Optimum Impulsive Hohmann Coplanar Elliptic Transfer (Aggregation of New Useful Relationships) Part I

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We present an elementary approach for the optimization problem relevant to the elliptic coplanar Hohmann type transfer arising from first principles. We assign the minimized increments sum of velocities at peri–apse and apo–apse by the application of the ordinary calculus optimum condition then resolving a simple second degree algebraic equation in the variable x which is the ratio of the velocities after and before the initial impulse. It is demonstrated that the classical elliptic Hohmann type transfer is the most economic one by this elementary representation. Moreover it is a generalized of the classical Hohmann type circular case transfer.

Keywords: Rocket dynamics, elliptic Hohmann transfer, optimization

1. Background

Orbit transfer is a major subject with regard to placing a spacecraft in an orbit around the Earth. The velocity increments are directly proportional to motor system thrusts of the rocket space vehicle. Consequently it is proportional to propellant fuel consumption. It is most convenient to regard the transfer problem as a problem of change of energy [1]. We utilize well known geometric properties of conic sections and ordinary calculus. The main two types of orbit transfer are the Hohmann and the Bi–Elliptic. For each type we confront the coplanar and the non coplanar cases. The criterion for optimality is the minimization of the characteristic velocity for the maneuver [2], [3]. The literature dealing with the optimal transfer is and so on extensive, we may recall the works by Prussing [3], Palmore [4], Edelbaum [5], Barrar [6], Marec [7], Lawden [8], Hiller [9]. It is established that the minimum total velocity increment solutions for specified trajectory end point conditions, are attainable directly by methods of the differential calculus.
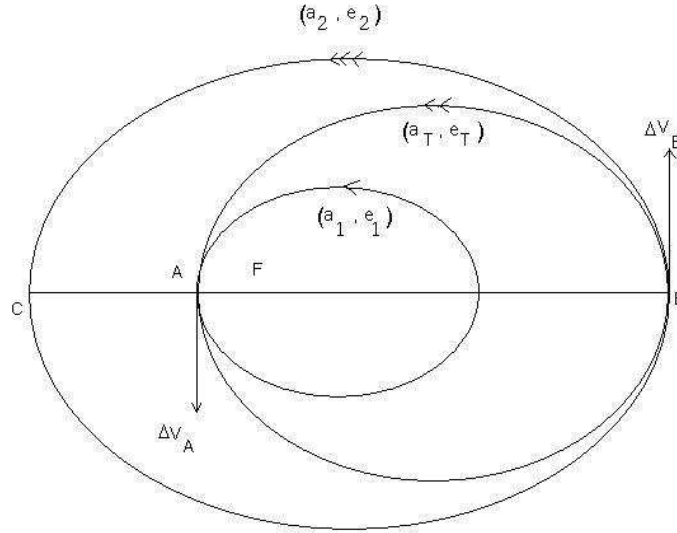
Figure 1

2.   Method and Results

2.1.   *Consideration of one single parameter* x

Herein we adopt only the first configuration for a two impulse Hohmann elliptic transfer of a space vehicle (Fig. 1). We consider the following relationships:

$$I_1 = \Delta v_1 = v_{A2} - v_{A1} = x v_{A1} - v_{A1} = (x - 1) v_{A1} \tag{1}$$

$$I_2 = \Delta v_2 = v_{B2} - v_{B1} \tag{2}$$

$$
\begin{aligned}
v_{A2} &= \sqrt{\frac{\mu(1+e_T)}{a_T(1-e_T)}} \\[2mm]
v_{A1} &= \sqrt{\frac{\mu(1+e_1)}{a_1(1-e_1)}} \\[2mm]
v_{B2} &= \sqrt{\frac{\mu(1-e_2)}{a_2(1+e_2)}} \\[2mm]
v_{B1} &= \sqrt{\frac{\mu(1-e_T)}{a_T(1+e_T)}}
\end{aligned}
\tag{3}
$$

where

$$x = \frac{v_{A2}}{v_{A1}} = \frac{velocity\ after\ peri-apse\ initial\ impulse}{velocity\ before\ peri-apse\ initial\ impulse}; \qquad x > 1$$

$$x = \sqrt{\frac{\frac{\mu(1+e_T)}{a_T(1-e_T)}}{\frac{\mu(1+e_1)}{a_1(1-e_1)}}} \tag{4}$$

From the geometry of Fig. 1, we have

$$a_T(1+e_T) = a_2(1+e_2) \tag{5}$$
$$a_T(1-e_T) = a_1(1-e_1) \tag{6}$$

From Eqs (5), (6), we get

$$\frac{1-e_T}{1+e_T} = \frac{a_1(1-e_1)}{a_2(1+e_2)} \tag{7}$$

whence, from Eqs (2), (7)

$$\Delta v_2 = \sqrt{\frac{\mu(1-e_2)}{a_2(1+e_2)}} - \sqrt{\frac{\mu a_1(1-e_1)}{a_T a_2(1+e_2)}} \tag{8}$$

From Eqs (4), (7), we acquire

$$x = \sqrt{\frac{1+e_T}{1+e_1}} > 1 \tag{9}$$
$$e_T = x^2(1+e_1) - 1 \tag{10}$$

From Eqs (5), (6), we find

$$a_T = \frac{a_1(1-e_1)}{1-e_T} = \frac{a_2(1+e_2)}{1+e_T}$$

Whence

$$a_T = \frac{a_1(1-e_1)}{2-x^2(1+e_1)} = \frac{a_2(1+e_2)}{x^2(1+e_1)} \tag{11}$$

We can easily derive

$$v_{B1} = \sqrt{\frac{\mu\{2-x^2(1+e_1)\}}{a_2(1+e_2)}}$$

Therefore

$$\Delta v_1 = \sqrt{\frac{\mu(1+e_1)}{a_1(1-e_1)}}(x-1)$$

$$\tag{12}$$

$$\Delta v_2 = \sqrt{\frac{\mu(1-e_2)}{a_2(1+e_2)}} - \sqrt{\frac{\mu\{2-x^2(1+e_1)\}}{a_2(1+e_2)}}$$

For the optimum condition:

$$\frac{d}{dx}\left(\Delta v_T\right) = \frac{d}{dx}\left(\Delta v_1\right) + \frac{d}{dx}\left(\Delta v_2\right) = 0 \tag{13}$$

Let

$$
\begin{aligned}
b_1 &= a_1\left(1 - e_1\right) \\[2mm]
b_2 &= a_1\left(1 + e_1\right) \\[2mm]
b_3 &= a_2\left(1 - e_2\right) \\[2mm]
b_4 &= a_2\left(1 + e_2\right)
\end{aligned}
\tag{14}
$$

Whence by differentiation w.r.t. the variable x

$$\frac{d}{dx}\left(\Delta v_1\right) = \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} = v_{A1} \tag{15}$$

$$\frac{d}{dx}\left(\Delta v_2\right) = \sqrt{\frac{\mu}{b_4}}\,\frac{x\left(1 + e_1\right)}{\sqrt{2 - x^2\left(1 + e_1\right)}} \tag{16}$$

$$\sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} + \sqrt{\frac{\mu}{b_4}}\,\frac{x\left(1 + e_1\right)}{\sqrt{2 - x^2\left(1 + e_1\right)}} = 0 \tag{17}$$

After some reductions and rearrangements, we get

$$\left(x\right)_{Min} = \pm\sqrt{\frac{2b_4}{\left(b_1 + b_4\right)\left(1 + e_1\right)}} \tag{18}$$

Or in explicit form

$$\left(x\right)_{Min} = \sqrt{\frac{2a_2\left(1 + e_2\right)}{\left(1 + e_1\right)\left\{a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)\right\}}} = constant \tag{19}$$

By substitution in Eqs (10), (11) for the value of $(x)_{Min.}$ Eq. (19), we get the unique values for $(a_T,\ e_T)_{Min.}$, namely

$$\left(a_T\right)_{Min} = \frac{1}{2}\left[a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)\right] \tag{20}$$

$$\left(e_T\right)_{Min} = \frac{-a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)}{a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)} \tag{21}$$

Which shows that the generalized Hohmann transfer is itself a minimum transfer system.

Now we evaluate the minimum characteristic velocity $(\Delta v_T = \Delta v_1 + \Delta v_2)_{Min}$, we have

$$(\Delta v_T)_{Min} = \sqrt{\frac{\mu(1+e_1)}{b_1}}((x)_{Min}-1) + \sqrt{\frac{\mu(1-e_2)}{b_4}}$$
$$-\sqrt{\frac{\mu}{b_4}\left\{2-(x^2)_{Min}(1+e_1)\right\}} \tag{22}$$

By substitution for $x = (x)_{Min}$, we find that

$$(\Delta v_T)_{Min} = \sqrt{\frac{2\mu b_4}{b_1(b_1+b_4)}} - \sqrt{\frac{2\mu b_1}{b_4(b_1+b_4)}} + \sqrt{\frac{\mu(1-e_2)}{b_4}} - \sqrt{\frac{\mu(1+e_1)}{b_1}}$$

in terms of the b's, or explicitly in terms of the elements a, e

$$(\Delta v_T)_{Min} = \sqrt{\frac{2\mu a_2(1+e_2)}{a_1(1-e_1)\left\{a_1(1-e_1)+a_2(1+e_2)\right\}}}$$
$$-\sqrt{\frac{\mu(1+e_1)}{a_1(1-e_1)}} + \sqrt{\frac{\mu(1-e_2)}{a_2(1+e_2)}} \tag{23}$$
$$-\sqrt{\frac{2\mu a_1(1-e_1)}{a_2(1+e_2)\left\{a_1(1-e_1)+a_2(1+e_2)\right\}}}$$

For the classical circular Hohmann transfer $e_1 = 0$ and $e_2 = 0$, whence we acquire the quite symmetric formula

$$(\Delta v_T)_{Min} = \sqrt{\frac{\mu}{a_1}}\left\{\sqrt{\frac{2a_2}{a_1+a_2}}-1\right\} + \sqrt{\frac{\mu}{a_2}}\left\{1-\sqrt{\frac{2a_1}{a_1+a_2}}\right\} \tag{24}$$

2.2. Consideration of two parameters x,y

Let

$$y = \frac{v_{B2}}{v_{B1}} = \frac{velocity\ after\ impulse\ at\ point\ B}{velocity\ before\ impulse\ at\ point\ B}$$

whence

$$y = \sqrt{\frac{1-e_2}{1-e_T}}$$

$$\Delta v_T = \Delta v_1 + \Delta v_2 = \left\{\sqrt{\frac{\mu(1+e_T)}{a_T(1-e_T)}} - \sqrt{\frac{\mu(1+e_1)}{a_1(1-e_1)}}\right\}$$
$$+ \left\{\sqrt{\frac{\mu(1-e_2)}{a_2(1+e_2)}} - \sqrt{\frac{\mu(1-e_T)}{a_T(1+e_T)}}\right\} \tag{25}$$

i.e.

$$\Delta v_T = \sqrt{\mu} \left[\frac{\sqrt{1+e_T} - \sqrt{1+e_1}}{\sqrt{a_1\,(1-e_1)}} + \frac{\sqrt{1-e_2} - \sqrt{1-e_T}}{\sqrt{a_2\,(1+e_2)}} \right]$$

We may write

$$\Delta v_T = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}}\,(x-1) + \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}}\left(1 - \frac{1}{y}\right) = f\,(x,\,y) \qquad (26)$$

From rules of partial differentiation of two variables

$$\frac{\partial}{\partial x} f\,(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} \qquad (27)$$

For optimum condition

$$\frac{\partial \Delta v_T}{\partial x} = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}} + \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}}\frac{1}{y^2}\frac{\partial y}{\partial x} = 0 \qquad (28)$$

where

$$y^2 = \frac{1-e_2}{2 - x^2\,(1+e_1)}$$

But

$$\frac{1}{y^2}\frac{\partial y}{\partial x} = \frac{x\,(1+e_1)}{\sqrt{(1-e_2)\,\{2 - x^2\,(1+e_1)\}}}$$

After some rearrangements and reductions, we get

$$\left(x^2\right)_{Min} = \frac{2a_2\,(1+e_2)}{(1+e_1)\,\{a_1\,(1-e_1) + a_2\,(1+e_2)\}} = constant \qquad (29)$$

Which is the same as Eq. (19).

Also, we may write

$$\frac{\partial}{\partial y} f\,(x,y) = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} \qquad (30)$$

By analogy we find

$$\left(y^2\right)_{Min} = \frac{(1-e_2)}{2} + \frac{a_2\,(1-e_2^2)}{2a_1\,(1-e_1)} = \frac{(1-e_2)}{2}\left\{1 + \frac{a_1\,(1+e_2)}{a_1\,(1-e_1)}\right\} \qquad (31)$$

Moreover, from definitions of x, y and from geometric properties of (Fig. 1), we may write

$$x^2 = \frac{1+e_T}{1+e_1}$$

$$y^2 = \frac{1-e_2}{1-e_T}$$

$$a_1\,(1-e_1) = a_T\,(1-e_T)$$

$$a_T\,(1+e_T) = a_2\,(1+e_2)$$

$$1+e_T = \frac{a_2\,(1+e_2)}{a_T}, 1-e_T = \frac{a_1\,(1-e_1)}{a_T}$$

$$a_T = \frac{a_1\,(1-e_1) + a_2\,(1+e_2)}{2}$$

whence

$$x^2 = \frac{2a_2\left(1+e_2\right)}{\left(1+e_1\right)\left\{a_1\left(1-e_1\right)+a_2\left(1+e_2\right)\right\}}$$

$$y^2 = \frac{1-e_2}{2} + \frac{a_2\left(1-e_2^2\right)}{2a_1\left(1-e_1\right)}$$

2.3. *Numerical check*

Now we consider the Earth – Mars Hohmann elliptic transfer to perform a check for the validity of the above calculations.

We evaluate

$$a_2 = \frac{a_T\left(1+e_T\right)+a_2\left(1-e_2\right)}{2}$$

$a_1 = 1 A.U.$
$e_1 = 0.0167$
$a_2 = 1.5237 A.U.$
$e_2 = 0.0934$

Where subscript 1 refers to the Earth and subscript 2 refers to planet Mars.[10]

We find that

$x = 1.1122$
$e_T = 0.2578$
$a_T = 1.3248 A.U.$
$a_2 = 1.5239 A.U.$

The above calculations could be repeated easily for the coplanar pairs Earth – Jupiter ; Earth – Saturn ; Earth – Neptune or Earth – Venus ; Earth – Mercury. We may easily draw graphs revealing the variation manner between the parameter x and the characteristic velocity (Δv_T). Also other variations may be plotted [9].

This will be executed in part II.

3. Concluding remarks

The choice of x as our variable leads to the most simple and exact formulae of the problem. After the resolution of the second degree equation in x arising from the optimum condition, we can determine the unique values $(e_T)_{Min}$, $(a_T)_{Min}$ from equations (10) and (11), knowing the given values of a_1, e_1 , a_2 , e_2 of the initial and final orbit. The minimum characteristic velocity $(\Delta v_T)_{Min}$ expressed by Eq. (23) is obviously expressed in terms of the initial and final orbital elements (the major axes and the eccentricities a_1, a_2, e_1, e_2).

The optimization procedure is based on formulas stemming from first principles considerations. It is not a special case arising from the general problem, when we assume non coplanar trajectories. We verified the correctness of the approach by the assignment of the approximative value of a_2 (the semi major axis of the final orbit), for the Earth – Mars system.

The simple numerical calculations yields a consistent real value of a_2. There are four feasible configurations for this transfer problem. The first one is represented in Art. 2, the other three will be dealt with in Part II. Two of the four configurations are relevant to the peri – apse perpendicular initial impulse, the other two

are relevant to initial perpendicular apo – apse impulse. It is obvious that this treatment shows that the elliptic Hohmann type transfer is the most economic in the expenditure of fuel, for the particular Earth – Mars transfer, because when we substitute $(x)_{Min}$ of Eq. (19), which is a constant in Eqs (10)–(11) we get the unique values of $(a_T)_{Min}$, $(e_T)_{Min}$ for this Earth – Mars elliptic Hohmann type of orbit transfer. $(\Delta v_T)_{Min}$ given by Eq. (23) is a measure of the extremum constant of the characteristic velocity. The above configuration (Fig. 1) corresponding to minimum fuel consumption $(\Delta v_1 + \Delta v_2)_{Min}$.

This approach is new, elementary, and straightforward. It avoids many complexities that appear in other works, thus it is advantageous for this particular transfer problem, and it is a proof that the generalized Hohmann transfer is itself a minimum orbit transfer system.

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Nomenclature:

x	ratio of velocities after and before initial impulse
a_1	semi–major axis of initial orbit
a_2	semi–major axis of final orbit
e_1	eccentricity of initial orbit
e_2	eccentricity of final orbit
a_T	semi–major axis of transfer orbit
e_T	eccentricity of transfer orbit
v_{A1}	peri–apse velocity in initial orbit at point A
v_{A2}	peri–apse velocity of transfer orbit at point A
v_{B1}	apo–apse velocity of transfer orbit at point B
v_{B2}	apo–apse velocity in final orbit at point B
Δv_1	increment of velocity at A
Δv_2	increment of velocity at B
Δv_T	characteristic velocity
μ	constant of gravitation