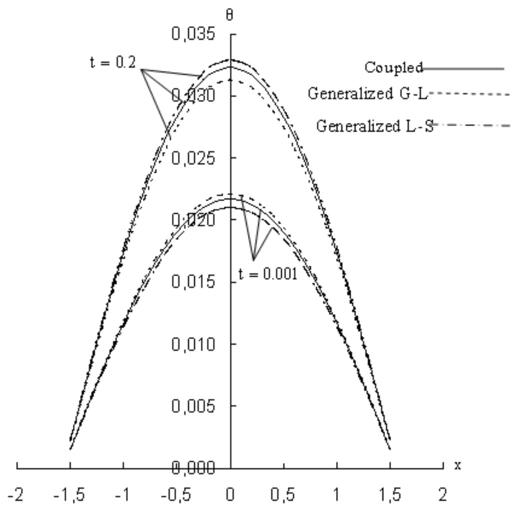

Figure 4 Temperature distribution on the surface plane for the problem II

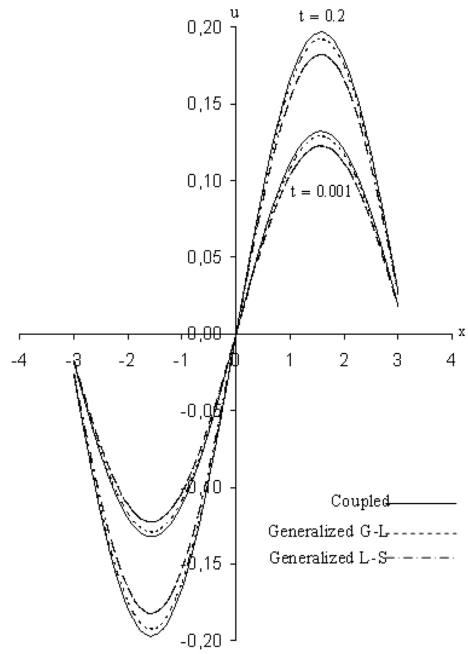

Figure 5 Horizontal displacement distribution u on the surface plane for the problem II

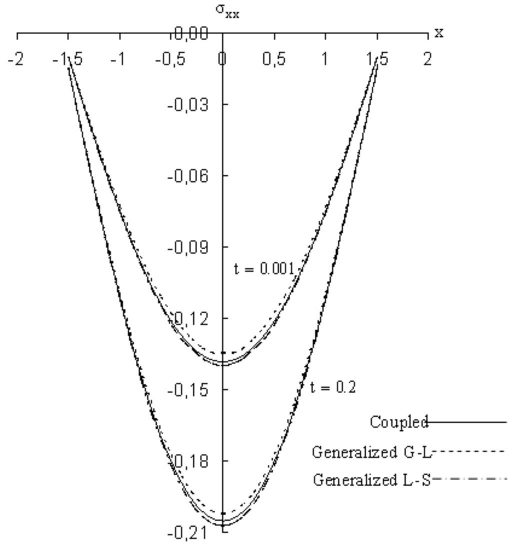

Figure 6 Stress distribution σ_{xx} on the surface plane for the problem I

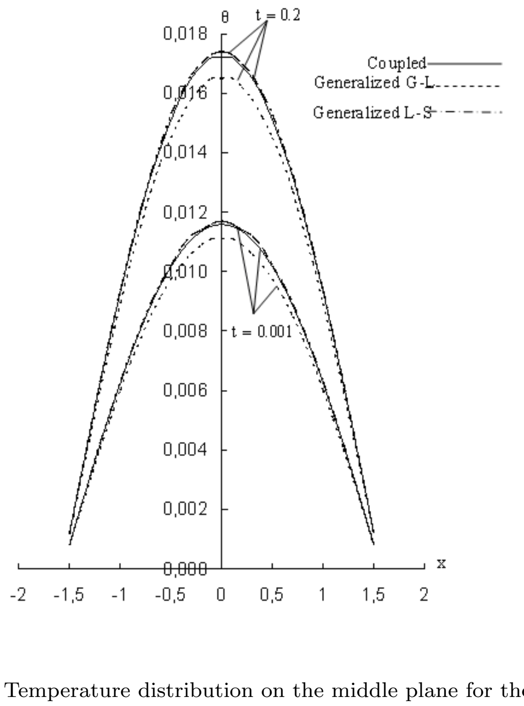

Figure 7 Temperature distribution on the middle plane for the problem II

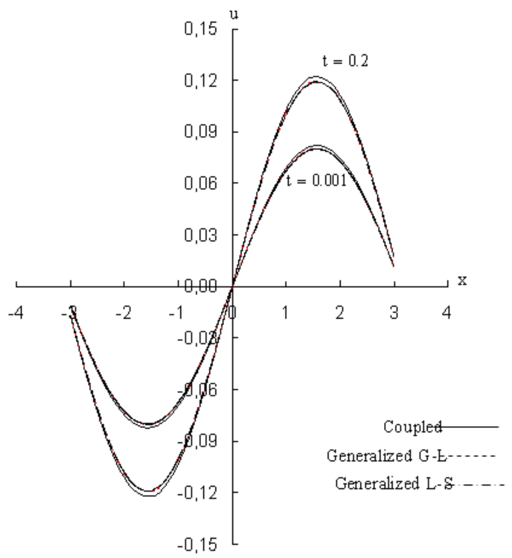

Figure 8 Horizontal displacement distribution on the middle plane for the problem II

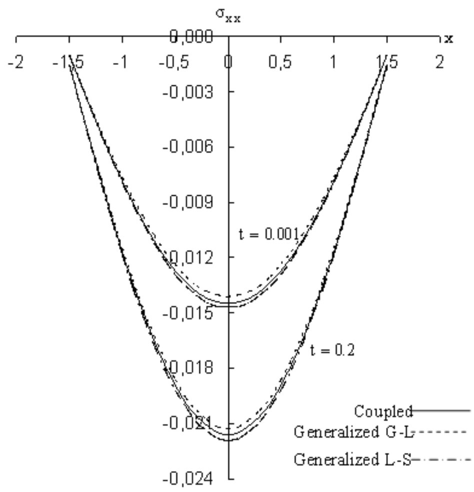

Figure 9 Stress distribution σ_{xx} on the middle plane for the problem I

It was found that near the surface of the solid where the boundary conditions dominate the coupled and the generalized theories give very close results. We notice also, that results for the temperature, displacement and stress distributions when the relaxation time is appeared in the heat equation are distinctly different from those when the relaxation time is not mentioned in the heat equation. This is due to the fact that thermal wave in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non–Fourier case. It is clear that for small values of time the solution is localized in a finite region. This region grows with increasing time and its edge is the location of the wave front. This region is determined only by the values of time t and the relaxation times τ and ν.

6. Concluding remarks

Owing to the complicated nature of the equations for the generalized thermo–viscoelasticity, few attempts have been made to solve problems in this field, theses attempts utilize approximate methods valid for only a specific range of some parameter [15].

In this work, the method of normal mode analysis is introduced for the solution of two–dimensional problems in generalized thermo–viscoelasticity and applied to two specific problems in which the temperature, displacement and stress are coupled. This method gives exact expressions without any assumed restrictions on either the temperature or displacement. This paper is considering the first study of the two–dimensional generalized thermo–viscoelasticity based on the Classical dynamical–coupled theorem, Green–Lindsay and Lord–Shulman theories simultaneously.

References

[1] **Gross, B**: Mathematical structure of the theories of viscoelasticity, Hermann, Paris, **(1953)**.

[2] **Staverman, A.J. and Schwarzl, F.**: Die physik der hochpolymeren, *Ed. H. A. Stuart*, vol. 4, Springer Verlag, Ch. 1, **(1956)**.

[3] **Alfrey T. and Gurnee, E.F.**: Rheology theory and applications, Ed. F. R. Eirich, vol. 1, *Academic Press*, New York, **(1956)**.

[4] **Ferry, J.D.**: Viscoelastic properties of polymers, *J. Wiley and Sons*, New York, **(1970)**.

[5] **Biot, M.A.**: Theory of stress–strain relations in an isotropic viscoelasticity, and relaxation phenomena, *J. Applied Physics*, 25, **(1954)**.

[6] **Biot, M.A.**: Variational principal in irreversible thermodynamics with application to viscoelasticity, *Physical Review*, 97, **(1955)**.

[7] **Morland, L.W. and Lee, E.H.**: *Trans Soc. Rheol.*, 4, **(1960)**.

[8] **Tanner, R.I.**: Engineering rheology, *Oxford Un. Press.*, **(1988)**.

[9] **Huilgol, R. and Phan–Thien, N.**: Fluid mechanics of viscoelasticity, Elsevier, **(1997)**.

[10] **Bland, D.R.**: The theory of linear viscoelasticity, *Pergamon Press*, Oxford, **(1960)**.

[11] **Gurtin, M.E. and Sternberg, E.**: On the linear theory of viscoelasticity, *Arch. Ration. Mech. Anal.*, 11, 182-191, **(1962)**.

[12] **Sternberg, E.**: On the analysis of thermal stresses in viscoelastic solids, *Brown Univ. Dir, Appl. Math. TR.*, 19, 213-219, **(1963)**.

[13] **Ilioushin, A.A.**: The approximation method of calculating the constructures by linear thermal viscoelastic theory, *Mekhanika Polimerov*, Riga, 2, 168–178, **(1968)**.

[14] **Ilioushin, A.A. and Pobedria, B.E.**: Mathematical theory of thermal visco–elasticity, *Nauka*, Moscow, **(1970)**.

[15] **Pobedria, B.E.**: Coupled problems in continuum mechanics, *J. Durabilility and plasticity*, No.1, Moscow State University, Moscow, **(1984)**.

[16] **Koltunov, M.A.**: Creeping and relaxation, Moscow, **(1976)**.

[17] **Lord, H. and Shulman, Y.**: A Generalized dynamical theory of thermo–elasticity, *J. Mech. Phys. Solid*, 15, 299–309, **(1967)**.

[18] **Dhaliwal, R. and Sherief, H.H.**: Generalized thermoelasticity for an isotropic media, *Quart. Appl. Math.*, 33, 1-8, **(1980)**.

[19] **Ezzat, M.A., Othman, M.I.A. and El Karamany, A.S.**: State space approach to two–dimensional generalized thermo–viscoelasticity with one relaxation time, *J. Thermal Stresses*, 25, 259–316, **(2002)**.

[20] **Othman, M.I.A.**: Generalized electromagneto–thermoviscoelastic in case of 2–D thermal shock problem in a finite conducting medium with one relaxation time, *Acta Mechanica*, 169, no. 1–4, 37–51, **(2004)**.

[21] **Müller, I.**: The Coldness, A Universal function in thermo–elastic Solids, *Arch. Rat. Mech. Anal.*, 41, 319, **(1971)**.

[22] **Green, A.E. and Laws, N.**: On the entropy production inequality, *Arch. Rat. Mech. Anal.*, 45, 47, **(1972)**.

[23] **Green, A.E. and Lindsay, K.A.**: Thermoelasticity, J. Elasticity, 2, 1, **(1972)**.

[24] **Şuhubi, E.S.**: Thermoelastic solids. In Continuum Physics II, Ch. 2 ed. A. C. Eringen., *Academic Press*, New York, **(1975)**.

[25] **Ezzat, M.A. and Othman, M.I.**: Electromagneto–thermoelastic plane waves with two relaxation times in a medium of perfect conductivity, *Int. J. Engng. Sci.*, 38, 107-120, **(2000)**.

[26] **Fung, Y.C.**: Foundation of solid mechanics, Prentice–Hall, Englewood Cliffs, NJ, **(1968)**.

Nomenclature

ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_o	reference temperature chosen so that $\lvert T - T_o \rvert << 1$
u_i	components of displacement vector
ε_{ij}	components of strain tensor
$e = \varepsilon_{kk}$	the dilatation
σ_{ij}	components of stress deviator
e_{ij}	components of strain deviator
k	thermal conductivity
$\tau,\ \nu$	two relaxation times
$\lambda,\ \mu$	Lamé's constants
$K = \lambda + \frac{2}{3}\mu$	
α_t	coefficient of linear thermal expansion
$\gamma = K\alpha_t$	
$\varepsilon = \frac{\gamma}{\rho C_E}$	
$\eta_o = \frac{\rho C_E}{k}$	
$c_o^2 = \frac{K}{\rho}$	
$\varepsilon_1 = \delta_o \varepsilon$	
$T_o = \frac{\delta_o\, \rho\, c_o^2}{\gamma} = \frac{\delta_o}{3\alpha_T}$	
δ_o	non–dimensional number
$\alpha^*,\ \beta,\ A$	are empirical constants