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The Thermal Relaxation Effect on 2–D Problems of the Generalized Linear Thermo–Viscoelasticity

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In this paper we introduced the normal mode analysis for two–dimensional problems of the generalized linear thermo–viscoelasticity with one relaxation time. The exact expressions for the temperature distribution, the displacement components and the stress are obtained. The resulting formulation is applied to three different concrete problems. The first deals with a thick plate subjected to a time–dependent heat source on each face. The second concerns to the case of a heated punch moving across the surface of a semi–infinite thermo–viscoelastic half–space subjected to appropriate boundary conditions and the third problem deals with a plate with thermo–isolated surfaces subjected to a time–dependent compression. Numerical results are given and illustrated graphically for each problem. Comparisons are made with the results predicted by the coupled theory.

Keywords: Thermo–visco–elasticity, normal mode, relaxation time and coupled theory

1. Introduction

The linear viscoelasticity remains an important area of research. Gross [1], Staverman and Schwarz [2], Alfery and Gurnee [3] and Ferry [4] investigated the mechanical model representation of linear viscoelastic behavior results. Solution of boundary value problems for linear viscoelastic materials including temperature variations in both quasistatic and dynamic problems made great strides in the last decades, in the work of Biot [5,6] and Huilgol and Phan–Thien [7]. Bland [8] linked the solution of linear–viscoelasticity problems to corresponding linear elastic solutions. A notable works in this field was the work of Gurtin and Sternberg [9], and Ilioushin [10] offered an approximation method for the linear thermal viscoelastic problems. One can refer to the book of Ilioushin and Pobedria [11] for a formulation of the mathematical theory of thermal viscoelasticity and the solutions of some boundary value problems, as well as, to the work of Pobedria [12] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book of Koltunov [13].

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves.

Biot [14] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical obser–vations. Lord and Shulman [15] introduced the theory of generalized thermo–elasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier law. This law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as relaxation time. The heat equation of this theory is of the wave–type, ensuring finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and the constitutive relations remain the same as those for the coupled and the uncoupled theories. Dhaliwal and Sherief [16] extended this theory to general anisotropic media in the presence of heat sources. Othman et al. [17] studied the model of two–dimensional generalized thermo-viscoelasticity with two relaxation times. Othman [18] introduced the equations of generalized thermo–viscoelasticity based on Lord–Shulman (L–S), Green and Lindsay (G–L) and Classical dynamical coupled (CD) theories, by using Laplace transforms, a uniqueness theorem for these equations is proved, also, a reciprocity theorem is obtained. Othman [19] studied the generalized electromagneto–thermoviscoelastic in case of two–dimensional thermal shock problem in a finite conducting medium with one relaxation time. Recently, Othman [20] investigated the effect of rotation and relaxation time on a thermal shock problem for a half–space in generalized thermoviscoelasticity.
In the present work we shall formulate the normal mode analysis to two–dimensional problems of thermo–viscoelasticity with one relaxation time. The resulting formulation is applied to three concrete problems. The exact expressions for temperature distribution, the displacement components and the stress are obtained for each problem.

2. Formulation of the problem

We assume that there are no external forces or heat sources acting on a viscoelastic solid region. The solid is assumed to obey the equations of generalized thermo–visco–elasticity with one relaxation time, which consists of:
The equation of motion

$$\sigma_{ij,j} = \rho\,\ddot{u}_i \tag{1}$$

The equation of generalized heat conduction

$$kT_{,ii} = \left(\frac{\partial}{\partial t} + \tau_0\,\frac{\partial^2}{\partial t^2}\right)(\rho\,C_E T + \gamma\,T_0\,e) \tag{2}$$

The constitutive equation (see, e.g. Pobedria [12] and Fung [21])

$$S_{ij} = \int_0^t R(t-\tau)\,\frac{\partial e_{ij}(\bar{x},\tau)}{\partial \tau}\,d\tau \;=\; \hat{R}(e_{ij}) \tag{3}$$

with the assumptions

$$\sigma_{ij}(\bar{x},t) = \frac{\partial \sigma_{ij}(\bar{x},t)}{\partial t} = 0\,, \quad \varepsilon_{ij}(\bar{x},t) = \frac{\partial \varepsilon_{ij}(\bar{x},t)}{\partial t} = 0\,, \qquad -\infty < t < 0 \tag{4}$$

where,

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\,\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\,\delta_{ij}\;, \quad e = \varepsilon_{kk}\;, \quad \sigma_{ij} = \sigma_{ji}\;,\; \bar{x} \equiv (x,y,z)$$

and R(t) is the relaxation function which can be taken (see e.g. Koltunov [13]) in the form:

$$R(t) = 2\,\mu\,[\,1 - A\int_0^t e^{-\beta\,t}\,t^{\alpha^*-1}\,dt\,] \tag{5}$$

where, $(0 < \alpha^* < 1,\, A > 0,\, \beta > 0)$.

Assuming that the relaxation effects of the volume properties of the material are ignored, one can write for the generalized theory of thermo–viscoelasticity with one relaxation time

$$\sigma \;= K\,[\,e - 3\,\alpha_T(\,T - T_0)\,]\,. \tag{6}$$

where, $\sigma = \sigma_{ii}/3$.

Substituting from equation (6) into equation (3) we obtain

$$\sigma_{ij} \;= \hat{R}\,(\,\varepsilon_{ij} - \;\frac{e}{3}\;\delta_{ij}) \;+\; K\,e\,\delta_{ij} - \;\gamma\,(\,T - T_0)\,\delta_{ij} \tag{7}$$

From equation (1) and (7), it follows that

$$\rho\,\ddot{u}_i \;= \hat{R}\,(\,\frac{1}{2}\nabla^2 u_i + \frac{1}{6}e_{,i}) \;+\; K\,e_{,\,i} - \gamma\,(\,T - T_0)_{,\,i} \tag{8}$$

We shall consider only the simplest case of the two–dimensional problem. We assume that all causes producing the wave propagation is independent of the variable z and that waves are propagated only in the xy–plane. Thus all quantities were appearing in equations (1)–(8) are independent of the variable z. Then the displacement vector has components $(u(x,\,y,\,t),\; v(x,\,y,\,t),\; 0)$ (plane strain problem).

Let us introduce the following non–dimensional variables

$$\begin{array}{lll} x' = c_0\eta_0 x & y' = c_0\eta_0 y & u' = c_0\eta_0 u \\ v' = c_0\eta_0 v & t' = c_0^2\eta_0 t & \tau'_0 = c_0^2\eta_0\tau_0 \\ \theta = \dfrac{\gamma\,(T - T_0)}{\rho\,c_0^2} & R' = \frac{2}{3K}R & \sigma'_{ij} = \dfrac{\sigma_{ij}}{K} \end{array}$$

In terms of these non–dimensional variables, equation (2), (7) and (8), taking the following form (dropping the dashes for convenience).

$$\frac{\partial^2 u}{\partial t^2} = \hat{R}(\zeta) + \frac{\partial e}{\partial x} - \frac{\partial \theta}{\partial x} \tag{9}$$

$$\frac{\partial^2 v}{\partial t^2} = \hat{R}(\psi) + \frac{\partial e}{\partial y} - \frac{\partial \theta}{\partial y} \tag{10}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\theta + \varepsilon_1 e) \tag{11}$$

$$\sigma_{xx} = \hat{R}\left(\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}\right) + e - \theta \tag{12}$$

$$\sigma_{yy} = \hat{R}\left(\frac{\partial v}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}\right) + e - \theta \tag{13}$$

$$\sigma_{zz} = -\frac{1}{2}\hat{R}(e) + e - \theta \tag{14}$$

$$\sigma_{xy} = \frac{3}{4}\hat{R}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{15}$$

where

$$\zeta = \frac{\partial^2 u}{\partial x^2} + \frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{1}{4}\frac{\partial^2 v}{\partial x\,\partial y} \tag{16}$$

$$\psi = \frac{\partial^2 v}{\partial y^2} + \frac{3}{4}\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial x\,\partial y} \tag{17}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{18}$$

3. Normal mode analysis

Equations (9)–(11) are simplified by decomposing the solution in terms of normal modes so that

$$[u,\, v,\, \theta,\, \zeta,\, \psi,\, e,\, \varepsilon_{ij},\, \sigma_{ij}](x,y,t) = [u^*, v^*, \theta^*, \zeta^*, \psi^*, e^*, {\varepsilon_{ij}}^*, {\sigma_{ij}}^*](y)\exp(\omega t + iax) \tag{19}$$

It can be proved that:

$$\hat{R}(f(x,\, y,\, t)) = \int_0^t R(t-\tau)\frac{\partial f(x,y,t)}{\partial \tau}d\tau = \omega \bar{R}(\omega) f^*(y) exp(\omega t + iax) \tag{20}$$

for any function $f(x,\, y,\, t)$ of class $C^{(1)}$, which satisfies the conditions:

$$f(x,\, y,\, t) = \frac{\partial f(x,y,t)}{\partial t} = 0, \qquad (-\infty < t < 0) \tag{21}$$

the function $f(x,\, y,\, t)$must belong to the original domain of the Laplace transform, i.e. the function must be additionally be assumed to be bounded growth with respect to the time variable, where,

$$\bar{R}(\omega) = \int_0^\infty e^{-\omega t} R(t)\, dt \tag{22}$$

and ω is the (complex) time constant and a is the wave number in the x–direction.

Differentiating partially each of equation (9) with respect to x and equation (10) with respect to y and adding them, this makes it possible to get

$$\theta^*(y) = \frac{1}{\alpha\,\omega^2}\left[D^2 - a^2 - \alpha\,\omega^2\right]\ \Phi^*(y) \tag{23}$$

where

$$\Phi^* = \frac{1}{\alpha}\, e^* - \theta^* \tag{24}$$

$$\alpha = \frac{1}{\omega\,\bar{R} + 1} \tag{25}$$

Equation (11) together with equation (24) simplifies to

$$\left[D^2 - a^2 - (1 + \alpha\,\varepsilon_1)\,\omega_1\right]\theta^*(y) = \varepsilon_1 \alpha\,\omega_1 \Phi^*(y) \tag{26}$$

where $D = d/dy$.

Eliminating $\theta^*(y)$ between equations (23) and (26), we get:

$$(D^4 - a_1 D^2 + a_2)\Phi^*(y) = 0 \tag{27}$$

where

$$a_1 = 2a^2 + \alpha\omega^2 + (1 + \alpha\,\varepsilon_1)\omega_1 \tag{28}$$

$$a_2 = (a^2 + \alpha\,\omega^2)(a^2 + \omega_1) + \alpha\,\varepsilon_1\omega_1 a^2 \tag{29}$$

Equation (27) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)\Phi^*(y) = 0 \tag{30}$$

where

$$k_{1,2}^2 = (a^2 + \omega_2) \pm \omega_3 \tag{31}$$

$$\omega_1 = \omega(1 + \tau_0\omega), \quad \omega_2 = \frac{1}{2}[\alpha\,\omega^2 + (1 + \alpha\,\varepsilon_1)\omega_1], \quad \omega_3 = \sqrt{\omega_2^2 - \alpha\,\omega^2\omega_1} \tag{32}$$

The solution of equation (30) is taken as:

$$\Phi^*(y) = A_1\cosh(k_1 y) + A_2\cosh(k_2 y) + A_3\sinh(k_1 y) + A_4\sinh(k_2 y) \tag{33}$$

where A_1,A_2,A_3 and A_4 are some parameters depending on a and ω.

Substituting equation (33) into the equation (23), we obtain:

$$\begin{aligned}\theta^*(y) = {} & [\frac{k_1^2 - a^2 - \alpha\,\omega^2}{\alpha\,\omega^2}][A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] \\ & +[\frac{k_2^2 - a^2 - \alpha\,\omega^2}{\alpha\,\omega^2}][A_2\cosh(k_2 y) + A_4\sinh(k_2 y)]\end{aligned} \tag{34}$$

Substituting from equation (33) and equation (34) into equation (24) one obtain

$$
\begin{aligned}
e^*(y) &= (\frac{k_1^2 - a^2}{\omega^2})[A_1 \cosh(k_1 y) + A_3 \sinh(k_1 y)] \\
&\quad + (\frac{k_2^2 - a^2}{\omega^2})[A_2 \cosh(k_2 y) + A_4 \sinh(k_2 y)]
\end{aligned} \tag{35}
$$

Introducing the function

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{36}$$

we obtain from equations (9) and (10) after some manipulations:

$$(D^2 - a^2 - \alpha_0 \, \omega^2)\Omega^* = 0 \tag{37}$$

then,

$$\Omega^*(y) = B_1 \sinh(my) + B_2 \cosh(my) \tag{38}$$

where,

$$m^2 = a^2 + \alpha_0 \, \omega^2, \qquad \alpha_0 = \frac{4\,\omega}{3\,\bar{R}} \tag{39}$$

Since, equations (18) and (36), in the normal mode form is

$$\Omega^* = i\,a\,v^* - D\,u^*, \qquad e^* = i\,a u^* + D\,v^* \tag{40}$$

From equation (35), (38) and (40), we obtain:

$$
\begin{aligned}
u^*(y) &= \frac{ia}{\omega^2}\,[A_1 \cosh(k_1 y) + A_3 \sinh(k_1 y) + A_2 \cosh(k_2 y) + A_4 \sinh(k_2 y)] \\
&\quad - \frac{m}{\alpha_0 \omega^2}\,[B_1 \cosh(my) + B_2 \sinh(my)]
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
v^*(y) &= \frac{k_1}{\omega^2}\,[A_1 \sinh(k_1 y) + A_3 \cosh(k_1 y)] + \frac{k_2}{\omega^2}\,[A_2 \sinh(k_2 y) + A_4 \cosh(k_2 y)] \\
&\quad + \frac{ia}{\alpha_0\,\omega^2}\,[B_1 \sinh(my) + B_2 \cosh(my)]
\end{aligned} \tag{42}
$$

where, B_1 and B_2 are some parameters depending on a and ω.

Equation (12)–(15), in the normal mode form is

$$\sigma^*_{xx} = \omega \bar{R}(iau^* - \frac{1}{2}Dv^*) + e^* - \theta^* \tag{43}$$

$$\sigma^*_{yy} = \omega \bar{R}(Dv^* - \frac{1}{2}iau^*) + e^* - \theta^* \tag{44}$$

$$\sigma^*_{zz} = (1 - \frac{1}{2}\omega \bar{R})e^* - \theta^* \tag{45}$$

$$\sigma^*_{xy} = \frac{3}{4}\omega \bar{R}(Du^* + iav^*) \tag{46}$$

Substituting from equation (34), (35), (41) and (42) into equation (43)–(46) we get

$$\begin{aligned}\sigma^*_{xx}(y) &= \frac{\beta_1}{\alpha\,\omega^2}[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] + \frac{\beta_2}{\alpha\,\omega^2}[A_2\cosh(k_2 y) \\ &\quad + A_4\sinh(k_2 y)] - \frac{2iam}{\alpha_0^2\omega^2}[B_1\cosh(my) + B_2\sinh(my)]\end{aligned} \tag{47}$$

$$\begin{aligned}\sigma^*_{yy}(y) &= -\frac{3a^2(\alpha-1)}{2\alpha\,\omega^2}[A_1\cosh(k_1 y) + A_3\sinh(k_1 y) + \ A_2\cosh(k_2 y) \\ &\quad + A_4\sinh(k_2 y)] + \frac{2iam}{\alpha_0^2\omega^2}[B_1\cosh(my) + B_2\sinh(my)]\end{aligned} \tag{48}$$

$$\begin{aligned}\sigma^*_{zz}(y) &= \frac{b_1}{\alpha\,\omega^2}[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] \\ &\quad + \frac{b_2}{\alpha\,\omega^2}[A_2\cosh(k_2 y) + A_4 sinh(k_2 y)]\end{aligned} \tag{49}$$

$$\begin{aligned}\sigma^*_{xy}(y) &= \frac{2ia}{\alpha_0}\,\{k_1[A_1\sinh(k_1 y) + A_3\cosh(k_1 y)] \\ &\quad + k_2[A_2\sinh(k_2 y) + A_4\cosh(k_2 y)]\} \\ &\quad - (m^2 + a^2)[B_1\sinh(my) + B_2\cosh(my)]\end{aligned} \tag{50}$$

where,

$$\beta_1 = (\alpha-1)(a^2 + \frac{3}{2}k_1^2) + \alpha\,\omega^2 \tag{51}$$

$$\beta_2 = (\alpha-1)(a^2 + \frac{3}{2}k_2^2) + \alpha\,\omega^2 \tag{52}$$

$$b_1 = \frac{3}{2}(\alpha-1)(k_1^2 - a^2) + \alpha\,\omega^2 \tag{53}$$

$$b_2 = \frac{3}{2}(\alpha-1)(k_2^2 - a^2) + \alpha\,\omega^2 \tag{54}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. Assuming that all the relations (temperature, etc.) are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Applications:

Problem I: A plate subjected to time–dependent heat sources on both sides [22].

We shall consider a homogeneous isotropic thermo–viscoelastic infinite thick flat plate of a finite thickness L occupying the region G given by

$$G = \left\{(x,\,y,\,z) \qquad x,\,y,\,z \in R, \qquad -\frac{L}{2} \le y \le \frac{L}{2}\right\}$$

with the middle surface of the plate coinciding with the plane $y = 0$.

The boundary conditions of the problem are taken as:

- The normal and tangential stress components are zero on both surfaces of the plate; thus,

$$\sigma_{xy} = 0 \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{55}$$

$$\sigma_{yy} = 0 \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{56}$$

- The thermal boundary condition

$$q_n + h_0\theta = r(x,\, y,\, t) \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{57}$$

  where q_n denotes the normal component of the heat flux vector, h_o is Biot's number and $r(x,\, y,\, t\,)$ represents the intensity of the applied heat sources.

Due to symmetry with respect to y–axis we can put $A_3 = A_4 = 0$ and $B_2 = 0$ in equation (33)–(50).

Equations (50), (48) together with equation (55), (56) give:

$$\frac{2ia}{\alpha_0}[A_1 k_1 \sinh(\frac{k_1 L}{2}) + A_2 k_2 \sinh(\frac{k_2 L}{2})] - (m^2 + a^2)B_1 \sinh(\frac{mL}{2}) = 0 \tag{58}$$

$$3a\alpha_0^2(\alpha - 1)[A_1 \cosh(\frac{k_1 L}{2}) + A_2 \cosh(\frac{k_2 L}{2})] - 4m\alpha B_1 \cosh(\frac{mL}{2}) = 0 \tag{59}$$

We now make use of the generalized Fourier's law of heat conduction in the non–dimensional form, (see e.g. Lord and Shulman [15]) namely,

$$q_n + \tau_0 \frac{\partial q_n}{\partial t} = -\frac{\partial \theta}{\partial n} \tag{60}$$

by using the normal mode we get

$$q_n^* = -\frac{1}{1 + \tau_0 \omega}\frac{\partial \theta^*}{\partial n} \tag{61}$$

Using equation (57) and (61) we arrive at

$$\omega_1 r^* = \omega_1 h_0 \theta^*(y) - \omega D\theta^*(y) \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{62}$$

Using equation (34) and (62) one obtains

$$\begin{aligned} &A_1\alpha_1 \left[\omega_1 h_0 \cosh(\tfrac{k_1 L}{2}) - \omega k_1 \sinh(\tfrac{k_1 L}{2})\right] \\ &+ A_2\alpha_2 \left[\omega_1 h_0 \cosh(\tfrac{k_2 L}{2}) - \omega k_2 \sinh(\tfrac{k_2 L}{2})\right] \quad = \quad \alpha\,\omega^2 \omega_1 r^* \end{aligned} \tag{63}$$

where,

$$\alpha_1 = k_1^2 - a^2 - \alpha\,\omega^2, \qquad \alpha_2 = k_2^2 - a^2 - \alpha\,\omega^2 \tag{64}$$

Equation (58), (59) and (63) can be solved for the four unknowns A_1, A_2 and B_1

$$A_1 = \frac{\alpha\,\omega_1 a_{11} r^*}{\Delta} \tag{65}$$

$$A_2 = -\frac{\alpha\,\omega_1 a_{12} r^*}{\Delta} \tag{66}$$

$$B_1 = \frac{6ia\alpha\,\alpha_0^2(\alpha-1)r^*}{\Delta}\,[a_{21}b_{11} + a_{22}b_{12}] \tag{67}$$

where,

$$a_{11} = b_{11}\cosh(\frac{k_2L}{2}) + k_2 b_{12}\sinh(\frac{k_2L}{2}) \tag{68}$$

$$a_{12} = b_{11}\cosh(\frac{k_1L}{2}) + k_1 b_{12}\sinh(\frac{k_1L}{2}) \tag{69}$$

$$a_{21} = \alpha_1\left[\omega_1 h_0\cosh(\frac{k_1L}{2}) - \omega\,k_1\sinh(\frac{k_1L}{2})\right] \tag{70}$$

$$a_{22} = \alpha_2\left[\omega_1 h_0\cosh(\frac{k_2L}{2}) - \omega\,k_2\sinh(\frac{k_2L}{2})\right] \tag{71}$$

$$b_{11} = 3\,\alpha_0^*(\,\alpha-1\,)(\,m^2+a^2)\sinh(\frac{mL}{2}) \tag{72}$$

$$b_{12} = 8\,\alpha\,m\cosh\,(\frac{mL}{2}) \tag{73}$$

$$b_{21} = a_{21}\cosh(\frac{k_2L}{2}) - a_{22}\cosh(\frac{k_1L}{2}) \tag{74}$$

$$b_{22} = a_{21}k_2\sinh(\frac{k_2L}{2}) - a_{22}\,k_1\sinh(\frac{k_1L}{2}) \tag{75}$$

$$\Delta = b_{11}\,b_{21} + b_{12}\,b_{22} \qquad \Delta \neq 0 \tag{76}$$

Problem II: A time–dependent heat punch across the surface of semi–infinite thermo–viscoelastic half–space [23].

We will consider a homogeneous isotropic thermo–viscoelastic solid occupying the region $G = \{(x,\,y,\,z) \quad x,\,y,\,z \in R, \quad y \leq 0\}$. In the physical problem, we shall suppress the positive exponential, which are unbounded at infinitely. Thus we should replace each sinh(ky) by $[\frac{1}{2}\,\exp(ky)]$ and each $\cosh(ky)$ by $[\frac{1}{2}\exp(ky)]$.

Then, equation (33), (34), (41), (42) and (47)-(50) can be written as:

$$\Phi^*(y) = A_1^*\exp(k_1 y) + A_2^*\exp(k_2 y) \tag{77}$$

$$\theta^*(y) = \frac{1}{\alpha\,\omega^2}[\alpha_1 A_1^*\exp(k_1 y) + \alpha_2 A_2^*\exp(k_2 y)] \tag{78}$$

where, α_1 *and* α_2 are given by equation (64)

$$u^*(y) = \frac{ia}{\alpha\,\omega^2}[A_1^*\exp(k_1 y) + A_2^*\exp(k_2 y)] - \frac{m}{\alpha_0\omega^2}B_1^*\exp(my) \tag{79}$$

$$v^*(y) = \frac{1}{\omega^2}[k_1 A_1^*\exp(k_1 y) + k_2 A_2^*\exp(k_2 y)] + \frac{i\,a}{\alpha_0\,\omega^2}B_1^*\exp(my) \tag{80}$$

$$\sigma_{xx}^*(y) = \frac{1}{\alpha\,\omega^2}[\beta_1 A_1^*\exp(k_1 y) + \beta_2 A_2^*\exp(k_2 y)] - \frac{2iam}{\alpha_0^2\omega^2}B_1^*\exp(my) \tag{81}$$

where, β_1 and β_2 are given by equations (51)and (52)

$$\begin{aligned}\sigma^*_{yy}(y) &= -\frac{3a^2(\alpha-1)}{2\alpha\,\omega^2}[A^*_1\exp(k_1y)+A^*_2\exp(k_2y)] \\ &\quad +\frac{2iam}{\alpha_0^2\,\omega^2}B^*_1\exp(my)\end{aligned} \tag{82}$$

$$\begin{aligned}\sigma^*_{xy}(y) &= \frac{2\,i\,a}{\alpha_0}[k_1A^*_1\exp(k_1y)+k_2A^*_2\exp(k_2y)] \\ &\quad -(m^2+a^2)B^*_1\exp(my)\end{aligned} \tag{83}$$

$$\sigma^*_{zz}(y) = \frac{1}{\alpha\,\omega^2}[b_1A^*_1\exp(k_1y)+b_2A^*_2\exp(k_2y)] \tag{84}$$

where, b_1 and b_2 are given by equations (53) and (54)

The boundary conditions on the surface $y=0$ are taken to be:

$$\theta(x,0,\,t) = n(x,0,\,t) \tag{85}$$

$$\sigma_{xy}(x,0,\,t) = 0 \tag{86}$$

$$\sigma_{yy}(x,\,0,\,t) = p(x,\,0,\,t) \tag{87}$$

where, n and p are given function of x and t.

Equation (78), (83) and (82) together with equations (85), (86) and (87) in the normal mode form give:

$$\alpha_1A^*_1+\alpha_2A^*_2=\alpha\,\omega^2n^* \tag{88}$$

$$2\,i\,a\,[\,k_1A^*_1+k_2A^*_2\,]-\alpha_0\omega^2\,(\,m^2+a^2\,)B^*_1=0 \tag{89}$$

$$3\,a\,\alpha_0^2\,(\alpha\;-1)\,[\,A^*_1+A^*_2\,]-4\,i\;m\;\alpha\;B^*_1=-2\,\alpha\,\alpha_0^2\,\omega^2p^* \tag{90}$$

$$A^*_1 = -\frac{\alpha\,\omega^2(\,\gamma_1\alpha_2p^*+\gamma_2n^*\,)}{a\,\Delta^*} \tag{91}$$

$$A^*_2 = \frac{\alpha\,\omega^2(\,\gamma_1\alpha_1p^*+\gamma_3n^*\,)}{a\,\Delta^*} \tag{92}$$

$$B^*_1 = \frac{2\,i\,\alpha\,\alpha_0}{\Delta^*}\;[\,\gamma_4p^*-\gamma_5n^*\,] \tag{93}$$

where,

$$\gamma_1 = 2\;\alpha_0^3\,\omega^2(m^2+a^2) \tag{94}$$

$$\gamma_2 = a\,[\,8\,m\,\alpha\,k_2+3\alpha_0^3\omega^2(\alpha-1)(m^2+a^2)] \tag{95}$$

$$\gamma_3 = a\,[\,8\,m\,\alpha\,k_1+3\alpha_0^3\omega^2(\alpha-1)(m^2+a^2)] \tag{96}$$

$$\gamma_4 = 2\;\alpha_0\;\omega^2(\alpha_1k_2-\alpha_2k_1) \tag{97}$$

$$\gamma_5 = 3\,a\,\alpha_0\;\omega^2(\,\alpha-1\,)\,(k_1-k_2) \tag{98}$$

$$\begin{aligned}\Delta^* &= 8m\alpha(\alpha_1k_2-\alpha_2k_1)+3\alpha_0^3\omega^2(\alpha-1)(m^2+a^2)(\alpha_1+\alpha_2)] \\ &\quad \Delta^*\neq 0\end{aligned} \tag{99}$$

Problem III: A plate with thermo–isolated surfaces $y = \pm\frac{L}{2}$, subjected to time dependent compression [17].

We shall consider the plate in problem I but with the boundary conditions:

$$\frac{\partial\,\theta}{\partial\,y} = 0, \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{100}$$

$$\sigma_{xy} = 0, \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{101}$$

$$\sigma_{yy} = -P_0(x,\, y,\, t), \qquad \text{on} \qquad y = \pm\frac{L}{2} \tag{102}$$

where $P_0(x, y, t)$,is a given function.

Equation (34) together with equation (100) gives:

$$\sum_{j=1}^{2} \alpha_j M_j = 0 \tag{103}$$

where, α_j, $j = 1, 2$ are given by equations (64).

$$M_j = \bar{A}_j\, k_j \sinh(\frac{k_j L}{2}), \qquad j = 1, 2 \tag{104}$$

where, $\bar{A}_j$ are parameters depending on a and ω.

Equation (50) and (48) together with equation (101), (102) gives:

$$2\,i\,a\alpha_0(\,M_1 + M_2) - (\,m^2 + a^2\,)\,\bar{B}_1 \sinh(\frac{mL}{2}) = 0 \tag{105}$$

$$3a^2\alpha_0^2(\alpha - 1)(\,L_1 M_1 + L_2 M_2\,) - 4\,i\,a\,\alpha\,m\,\bar{B}_1 cosh(\frac{mL}{2}) \;\; = \;\; 2\,\alpha\,\alpha_0^2\omega^2\,P_0^* \tag{106}$$

where, $\bar{A}_j$, $\quad j = 1, 2$ are parameters depending on a and ω and

$$L_j = \frac{1}{k_j\, tanh(\frac{k_j L}{2})} \;\; , \quad j = 1, 2, \tag{107}$$

Equation (103), (105) and (106) can be solved for the three unknowns M_j and $\bar{B}_1$

$$M_1 = \frac{-2\alpha\,\alpha_0^3\,\alpha_2\,\omega^4(m^2 + a^2)\,P_0^*\sinh(\frac{m\,L}{2})}{[\,N_1\,(\,\alpha_1 L_2 - \;\alpha_2 L_1) + \;N_2(\,\alpha_1 - \alpha_2)\,]} \tag{108}$$

$$M_2 = \frac{2\alpha\,\alpha_0^3\,\alpha_1\,\omega^4(m^2 + a^2)\,P_0^*\sinh(\frac{m\,L}{2})}{[\,N_1\,(\,\alpha_1 L_2 - \;\alpha_2 L_1) + \;N_2(\,\alpha_1 - \alpha_2)\,]} \tag{109}$$

$$\bar{B}_1 = \frac{4\,i\,a\,\alpha\,\alpha_0^2\,\omega^2(\alpha_1 - \alpha_2)\,P_0^*}{[\,N_1\,(\,\alpha_1 L_2 - \;\alpha_2 L_1) + \;N_2(\,\alpha_1 - \alpha_2)\,]} \tag{110}$$

Using equation (104) one obtains

$$\bar{A}_1 = \;\frac{-2\alpha\,\alpha_0^3\,\alpha_2\,\omega^4(m^2 + a^2)\,P_0^*\sinh(\frac{m\,L}{2})}{k_1\sinh(\frac{k_1 L}{2})[\,N_1\,(\,\alpha_1 L_2 - \;\alpha_2 L_1) + \;N_2(\,\alpha_1 - \alpha_2)\,]} \tag{111}$$

$$\bar{A}_2 = \frac{2\alpha\,\alpha_0^3\alpha_1\,\omega^4(m^2+a^2)\,P_0^*\sinh(\frac{m\,L}{2})}{k_2\sinh(\frac{k_2\,L}{2})[\,N_1\,(\,\alpha_1 L_2 - \,\alpha_2 L_1) + \,N_2(\,\alpha_1 - \alpha_2)\,]} \tag{112}$$

$$N_1 = 3a^2\alpha_0^3\,\omega^2(\alpha-1)\,(m^2+a^2)\sinh(\frac{m\,L}{2}) \tag{113}$$

$$N_2 = 8a^2\alpha\,m\cosh(\frac{m\,L}{2}) \tag{114}$$

5. Numerical Results

As a numerical example we have considered polymethyl methacrylate that has a wide applications in industry and medicine. Since we have $\omega = \omega_0 + i\,\zeta$, where i is imaginary unit, $e^{\omega\,t} = e^{\omega_0 t}(\cos\zeta\,t + i\sin\zeta\,t)$ and for small values of time, we can take $\omega = \omega_0$ (real). Taking $\alpha^* = 0.5$ in equation (4) and using equation (22) we get:

$$\bar{R}(\omega) = \frac{4\,\mu}{3\,K}\left[\frac{1}{\omega_0} - \frac{A\sqrt{\pi}}{\omega_0\sqrt{\omega_0+\beta}}\right] \tag{115}$$

The numerical constants are taken as:

$$\frac{4\,\mu}{3\,K} = 0.8 \quad A = 0.106, \quad \varepsilon_1 = 0.045, \quad \beta = 0.005, \quad T_0 = 773K, \quad \tau_0 = 0.02,$$

$$L = 10, \quad \omega_0 = 2, \quad h_0 = 0.5, \quad r^* = 1, \quad n^* = 5, \quad P^* = 50, \quad P_0^* = 100.$$

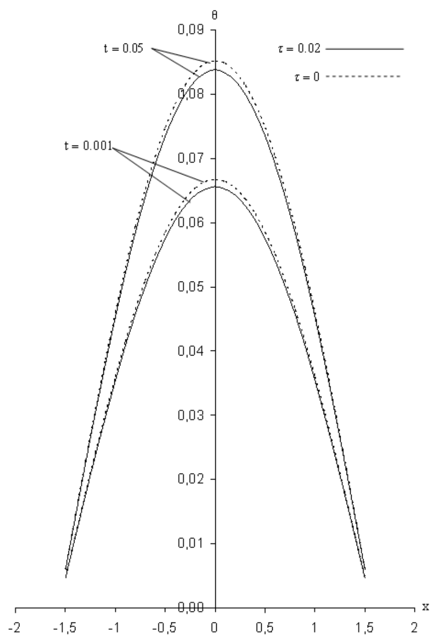

Figure 1 Temperature distribution θ for the problem I

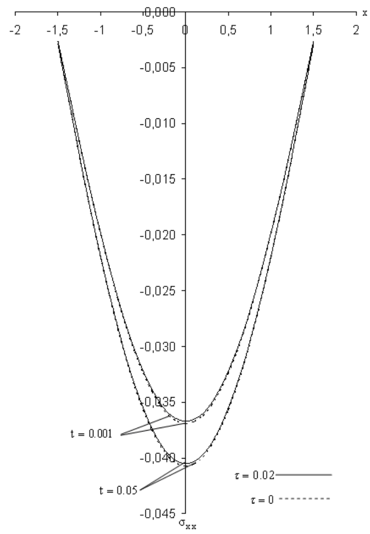

Figure 2 Horizontal displacement distribution u for the problem I

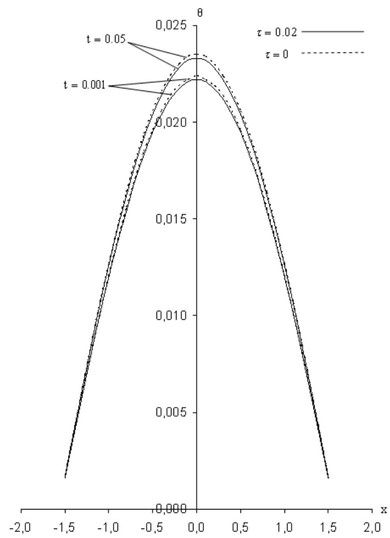

Figure 3 Temperature distribution θ for the problem I

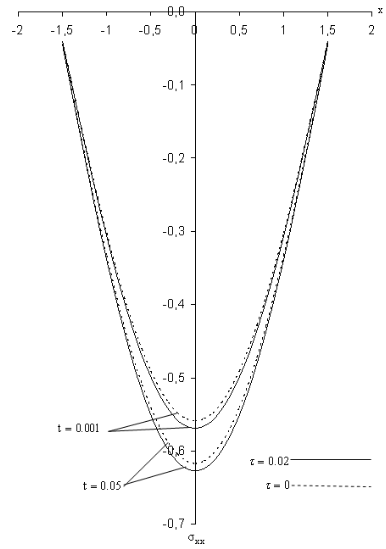

Figure 4 Horizontal displacement distribution u for the problem I

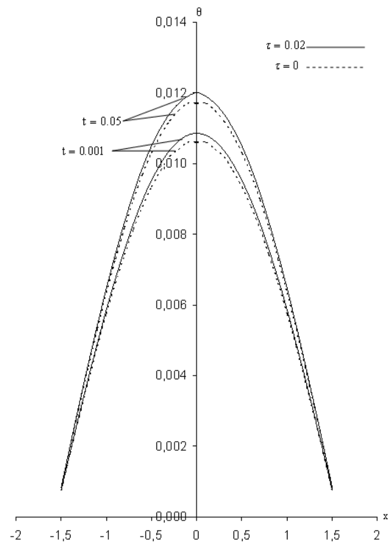

Figure 5 Temperature distribution θ for the problem I

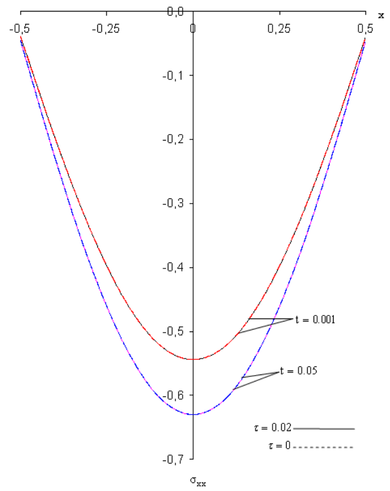

Figure 6 Horizontal displacement distribution u for the problem I

The real part of the function $\theta\ (x,\ y,\ t)$ and stress component $\sigma_{xx}(x,\ y,\ t)$, on the plane $(y = 5)$ for problems I and III while for problem II on $(y = -3)$, are evaluated for the two different values of time namely $t = 0.05$ and $t = 0.001$.

These results are shown in Figs 1–6. The graph shows the four curves predicted by the different theories of thermoelasticity. In these figures the solid lines represent the solution for Lord–Shulman theory and the dotted lines represent the solution corresponding to using the coupled equation of heat conduction $(\tau_0 = 0)$.

It was found that near the surface of the solid where the boundary conditions dominate the coupled and the generalized theories give very close results. We notice also, that results for the temperature and stress distributions when the relaxation time is appeared in the heat equation are distinctly different from those when the relaxation time is not mentioned in the heat equation. This is due to the fact that thermal wave in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non–Fourier case. It is clear that for small values of time the solution is localized in a finite region. This region grows with increasing time and its edge is the location of the wave front. This region is determined only by the values of time t and the relaxation time τ_0.

6. Concluding Remarks

Owing to the complicated nature of the governing equations for the generalized thermo–viscoelasticity, few attempts have been made to solve problems in this field, theses attempts utilize approximate methods valid for only a specific range of some parameters.

In this work, the method of normal mode analysis is introduced in the field of thermo–viscoelasticity and applied to three specific problems in which the displacement, temperature and stress are coupled. This method gives exact expressions without any assumed restrictions on either the temperature or displacement.

The normal mode analysis is applied to a wide range of problems in different branches [17, 19, 20, 24]. It can be applied to boundary layer problems, which are described by the linearized Navier-Stokes equations in hydrodynamics [25–27].

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Nomenclature

λ, μ	Lamé's constants
$K = \lambda + \frac{2}{3}\mu$	
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_o	reference temperature chosen so that $\left\lvert \frac{T-T_o}{T_o} \right\rvert << 1$
u_i	components of displacement vector
ε_{ij}	components of strain tensor
$e,\ \varepsilon_{kk}$	the dilatation
σ_{ij}	components of stress deviator
e_{ij}	components of strain deviator
k	thermal conductivity
τ_o	one relaxation time
α_t	coefficient of linear thermal expansion
$\gamma = 3\,K\ \alpha_t$	
$\varepsilon = \frac{\gamma}{\rho\, C_E}$	
$\eta_o = \frac{\rho\, C_E}{k}$	
$c_o^2 = \frac{K}{\rho}$	
$\varepsilon_1 = \delta_o \varepsilon$	
$T_o = \frac{\delta_o\, \rho\, c_o^2}{\gamma} = \frac{\delta_o}{3\alpha_T}$	
δ_o	non–dimensional number
α^*, β, A	are empirical constants