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# Optimal Generalized Coplanar Bi–Elliptic Transfer Orbit
# Part I

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We have four feasible simple Bi–elliptic configurations for the transfer problem, for a central gravitational field. We restrict our selves to the first one in this part. We apply three impulses at the points A, C, B. $x$, $z$ are our independent variables and are equal to the ratio between values of the velocities after and before the application of the impulses at points A, B respectively. Similarly $y$ is defined as the corresponding parameter for the point C. We utilize the optimum condition of ordinary calculus for algebraic functions, to evaluate minimum values of *x, z, y.* We adopt the Earth – Mars bi–elliptic coplanar transfer system as an example, for the first configuration, to evaluate the numerical minimum values of *x, z, y.* In part II, we shall consider the other three configurations and expand to domain of application of the numerical results.

*Keywords*: Rocket dynamics, orbital mechanics, bi–elliptic transfer, optimization

## 1. Introduction

Most generally, the change of kinematic conditions represented by $t_i, r_i, v_i \rightarrow t_f, r_f, v_f$ is the definition of a "transfer", where $t$ the physical time $r$ the radius, and $v$ the velocity. Deterministic aspects of optimization of rendezvous orbital transfer is an essential application. Among all three impulse transfers, applying the gradient method, the simple bi–elliptic transfer is the most economic or equivalently the optimal transfer.

If $11.94 < R < 15.58$ and midcourse impulse location $r_i$ ( $r_i > r_2$ ) is sufficiently large, then the bi–elliptic transfer is more economic than the Hohmann transfer[(1)]. L. Ting demonstrated that for optimality the terminal and transfer trajectories should be coplanar[(2)]. Billik and Roth discussed, in quite a general manner,

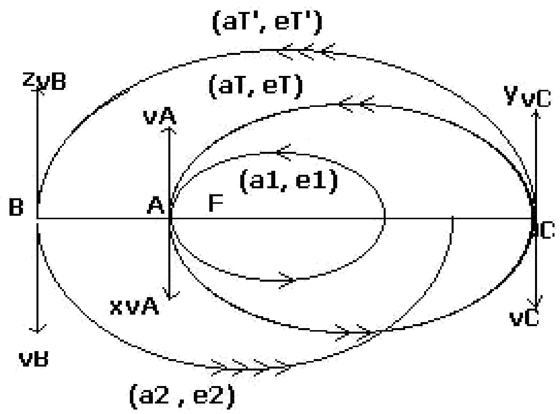


**Figure 1**

the two dimensional simple bi–elliptic transfer, with or without parking in one of the two transfer ellipses. They concluded that the bi–elliptic transfer is and alternative of the Hohmann transfer where $r_f/r_i \approx 1^{(3)}$.

## 2. Methods and Results

From Fig. 1, we have

$$
\begin{aligned}
&a_{T'}\left(1-e_{T'}\right)=a_2\left(1-e_2\right) \\
&v_A=\sqrt{\frac{\mu\left(1+e_1\right)}{a_1\left(1-e_1\right)}} \qquad xv_A=\sqrt{\frac{\mu\left(1+e_T\right)}{a_T\left(1-e_T\right)}} \\
&v_C=\sqrt{\frac{\mu\left(1-e_T\right)}{a_T\left(1+e_T\right)}} \qquad yv_C=\sqrt{\frac{\mu\left(1-e_{T'}\right)}{a_{T'}\left(1+e_{T'}\right)}} \\
&v_B=\sqrt{\frac{\mu\left(1+e_{T'}\right)}{a_{T'}\left(1-e_{T'}\right)}} \qquad zv_B=\sqrt{\frac{\mu\left(1+e_2\right)}{a_2\left(1-e_2\right)}}
\end{aligned}
$$

$$
\begin{aligned}
x&=\frac{xv_A}{v_A}=\sqrt{\frac{1+e_T}{1+e_1}} \quad \text{where} \quad a_T\left(1-e_T\right)=a_1\left(1-e_1\right) \\
y&=\frac{yv_C}{v_C}=\sqrt{\frac{1-e_{T'}}{1-e_T}} \quad \text{where} \quad a_{T'}\left(1+e_{T'}\right)=a_T\left(1+e_T\right) \\
z&=\frac{zv_B}{v_B}=\sqrt{\frac{1+e_2}{1+e_{T'}}} \quad \text{where} \quad a_{T'}\left(1-e_{T'}\right)=a_2\left(1-e_2\right)
\end{aligned}
$$

Then

$$\begin{aligned} 1+e_T &= x^2\left(1+e_1\right) \\ 1-e_T &= 2-x^2\left(1+e_1\right) \\ 1+e_{T'} &= \frac{1+e_2}{z^2} \\ 1-e_{T'} &= 2-\frac{1+e_2}{z^2} \end{aligned}$$

Therefore

$$y^2=\frac{2-\frac{1+e_2}{z^2}}{2-x^2\left(1+e_1\right)}=\frac{2z^2-\left(1+e_2\right)}{z^2\left\{2-x^2\left(1+e_1\right)\right\}} \tag{1}$$

Differentiating Eq. (1) partially w.r.t. $x$

$$\frac{\partial y}{\partial x}=\frac{x\left(1+e_1\right)\sqrt{2z^2-\left(1+e_2\right)}}{z\left\{2-x^2\left(1+e_1\right)\right\}^{3/2}} \tag{2}$$

And differentiating Eq. (1) partially w.r.t. $z$

$$\frac{\partial y}{\partial z}=\frac{1+e_2}{z^2\sqrt{\left\{2-x^2\left(1+e_1\right)\right\}\left\{2z^2-\left(1+e_2\right)\right\}}} \tag{3}$$

But we have

$$\begin{aligned} &\Delta v_T=\Delta v_A+\Delta v_C+\Delta v_B \\ &\Delta v_T=v_A\left(x-1\right)+v_C\left(y-1\right)+v_B\left(z-1\right) \end{aligned}$$

Whence, by optimum condition

$$\frac{\partial \Delta v_T}{\partial x}=\frac{\partial \Delta v_T}{\partial y}=\frac{\partial \Delta v_T}{\partial z}=0 \tag{4}$$

Namely

$$\frac{\partial \Delta v_T}{\partial x}=v_A+\frac{\partial v_C}{\partial x}\left(y-1\right)+v_C\frac{\partial y}{\partial x}=0 \tag{5}$$

But

$$v_C=\frac{2-x^2\left(1+e_1\right)}{x}\sqrt{\frac{\mu}{b_1\left(1+e_1\right)}}; with b_1=a_1\left(1-e_1\right)$$

and

$$\frac{\partial v_C}{\partial x}=-\frac{2+x^2\left(1+e_1\right)}{x^2}\sqrt{\frac{\mu}{b_1\left(1+e_1\right)}} \tag{6}$$

whence Eq. (5) can be written as :

$$\sqrt{1+e_1}-\frac{1}{\sqrt{1+e_1}}\left[\frac{\left\{2+x^2\left(1+e_1\right)\right\}\left\{\sqrt{2z^2-\left(1+e_2\right)}-z\sqrt{2-x^2\left(1+e_1\right)}\right\}}{zx^2\sqrt{2-x^2\left(1+e_1\right)}}\right]$$
$$+\frac{1}{\sqrt{1+e_1}}\left[\frac{\left(1+e_1\right)\sqrt{2z^2-\left(1+e_2\right)}}{z\sqrt{2-x^2\left(1+e_1\right)}}\right]=0$$

i.e.

$$zx^2\left(1+e_1\right)\sqrt{2-x^2\left(1+e_1\right)}-\sqrt{2z^2-\left(1+e_2\right)}+z\sqrt{2-x^2\left(1+e_1\right)}=0$$

or

$$z^2\left\{2-x^2\left(1+e_1\right)\right\}\left\{x^4\left(1+e_1\right)^2+2x^2\left(1+e_1\right)+1\right\}=2z^2-\left(1+e_2\right)$$

Let

$$c=1+e_1 \qquad ; c_1=1+e_2$$

Whence

$$z^2\left\{3x^2c-x^6c^3\right\}+c_1=0 \tag{7}$$

We may write

$$\frac{\partial \Delta v_T}{\partial z}=v_C\frac{\partial y}{\partial z}+(z-1)\frac{\partial v_B}{\partial z}+v_B=0 \tag{8}$$

where

$$v_B=\sqrt{\frac{\mu\frac{1+e_2}{z^2}}{a_2\left(1-e_2\right)}}=\sqrt{\frac{\mu\left(1+e_2\right)}{b_3z^2}} \quad \text{with} \quad b_3=a_2\left(1-e_2\right)$$

$$\frac{\partial v_B}{\partial z}=-\frac{1}{z^2}\sqrt{\frac{\mu\left(1+e_2\right)}{b_3}} \tag{9}$$

From Eqs. (3), (8), we may write

$$\frac{1}{\sqrt{b_1\left(1+e_1\right)}}\left\{\frac{\left(1+e_2\right)\sqrt{2-x^2\left(1+e_1\right)}}{xz\sqrt{2z^2-\left(1+e_2\right)}}\right\}+\frac{(1-z)}{z}\sqrt{\frac{1+e_2}{b_3}}+\sqrt{\frac{1+e_2}{b_3}}=0$$

After little reductions, we have

$$\frac{c_1\left(2-x^2c\right)}{x^2b_1c\left(2z^2-c_1\right)}=\frac{1}{b_3}$$

i.e.

$$z^2=\frac{2c_1b_3+x^2\left(b_1cc_1-b_3cc_1\right)}{2x^2b_1c}$$

Let

$$2c_1b_3=c_2 \qquad cc_1\left(b_1-b_3\right)=c_3 \qquad 2b_1c=c_4$$

whence

$$z^2=\frac{c_2+x^2c_3}{x^2c_4} \tag{10}$$

From Eqs. (7), (10), we get

$$\left(c_2+x^2c_3\right)\left(3x^2c-x^6c^3\right)+x^2c_1c_4=0$$

and after some rearrangements

$$x^8+\frac{c_2}{c_3}x^6+\frac{3}{-c^2}x^4+\frac{c_1c_4+3cc_2}{-c_3c^3}x^2=0$$

Let

$$\frac{c_2}{c_3} = c_5 \qquad \frac{3}{-c^2} = c_6 \qquad \frac{c_1 c_4 + 3cc_2}{-c_3 c^3} = c_7$$

Then, we have an equation of degree six in x on the form

$$x^6 + c_5 x^4 + c_6 x^2 + c_7 = 0 \tag{11}$$

For Earth – Mars generalized bi–elliptic system, we have[5]

$$\begin{aligned} a_1 &= a_E = 1.00000011 \\ a_2 &= a_M = 1.52366231 \\ e_1 &= e_E = 0.01671022 \\ e_2 &= e_M = 0.09341233 \end{aligned}$$

By solving this last equation numerically, we get

$$\begin{aligned} c_5 &= -6.8261 \\ c_6 &= -2.9023 \\ c_7 &= 24.5119 \\ x_{1,2} &= \pm 1.40255 \\ x_{3,4} &= \pm 2.59127 \\ x_{5,6} &= -3.00698 \times 10^{-17} \quad \pm 1.36225 I \end{aligned}$$

i.e. the consistent values are

$$\begin{aligned} x &= 1.4026 \\ z &= \frac{1}{x}\sqrt{\frac{c_2 + x^2 c_3}{\frac{c}{4}}} = 0.7394 \\ y &= \frac{1}{z}\sqrt{\frac{2z^2 - c_1}{2 - x^2 c}} = 1.1853 \end{aligned}$$

## 3. Concluding Remarks:

It is possible to reduce the impulsive optimal transfer problem to a parametric optimization one with constraints. A numerical solution, or even analytical one in some simple cases could be acquired. In addition we may have the semi–analytical resolution as shown in this article [4]. For the first configuration we assigned the values of $(x,\, z,\, y)_{Min}$ for the generalized Earth – Mars bi–elliptic transfer. Our procedure is elementary and straightforward, using only the properties of the elliptic conic section, and the minimum – partial ordinary calculus – conditions. Our choice of the independent parameters $x, z$ proved simplicity and efficiency of this analysis when compared with other approaches. The parameter $x$ is determined from a numerical solution of a sixth degree polynomial. The numerical results may be repeatedly acquired for subsystems with exterior member as one of the outer planets Jupiter, Saturn, Uranus and Neptune, or even more the inner planets Venus and Mercury.

## References

[1] **Prussing, J. and Conway, B.**: Orbital Mechanics, *Oxford University Press*, **(1993)**.

[2] **Ting, L.**: *ARS J.*, PP. 1013–1018, **(1960)**.

[3] **Billik, B.H. and Roth, H.L.**: *Astronautica Acta*, Vol. 13, PP. 23–36, **(1966)**.

[4] **Marec, J.P.**: Space–Vehicle Trajectories : Optimization, Reprinted from Systems & Control Encyclopedia, Theory, Technology, Applications, *Pergamon Press*, **(1988)**.

[5] **Murray, C.D. and Dermott, S.F.**: Solar System Dynamics, *Cambridge University Press*, **(1999)**.