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Optimum Impulsive Hohmann Coplanar Elliptic Transfer Using Energy Change Concept Part II

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We present an elementary approach for the optimization of the elliptic coplanar coaxial Hohmann type transfer arising from the first principles. We assign the minimized increments of velocities at peri–apse and apo–apse by equating to zero the gradient of $\Delta_{v1} + \Delta_{v2}$, then resolving a second degree algebraic equation in the variable x (the ratio of the velocities before and after the initial impulse). We consider the four feasible configurations, and we assign the most economic one. By setting $e_1 = 0$, $e_2 = 0$ for the terminal orbits, we confront the original circular Hohmann transfer case promptly.

Keywords: Rocket dynamics, ellitpic Hohmann transfer, optimization

1. Background

Orbit transfer is a major subject with regard to placing a spacecraft in an orbit around the Earth. The velocity increments are directly proportional to motor system thrusts of the rocket space vehicle. Consequently it is proportional to propellant fuel consumption. It is most convenient to regard the transfer problem as a problem of change of energy [1]. We utilize well known geometric properties of conic sections and ordinary calculus. The main two types of orbit transfer are the Hohmann and the bi–elliptic. For each type we face the coplanar and the non coplanar cases. The criterion for optimality is the minimization of the characteristic velocity for the maneuver [2], [3]. The literature of optimal transfer is and so on extensive, we may recall the works by Prussing [3], Palmore [4], Edelbaum [5], Barrar [6], Marec [7], Lawden [8], Hiller [9] and Altman and Pistiner [10]. It is established that the minimum total velocity increment solutions for specified trajectory end point conditions, are attainable directly by methods of the differential calculus.

2. Method and Results

We begin by the first configuration for a two impulse Hohmann elliptic transfer of a space vehicle (Fig. 1). We consider the following relationships:

$$I_1 = \Delta v_1 = v_{A2} - v_{A1} = xv_{A1} - v_{A1} = (x-1)\, v_{A1} \tag{1}$$

$$I_2 = \Delta v_2 = v_{B2} - v_{B1} \tag{2}$$

$$\begin{aligned} v_{A2} &= \sqrt{\frac{\mu\,(1+e_T)}{a_T\,(1-e_T)}} \qquad v_{A1} = \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}} \\ v_{B2} &= \sqrt{\frac{\mu\,(1-e_2)}{a_2\,(1+e_2)}} \qquad v_{B1} = \sqrt{\frac{\mu\,(1-e_T)}{a_T\,(1+e_T)}} \end{aligned} \tag{3}$$

where

$$x = \frac{xv_{A1}}{v_{A1}} = \frac{velocity\, afterperi - apse\, initial\, impulse}{velocity\, beforeperi - apse\, initil\, impulse} \qquad x > 1$$

$$x = \sqrt{\frac{\frac{\mu(1+e_T)}{a_T(1-e_T)}}{\frac{\mu(1+e_1)}{a_1(1-e_1)}}} \tag{4}$$

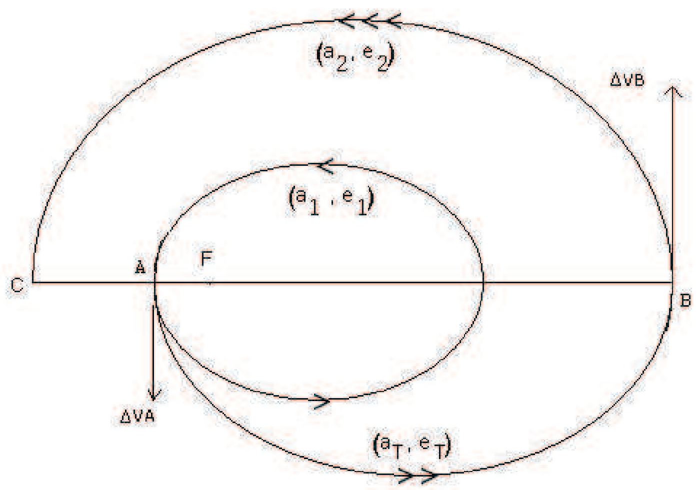

Figure 1 Apogee of transfer orbit coincides with apogee of final orbit.(Initial impulse at perigee)

From the geometry of Fig. 1, we have

$$a_T\left(1+e_T\right)=a_2\left(1+e_2\right) \tag{5}$$

$$a_T\left(1-e_T\right)=a_1\left(1-e_1\right) \tag{6}$$

From Eqs. (5), (6), we get

$$\frac{1-e_T}{1+e_T}=\frac{a_1\left(1-e_1\right)}{a_2\left(1+e_2\right)} \tag{7}$$

whence, from Eqs (2), (7)

$$\Delta v_2=\sqrt{\frac{\mu\left(1-e_2\right)}{a_2\left(1+e_2\right)}}-\sqrt{\frac{\mu a_1\left(1-e_1\right)}{a_T a_2\left(1+e_2\right)}} \tag{8}$$

From Eqs (4), (7), we acquire

$$x=\sqrt{\frac{1+e_T}{1+e_1}}>1 \tag{9}$$

i.e.

$$e_T=x^2\left(1+e_1\right)-1$$

From Eqs (5), (6), we find

$$a_T=\frac{a_1\left(1-e_1\right)}{1-e_T}=\frac{a_2\left(1+e_2\right)}{1+e_T}$$

Whence

$$a_T=\frac{a_1\left(1-e_1\right)}{2-x^2(1+e_1)}=\frac{a_2\left(1+e_2\right)}{x^2(1+e_1)} \tag{10}$$

We can easily derive

$$v_{B1}=\sqrt{\frac{\mu\left\{2-x^2\left(1+e_1\right)\right\}}{a_2\left(1+e_2\right)}}$$

Therefore

$$\begin{aligned}\Delta v_2&=\sqrt{\frac{\mu\left(1-e_2\right)}{a_2\left(1+e_2\right)}}-\sqrt{\frac{\mu\left\{2-x^2\left(1+e_1\right)\right\}}{a_2\left(1+e_2\right)}}\\ \Delta v_1&=\sqrt{\frac{\mu\left(1+e_1\right)}{a_1\left(1-e_1\right)}}\left(x-1\right)\end{aligned} \tag{11}$$

For the optimum condition:

$$\frac{d}{dx}\left(\Delta v_T\right)=\frac{d}{dx}\left(\Delta v_1\right)+\frac{d}{dx}\left(\Delta v_2\right)=0 \tag{12}$$

Let

$$b_1 = a_1\left(1 - e_1\right) \qquad b_2 = a_1\left(1 + e_1\right)$$
$$b_3 = a_2\left(1 - e_2\right) \qquad b_4 = a_2\left(1 + e_2\right) \tag{13}$$

Whence by differentiation w.r.t. the variable x

$$\frac{d}{dx}\left(\Delta v_1\right) = \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} = v_{A1} \tag{14}$$

$$\frac{d}{dx}\left(\Delta v_2\right) = \sqrt{\frac{\mu}{b_4}}\frac{x\left(1 + e_1\right)}{\sqrt{2 - x^2\left(1 + e_1\right)}} \tag{15}$$

i.e.

$$\sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} + \sqrt{\frac{\mu}{b_4}}\frac{x\left(1 + e_1\right)}{\sqrt{2 - x^2\left(1 + e_1\right)}} = 0 \tag{16}$$

After some reductions and rearrangements, we get

$$(x)_{Min} = \pm\sqrt{\frac{2b_4}{\left(b_1 + b_4\right)\left(1 + e_1\right)}} \tag{17}$$

Or in explicit form

$$(x)_{Min} = \sqrt{\frac{2a_2\left(1 + e_2\right)}{\left(1 + e_1\right)\left\{a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)\right\}}} \tag{18}$$

By substitution in Eqs (9), (10) for the value of $(x)_{Min.}$ Eq. (18), we get the unique values for $(a_T,\, e_T)$, namely

$$\left(a_T\right)_{Min} = \frac{1}{2}\left[a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)\right] \tag{19}$$

$$\left(e_T\right)_{Min} = \frac{-a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)}{a_1\left(1 - e_1\right) + a_2\left(1 + e_2\right)} \tag{20}$$

Which shows that the generalized Hohmann transfer is itself a minimum transfer system.

Now we evaluate the minimum characteristic velocity $\left(\Delta v_T = \Delta v_1 + \Delta v_2\right)_{Min}$, we have

$$\Delta v_T = \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}}\left(x - 1\right) + \sqrt{\frac{\mu\left(1 - e_2\right)}{b_4}} - \sqrt{\frac{\mu}{b_4}\left\{2 - x^2\left(1 + e_1\right)\right\}} \tag{21}$$

By substitution for $x = (x)_{Min}$, we find that

$$\left(\Delta v_T\right)_{Min} = \sqrt{\frac{2\mu b_4}{b_1\left(b_1 + b_4\right)}} - \sqrt{\frac{2\mu b_1}{b_4\left(b_1 + b_4\right)}} + \sqrt{\frac{\mu\left(1 - e_2\right)}{b_4}} - \sqrt{\frac{\mu\left(1 + e_1\right)}{b_1}} \tag{22}$$

In terms of the b's or explicitly in terms of the elements a, e

$$(\Delta v_T)_{Min} = \sqrt{\frac{2\mu a_2 (1+e_2)}{a_1 (1-e_1) \{a_1 (1-e_1) + a_2 (1+e_2)\}}} - \sqrt{\frac{\mu (1+e_1)}{a_1 (1-e_1)}}$$
$$+ \sqrt{\frac{\mu (1-e_2)}{a_2 (1+e_2)}} - \sqrt{\frac{2\mu a_1 (1-e_1)}{a_2 (1+e_2) \{a_1 (1-e_1) + a_2 (1+e_2)\}}} \tag{23}$$

For the classical circular Hohmann transfer $e_1 = 0$ and $e_2 = 0$, whence we acquire the quite symmetric formula

$$(\Delta v_T)_{Min} = \sqrt{\frac{\mu}{a_1}} \left\{ \sqrt{\frac{2a_2}{a_1 + a_2}} - 1 \right\} + \sqrt{\frac{\mu}{a_2}} \left\{ 1 - \sqrt{\frac{2a_1}{a_1 + a_2}} \right\} \tag{24}$$

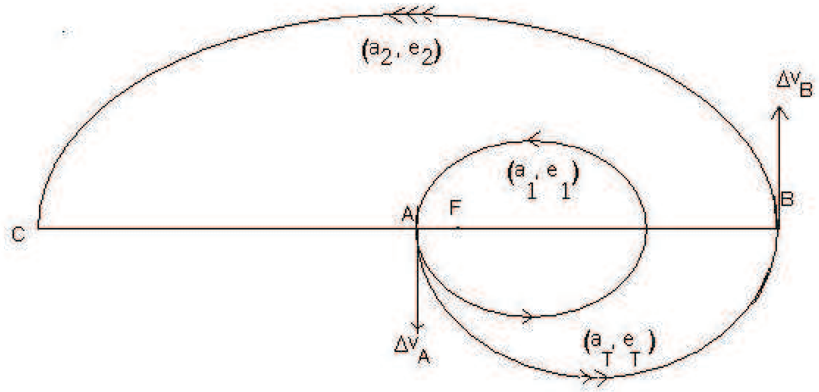

Figure 2 Apogee of transfer orbit coincides with perigee of final orbit. (Initial impulse at perigee)

For the second configuration (Fig. 2), we have the following relationships:

$$a_1 (1-e_1) = a_T (1-e_T) \qquad a_T (1+e_T) = a_2 (1-e_2)$$
$$v_{A1} = \sqrt{\frac{\mu (1+e_1)}{a_1 (1-e_1)}} \qquad v_{A2} = \sqrt{\frac{\mu (1+e_T)}{a_T (1-e_T)}}$$

Let

$$x = \frac{v_{A2}}{v_{A1}} = \sqrt{\frac{1+e_T}{1+e_1}} > 1$$

i.e.

$$e_T = x^2 (1 + e_1) - 1$$
$$a_T = \frac{a_1 (1 - e_1)}{1 - e_T} = \frac{a_2 (1 - e_2)}{1 + e_T} \tag{25}$$

From Eq. (13),

$$a_T = \frac{b_1}{1 - e_T} = \frac{b_3}{1 + e_T}$$

Or

$$a_T = \frac{b_1}{2 - x^2 (1 + e_1)} = \frac{b_3}{x^2 (1 + e_1)}$$

Now,

$$\Delta v_A = v_{A2} - v_{A1} = x v_{A1} - v_{A1} = (x - 1) v_{A1}$$
$$\Delta v_A = (x - 1) \sqrt{\frac{\mu (1 + e_1)}{b_1}} \tag{26}$$
$$v_{B1} = \sqrt{\frac{\mu (1 - e_T)}{a_T (1 + e_T)}} \qquad v_{B2} = \sqrt{\frac{\mu (1 + e_2)}{a_2 (1 - e_2)}}$$
$$\Delta v_B = v_{B2} - v_{B1} = \sqrt{\frac{\mu (1 + e_2)}{b_3}} - \sqrt{\frac{\mu \{2 - x^2 (1 + e_1)\}}{b_3}} \tag{27}$$

Optimum condition is:

$$\frac{d}{dx} (\Delta v_T) = \frac{d}{dx} (\Delta v_A) + \frac{d}{dx} (\Delta v_B) = 0$$

Then, from Eqs (26), (27) we get

$$\sqrt{\frac{\mu (1 + e_1)}{b_1}} + \sqrt{\frac{\mu}{b_3}} \frac{x (1 + e_1)}{\sqrt{2 - x^2 (1 + e_1)}} = 0$$

After some reductions, we obtain the value of $(x)_{Min.}$on the form

$$(x)_{Min} = \pm \sqrt{\frac{2 b_3}{(1 + e_1)(b_1 + b_3)}}$$

Finally, $(\Delta v_T)_{Min} = (\Delta v_A)_{Min} + (\Delta v_B)_{Min}$

$$(\Delta v_T)_{Min} = \sqrt{\frac{2 \mu b_3}{b_1 (b_1 + b_3)}} - \sqrt{\frac{\mu (1 + e_1)}{b_1}} + \sqrt{\frac{\mu (1 + e_2)}{b_3}} - \sqrt{\frac{2 \mu b_1}{b_3 (b_1 + b_3)}} \tag{28}$$

Or explicitly in terms of the elements a, e:

$$(\Delta v_T)_{Min} = \sqrt{\frac{2\mu a_2 (1-e_2)}{a_1 (1-e_1) \{a_1 (1-e_1) + a_2 (1-e_2)\}}} - \sqrt{\frac{\mu (1+e_1)}{a_1 (1-e_1)}}$$
$$+\sqrt{\frac{\mu (1+e_2)}{a_2 (1-e_2)}} - \sqrt{\frac{2\mu a_1 (1-e_1)}{a_2 (1-e_2) \{a_1 (1-e_1) + a_2 (1-e_2)\}}} \tag{29}$$

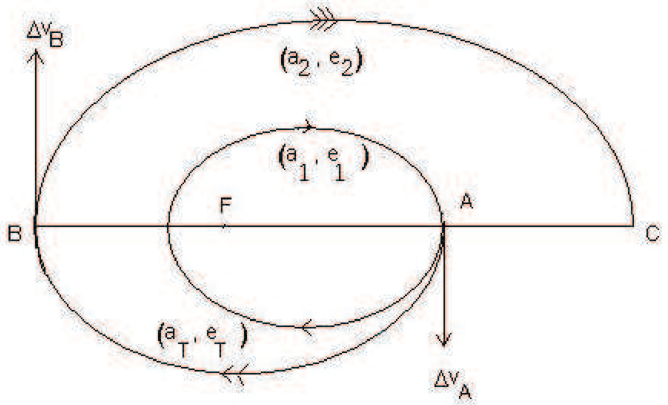

Figure 3 Perigee of transfer orbit coincides with perigee of final orbit. (Initial impulse at apogee)

For the third configuration (Fig. 3), we have

$$a_1 (1+e_1) = a_T (1+e_T) \,; a_T (1-e_T) = a_2 (1-e_2) \tag{30}$$
$$v_{A1} = \sqrt{\frac{\mu (1-e_1)}{b_2}} \qquad v_{A2} = \sqrt{\frac{\mu (1-e_T)}{b_2}}$$

Let

$$x = \frac{v_{A2}}{v_{A1}} = \sqrt{\frac{1-e_T}{1-e_1}} > 1$$
i.e.
$$e_T = 1 - x^2\left(1-e_1\right) \tag{31}$$
$$a_T = \frac{a_1\left(1+e_1\right)}{1+e_T} = \frac{a_2\left(1-e_2\right)}{1-e_T}$$
i.e.
$$a_T = \frac{b_2}{1+e_T} = \frac{b_3}{1-e_T}$$
$$a_T = \frac{b_2}{2-x^2\left(1-e_1\right)} = \frac{b_3}{x^2\left(1-e_1\right)}$$
$$\Delta v_A = v_{A2} - v_{A1} = x v_{A1} - v_{A1} = \left(x-1\right) v_{A1}$$
$$\Delta v_A = \left(x-1\right)\sqrt{\frac{\mu\left(1-e_1\right)}{b_2}} \tag{32}$$
$$v_{B1} = \sqrt{\frac{\mu\left(1+e_T\right)}{a_T\left(1-e_T\right)}} \qquad v_{B2} = \sqrt{\frac{\mu\left(1+e_2\right)}{a_2\left(1-e_2\right)}}$$
$$\Delta v_B = v_{B2} - v_{B1} = \sqrt{\frac{\mu\left(1+e_2\right)}{b_3}} - \sqrt{\frac{\mu\left\{2-x^2\left(1-e_1\right)\right\}}{b_3}} \tag{33}$$

Optimum condition is:

$$\frac{d}{dx}\left(\Delta v_T\right) = \frac{d}{dx}\left(\Delta v_A\right) + \frac{d}{dx}\left(\Delta v_B\right) = 0$$

Then from Eqs (32), (33) we get

$$\sqrt{\frac{\mu\left(1-e_1\right)}{b_2}} + \sqrt{\frac{\mu}{b_3}}\frac{x\left(1-e_1\right)}{\sqrt{2-x^2\left(1-e_1\right)}} = 0$$

After some simple algebraic reductions, we get

$$\left(x\right)_{Min} = \pm\sqrt{\frac{2b_3}{\left(1-e_1\right)\left(b_2+b_3\right)}}$$
$$\left(\Delta v_T\right)_{Min} = \sqrt{\frac{2\mu b_3}{b_2\left(b_2+b_3\right)}} - \sqrt{\frac{\mu\left(1-e_1\right)}{b_2}} + \sqrt{\frac{\mu\left(1+e_2\right)}{b_3}} - \sqrt{\frac{2\mu b_2}{b_3\left(b_2+b_3\right)}} \tag{34}$$

Or explicitly,

$$(\Delta v_T)_{Min} = \sqrt{\frac{2\mu a_2 (1-e_2)}{a_1 (1+e_1)\{a_1 (1+e_1) + a_2 (1-e_2)\}}} - \sqrt{\frac{\mu (1-e_1)}{a_1 (1+e_1)}}$$
$$+ \sqrt{\frac{\mu (1+e_2)}{a_2 (1-e_2)}} - \sqrt{\frac{2\mu a_1 (1+e_1)}{a_2 (1-e_2)\{a_1 (1+e_1) + a_2 (1-e_2)\}}} \tag{35}$$

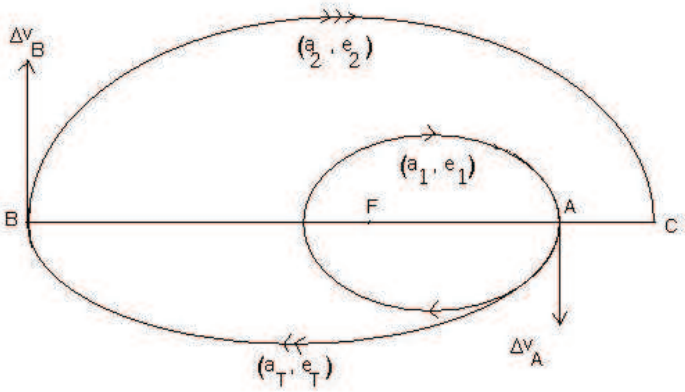

Figure 4 Apogee of transfer orbit coincides with apogee of final orbit. (Initial impulse at apogee)

For the fourth configuration (Fig. 4), we have

$$\begin{aligned}
&a_1 (1+e_1) = a_T (1-e_T) \qquad a_T (1+e_T) = a_2 (1+e_2) \\
&v_{A1} = \sqrt{\frac{\mu (1-e_1)}{b_2}} \qquad v_{A2} = \sqrt{\frac{\mu (1+e_T)}{b_2}} \\
&x = \frac{v_{A2}}{v_{A1}} = \sqrt{\frac{1+e_T}{1-e_1}} > 1
\end{aligned} \tag{36}$$

i.e.

$$\begin{aligned}
&e_T = x^2 (1 - e_1) - 1 \\
&a_T = \frac{a_1 (1 + e_1)}{1 - e_T} = \frac{a_2 (1 + e_2)}{1 + e_T}
\end{aligned} \tag{37}$$

$$a_T = \frac{b_2}{2 - x^2 (1 - e_1)} = \frac{b_4}{x^2 (1 - e_1)}$$

$$\Delta v_A = v_{A2} - v_{A1} = x v_{A1} - v_{A1} = (x - 1) v_{A1}$$

$$\Delta v_A = (x - 1) \sqrt{\frac{\mu (1 - e_1)}{b_2}} \tag{38}$$

$$v_{B1} = \sqrt{\frac{\mu (1 - e_T)}{b_4}} \qquad v_{B2} = \sqrt{\frac{\mu (1 - e_2)}{b_4}}$$

$$\Delta v_B = v_{B2} - v_{B1} = \sqrt{\frac{\mu (1 - e_2)}{b_4}} - \sqrt{\frac{\mu \{2 - x^2 (1 - e_1)\}}{b_4}} \tag{39}$$

$$\Delta v_T = \Delta v_A + \Delta v_B$$

Optimum condition is

$$\frac{d}{dx} (\Delta v_T) = \frac{d}{dx} (\Delta v_A) + \frac{d}{dx} (\Delta v_B) = 0$$

Then from Eqs (37), (38) we get

$$\sqrt{\frac{\mu (1 - e_1)}{b_2}} + \sqrt{\frac{\mu}{b_4}} \frac{x (1 - e_1)}{\sqrt{2 - x^2 (1 - e_1)}} = 0$$

$$(x)_{Min} = \pm \sqrt{\frac{2 b_4}{(1 - e_1)(b_2 + b_4)}}$$

$$(\Delta v_T)_{Min} = (\Delta v_A)_{Min} + (\Delta v_B)_{Min}$$

Finally,

$$(\Delta v_T)_{Min} = \sqrt{\frac{2 \mu b_4}{b_2 (b_2 + b_4)}} - \sqrt{\frac{\mu (1 - e_1)}{b_2}} + \sqrt{\frac{\mu (1 - e_2)}{b_4}} - \sqrt{\frac{2 \mu b_2}{b_4 (b_2 + b_4)}} \tag{40}$$

Or explicitly

$$\begin{aligned}
(\Delta v_T)_{Min} = &\sqrt{\frac{2 \mu a_2 (1 + e_2)}{a_1 (1 + e_1) \{a_1 (1 + e_1) + a_2 (1 + e_2)\}}} - \sqrt{\frac{\mu (1 - e_1)}{a_1 (1 + e_1)}} \\
&+ \sqrt{\frac{\mu (1 - e_2)}{a_2 (1 + e_2)}} - \sqrt{\frac{2 \mu a_1 (1 + e_1)}{a_2 (1 + e_2) \{a_1 (1 + e_1) + a_2 (1 + e_2)\}}}
\end{aligned} \tag{41}$$

3. Numerical calculations

We consider the Earth – Mars Hohmann elliptic transfer to perform an approximative check for the validity of the above calculations.

We have $a_1 = 1$ AU; $a_2 = 1.5237$ AU; $e_1 = 0.0167$; $e_2 = 0.0934$ where subscript 1 refers to the Earth and subscript 2 refers to the Mars.

We have the following table for the four configurations:

Fig.	$(a_T)_{Min}$	$(e_T)_{Min}$	$(x)_{Min}$	$(\Delta v_T)_{Min}$	a_2
1	1.3247	0.2577	1.1122	0.1843	1.5237
2	1.1823	0.1683	1.0720	0.1870	1.5236
3	1.1990	-0.1521	1.0824	0.1873	1.5237
4	1.3414	0.2420	1.1239	0.1850	1.5237

We note that $(e_T)_{Min}$ for Fig. 3 is negative value, according to the sketch of this figure and the Eq. (30).

We assume that a_1, a_2 the semi–major axes of the Earth and Mars are equal to the mean distances of the two planets from the primary (the Sun). Evidently $(\Delta V_T)_{Min}$ of Fig. 1 is the most economic.

4. Concluding Remarks

The choice of x as our variable leads to the most simple and exact formulate of the problem. After the resolution of the second degree equation in x arising from the optimum condition, we can determine the unique values $(e_T)_{Min}$, $(a_T)_{Min}$ from Eqs (9), (10), (25),(31),(36) knowing the given values of a_1, e_1 , a_2 , e_2 of the initial and final orbit. The minimum characteristic velocity $(\Delta v_T)_{Min}$ Eqs (22), (28), (34), (39) are obviously expressed in terms of the initial and final orbital elements (the major axes and the eccentricities a_1, a_2, e_1, e_2). The optimization procedure is based on formulas stemming from first principles considerations. It is not a special case arising from the general problem, when we assume non coplanar trajectories. We verified the correctness of the approach by the assignment of the approximative value of a_2 (the semi major axis of the final orbit),

from the formula $2a_2 = \{a_T(1 + e_T) + a_2(1 - e_2)\}$, for Fig. 1 and Fig. 4,

from the formula $2a_2 = \{a_T(1 + e_T) + a_2(1 + e_2)\}$, for Fig. 2

from the formula $2a_2 = \{a_T(1 - -e_T) + a_2(1 + e_2)\}$, for Fig. 3.

In this paper, we consider four feasible configurations for this transfer problem, two of them are relevant to the peri–apse perpendicular initial impulse (Fig. 1 and Fig. 2), the other two are relevant to initial perpendicular apo-apse impulse (Fig. 3 and Fig. 4).

This approach is new, elementary, and straightforward. It avoids many complexities that appear in other works, thus it is advantageous for this particular transfer problem, and it is a proof that the generalized Hohmann transfer is itself a minimum orbit transfer system.

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Nomenclature

x	ratio of velocities after and before initial impulse
a_1	semi–major axis of initial orbit
a_2	semi–major axis of final orbit
e_1	eccentricity of initial orbit
e_2	eccentricity of final orbit
a_T	semi–major axis of transfer orbit
e_T	eccentricity of transfer orbit
v_{A1}	peri–apse velocity in initial orbit at point A
v_{A2}	peri–apse velocity of transfer orbit at point A
v_{B1}	apo–apse velocity of transfer orbit at point B
v_{B2}	apo–apse velocity in final orbit at point B
$\Delta \mathrm{v}_1$	increment of velocity at A
$\Delta \mathrm{v}_2$	increment of velocity at B
$\Delta \mathrm{v}_T$	characteristic velocity
μ	constant of gravitation