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Stability of Piezoelectric Circular Plates

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A general formulation that can be used for the stability analysis of axisymmetric circular plates with piezoelectric (PZT) is presented in the paper. Demonstrated approach is based on the Rayleigh–Ritz method that is applied for functional of electric energy enthalpy. Numerical examples deal with compressed circular plates. The computations are conducted both for Love–Kirchhoff and first order shear deformation plate theory (SDT). Destabilization effects of electric voltages and piezoelectric widths are studied.

Keywords: Stability, piezoelectric circular plate, free vibrations, Rayleigh–Ritz method

1. Introduction

In recent years, there has been an increasing interest in the development of lightweight smart or intelligent structures for engineering applications to control e.g. deformations. Smart structures (e.g., piezoelectric coupled plates) may be used as sensors or/and actuators in various applications, including vibration control, acoustic noise suppression, active damping and so on.

Mindlin [1, 2] initiated the study based on power series expansions of the mechanical displacements and the electric potential along the thickness of the plate and formulated the variational principle. He proposed the plate theory [3, 4] which included the effects of rotatory inertia and shear deformation. The first contributions to the field concerning vibration of single–layer piezoelectric plates are summarized in the work written by Tiersten [5].

The analysis of circular plates with piezoelectric layers deals mainly with the evaluation of eigenfrequencies (free vibrations) that is almost similar to the buckling analysis. Haojianga et al. [6] developed an exact solution for axisymmetric vibration of piezoelectric circular plates under certain types of boundary conditions. To investigate the free vibration of piezoelectric laminate circular plates, Heyliger and Ramirez [7] combined approximations of one–dimensional finite elements in the thickness direction and analytic functions in the plane within the context of the Ritz method.

Using classical theory of plates and the theory of electric potential Zhanga et al. [8] investigated the transient bending characteristics of a thin, piezoelectric circular plate under axisymmetric, mechanical loading, electrically grounded over the whole surface and built–in or simply supported at the edge. The space–dependent terms of the transient solution were expressed in terms of a single, elementary Bessel function, whose analytic behaviour and numerical evaluation were readily available, and the explicit time history of the solution was obtained by precise inverse Laplace transformation.

Ebrahimi, Abbas Rastgo [9] presented analytical investigation of the free vibration behavior of thin circular functionally graded plates integrated with two uniformly distributed actuator layers made of piezoelectric material based on the classical plate theory.

One-term space mode approximate analytical solutions for geometrically nonlinear response of elastic circular thin plates have been shown in the literature to yield quite accurate results for static, transient, buckling, postbuckling and nonlinear vibration response. Tzou et al. [10] presented static and dynamic control of a circular plate with geometric nonlinearity. Tzou and Zhou [11] provided a polynomial series solution for nonlinear static and dynamic response of circular plates under thermoelectromechanical excitations. Kapuria and Dumir [12] presented an approximate analytical one–term Galerkin solution for the problem of nonlinear deflection, thermal buckling and natural frequencies of a three–layer thin circular plate made of an isotropic elastic core with piezoelectric layers bonded to its faces. The analysis was restricted to axisymmetric moderately large deflection of the plate subjected to a thermal load, radial edge load or edge displacement and actuated by applying an electric potential across a piezoelectric layer. The rotational and inplane inertia and the shear deformation were neglected in the Von Karman type classical thin plate theory used in the analysis.

Both free vibration and buckling analysis were mainly based on the use of the classical plate theory (the Love–Kirchhoff approximation). First–order shear deformation plate theory (SDT) of Mindlin [3], including the effects of shear deformation and rotary inertia, was a natural extension of the classical plate theory in order to analyze vibration behavior of moderately thick plates. Liu et al. [13] applied a first–order shear deformation plate theory, which considered transverse shear deformation and rotary inertia in some way, to investigate vibration of piezoelectric coupled moderately thick circular plates.

Based on three–dimensional elastic theory of piezoelectric materials, the axisymmetric state space for deformation of piezoelectric laminated circular plates was derived by Haojianga et al. [6]. Solutions for transversely isotropic circular laminates were obtained.

Different higher–order shear deformation plate theories (HSDT) were also proposed, including the second-order shear deformation formulation of Whitney and Sun [14] and the third–order shear deformation theory of Lo et al. [15] with 11 unknowns; Reddy [16] with five unknowns and Hanna and Leissa [17] with four unknowns. The HSDT does not need to use any shear correction coefficient since its third–order displacement field assumption satisfies the zero shear stress condition at the free surfaces [18]. Therefore, the HSDT approximating radial and circumferential displacements up to the cubic order produces better in–plane responses when compared with the SDT. In return, their governing equations are much more complicated than those of the SDT. There are few works on the free vibration of isotropic homogeneous circular or annular plates on the basis of the HSDT. Chen and Hwang [19] utilized the average stress and Galerkin methods to obtain natural frequencies of axisymmetric initially stressed circular and annular plates using the HSDT.

3–D vibration analysis of isotropic homogeneous plates having different shapes (e.g., circular and annular plates) and different boundary conditions were widely carried out via numerical approaches, including the finite element method [20] and the Ritz method.

In the present paper our attention is focused on the formulation of stability analysis (it may be treated as similar to free vibration problem) of circular axisymmetric isotropic plates with piezoelectric layers bounded symmetrically on both sides of the plates. The length of the piezoelectric layer is variable. The presented results and formulation is based on the Rayleigh–Ritz method and approximated form of deflections. The fundamental relations are introduced for the first order shear deformation plate theory and the classical plate theory.

2. The generalized energy functional

The Rayleigh–Ritz method is commonly used for the computation of approximate solutions of operator eigenvalue equations (buckling loads or natural frequencies) and partial differential equations. The method is based on a linear expansion of the solution and determines the expansion coefficients by a variational procedure, which is why the method is also known as linear variation method. Usually, the eigenvalue problem can be found by variations of the Lagrange (displacement) functional (or Hamilton for free vibration). In the case of piezoelectric materials classical energy density is extended through introduction of density of electric enthalpy (see e.g. Mindlin [21]) which takes into the consideration the electric field. In the global coordinate system x, y, z the density of electric enthalpy (3-D formulation) is defined in the following way:

$$\Pi_{int} = \frac{1}{2}\,[\sigma]\,[\varepsilon] \tag{1}$$

where $[\sigma]$ and $[\varepsilon]$ are quantities understand in the broader sense than classical (i.e. stress and strain, respectively) and are expressed in the so–called generalized form, i.e.

$$[\sigma] = \begin{bmatrix} [\bar{\sigma}] \\ [D] \end{bmatrix} = [C]\,[\varepsilon] = \begin{bmatrix} [\bar{C}] & [-e] \\ [e]^T & [\mu] \end{bmatrix} \begin{bmatrix} [\bar{\varepsilon}] \\ [E] \end{bmatrix} \tag{2}$$

Symbols with the dash have traditional mechanical interpretation, i.e. stress, stiffness matrix and strain. $[D]$ denotes the displacement vector equivalent to electrostatic induction vector generated by the electric field $[E]$ – each of these quantities have three components. $[e]$ matrix (6x3) defines the permittivity coefficients of the medium. $[\mu]$ (3x3 dimension) means dielectric conductivity. The electric field is usually defined by electric potential gradient Φ_{el} ($[E]$=-grad Φ_{el}) that satisfies Maxwell law *rot* $[E]$=0. The energy density expressed by formula (1) is supplemented by the energy of body forces and surface forces.

In view of solving variational problems it is necessary to consider independently two cases:

- problem of mechanical excitation measurement: in this case displacement field and electric field vector are independent variables – six components so–called sensor problem;
- problem of electric activation: searching for components of displacement field and electric input function (electric potential) are treated in the analogous manner as thermal input function (so–called actuator problem).

We observe that in the problem of electric activation the density of enthalpy (1) is the functional having four independent variables: three components of displacement vector and the electric potential Φ_{el}. Therefore, in the case of transformation from variational form to differential equations we obtain classical equations of equilibrium for 3-D body and the Maxwell equations in the form $div[D]$=0 and the relation $[E]$=-*grad* Φ_{el} which are supplemented by physical relation (2) and boundary conditions.

In this work we intend to express the relationship between the electric potential and the displacement vector in differential form. Next knowing distribution of potential we can insert it to the relation for electric enthalpy (1) and find value of critical load from that functional using classical Rayleigh-Ritz approach.

3. Relation between electric potential and displacement field

We assume that the potential of the electric field Φ_{el} has the following distribution in the z direction (see Wang et al. [22]):

$$\Phi_{el} = \left[1 - \left(\frac{2z - t_{PZT} - t}{t_{PZT}}\right)^2\right] \varphi\left(x, y\right) \tag{3}$$

where $\varphi(x, y)$ denotes value of potential on mid–surface of piezoelectric layer, whereas t_{pzt} means the piezoelectric layers thickness, and t is a thickness of the host plate (Fig. 1.). Using the Maxwell law, for which density of divergence of electrostatic induction is equal to zero at each point of body [23], one can find

$$\int\limits_{t/2}^{t/2+t_{PZT}} div\,[D] dz \;=\; 0 \tag{4}$$

Using the relation (2) and assuming the distribution of the electric potential (3) the equation (4) can be presented in the explicit form as follows:

$$\frac{t_{PZT}^2 \mu_{11}}{12\mu_{33}} \Delta\varphi - \varphi + \frac{e_{31}}{\mu_{33}} \int\limits_{t/2}^{t/2+t_{PZT}} dz\left(\bar{\varepsilon}_{xx} + \bar{\varepsilon}_{yy}\right) = 0 \tag{5}$$

It gives the explicit relation between potential and the mechanical strains.

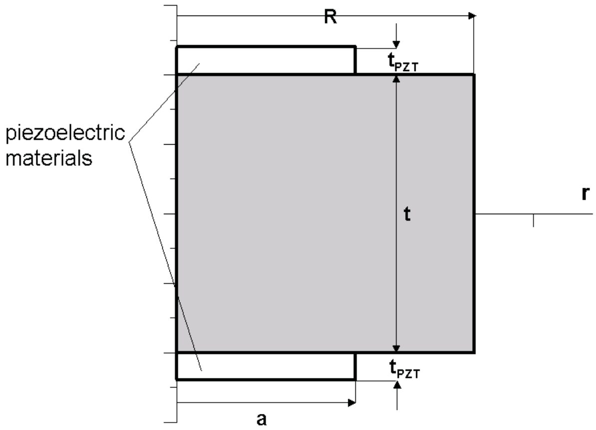

Figure 1 Geometry of piezoelectric circular plate (cross–section)

4. Stability of circular plate

Assuming the axisymmetry of the problems using the first order shear deformation theory and von Karman nonlinear geometrical theory the strains can be presented in the following way:

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + z\frac{d\psi}{dr} \qquad \varepsilon_\psi = \frac{u}{r} + \frac{z\psi}{r} \qquad \varepsilon_{rz} = \psi + \frac{dw}{dr} \tag{6}$$

where u and w mean displacement components in the radial (r) direction and the transverse (z) directions, respectively (Fig. 1.), ψ denotes angle of rotation of the normal to the plate mid–surface. For the classical Love–Kirchhoff theory the appropriate strains are determined assuming transverse shear strain ε_{rz} to be equal to zero, what allows us to find the unknown variable ψ. In the relation (6) strains are presented in the 3–D form. In the relation (6) all terms multiplied by the z coordinate describe the change of curvature, and other terms to membrane strains.

Substituting relation (6) to equation (1), using the relationship (2) and assuming electrical field as potential field for the plate in the case of axi-symmetrical strain loaded by compressive radial axial forces N a density of electrical enthalpy (1) takes the following form

$$
\begin{aligned}
&\Pi = \Pi_{plate} + \Pi_{PZT} + \Pi_{load} + \Pi_{elect} \\
&\Pi_{plate} = \frac{1}{2}\int_0^R \left\{D_{plate}\left[\left(\kappa_r + \nu\kappa_\theta\right)\kappa_r + \left(\kappa_\theta + \nu\kappa_r\right)\kappa_\theta\right]\right. \\
&\left.+k_{TS}G_{plate}t_{plate}\varepsilon_{rz}^2\right\} rdr \\
&\Pi_{PZT} = \frac{1}{2}\int_0^a \left\{D_{PZT}\left[\left(\kappa_r + \nu\kappa_\theta\right)\kappa_r + \left(\kappa_\theta + \nu\kappa_r\right)\kappa_\theta\right]\right. \\
&\left.-M_{PZT}\left(\kappa_r + \kappa_\theta\right) + k_{TS}G_{PZT}t_{PZT}\varepsilon_{rz}^2\right\} rdr \\
&M_{PZT} = e_{31}\left(t + t_{PZT}/2\right)\Phi_{el} \\
&\Pi_{load} = \frac{1}{2}\int_0^R N\left(\frac{dw}{dr}\right)^2 rdr \\
&\Pi_{elect} = -\frac{1}{2}\int_0^R dr\left[\mu_{11}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \mu_{33}\varphi^2\right]
\end{aligned}
\tag{7}
$$

In the above relations D denotes the bending rigidity, G means shear modulus. Subscripts *plate* and PZT relate to plate and piezoelectric layers, respectively. k_{TS} describes transverse shear coefficient and is equal to 5/6 for isotropic structures. Symbol κ denotes the parameter of change of curvature in the radial direction (r) and circumferential (θ) directions, respectively – they are defined by equations $(6)_{1,2}$ as the multipliers of the z variable. Piezoelectric element increases plate stiffness and produces additional moment M_{PZT} following from coupling between electrical field and mechanical field.

5. Numerical results

In order to illustrate the piezoelectric effects for plates described with the use of the Love–Kirchhoff theory a normal deflection is assumed in the following form:

$$
w\left(r\right) = \sum_{m=1}^{\infty} B_m r^{2m-2}\left(r^2 - R^2\right)^2 \tag{8}
$$

Taking into a count the first term (m=1) in Eq. (8) substituting it to Eq. (7) an approximate value of critical force can be found:

$$
N_{cr}^{L-K} = 16\frac{D_{plate} + D_{PZT}}{R^2} \tag{9}
$$

This quantity is higher about 9% than exact value. On the basis of the first order shear deformation theory the value of critical force is characterized by the following relation:

$$
N_{cr}^{SDT} = \frac{N_{cr}^{L-K}}{1 + N_{cr}^{L-K}/\left[k_{TS}^2\left(G_{plate}t + 2G_{PZT}t_{PZT}\right)\right]} \tag{10}
$$

Now, let us take into account the influence of applied voltage on the value of critical (buckling) force. In numerical computation it is assumed that material constants are defined in the following:

- plate: E=200 [GPa], ν=0.3, t = 10 [mm], R =500 [mm],
- *PZT*: E=132 [GPa], ν=0.3, t_{PZT} = 2 [mm], e_{31} = -4.1 [C/m^2], μ_{11} = 7.124 [nF/m], μ_{33} = 5.841 [nF/m].

Substituting relations (6) to Eq. (5), neglecting displacement u and assuming the following form of the potential:

$$\varphi = A_1 Y_1 (r) + A_2 Y_2 (r) \tag{11}$$

one can find the analytical form of the function $\psi(r)$. $Y(r)$ denotes Bessel function of the first kind and A are unknown constants. Unknown distribution $w(r)$ is assumed as power series which maximal index exponent is higher than indexes in series $\psi(r)$. We have assumed that $w(r)$ and $\psi(r)$ functions satisfy the following boundary conditions (clamped edges):

$$\psi(0) = 0 \qquad \psi(R) = 0 \qquad w(R) = 0 \tag{12}$$

Finally, using the relations (7), (8) (taking into a count to terms m=1 and m=2) and (11) we can express the explicit form for buckling loads as follows

$$N_{cr} = f(A_1, A_2, B_1, B_2) \tag{13}$$

Using symbolic package Mathematica 7 calculations can be easily performed. In order to verify convergence the compute from procedure is repeat increasing number of term in the expression (11).

Depending on voltage sign one can observe stabilizing or destabilizing effects. In the present work the decrease of buckling force is discussed in details. Results are presented in Fig. 2 and Fig. 3. For constant voltage equal to 100 [V] one can observe the reduction critical force with the variation of the piezoelectric width (Fig. 2).

Destabilizing effect grows with the increase of the width parameter a/R. The next example (Fig. 3) illustrates the influence of the voltage value on lost of stability. It is assumed that width parameter a/R is equal to 0.5. It is obvious that the increase of the voltage affects significantly the value of critical loads.

Both plots (Fig. 2 and Fig. 3) demonstrate the variation of dimensionless buckling loads and referred to the value given by equation (9). As it may be seen first order shear deformation theory results in the reduction of the buckling loads in the comparison with the classical theory. In general the reduction ratio may be approximated with the relation (10).

The proposed formulation and methodology of analysis can be easily extended for the stability/free vibration analysis of multilayered composite plates.

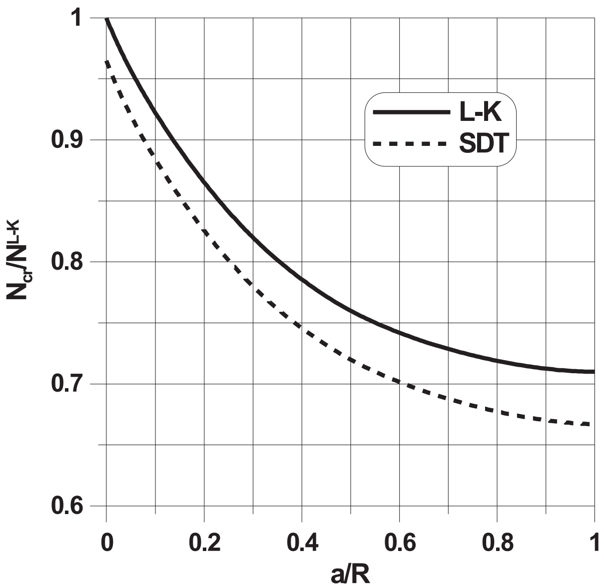

Figure 2 Influence of width of piezoelectric layer on value of critical force (L–K – Love-Kirchhoff theory, SDT – first order shear deformation theory)

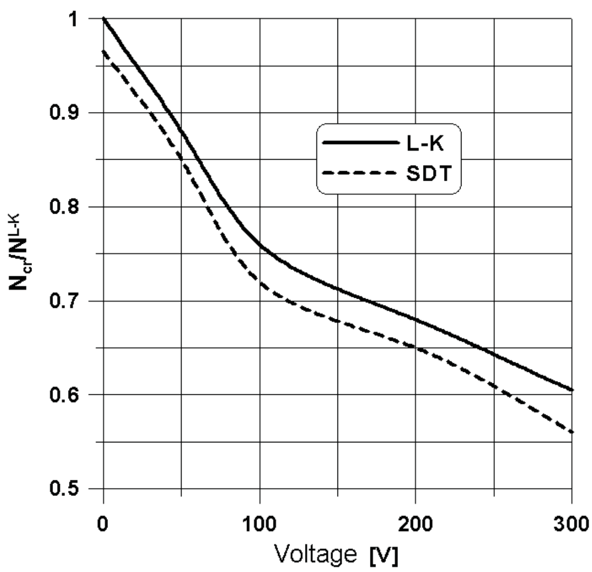

Figure 3 Influence of voltage on value of critical force (L–K – Love–Kirchhoff theory, SDT – first order shear deformation theory)

6. Conclusions

An efficient and effective approach to the analysis of stability problems of circular plates with piezoelectric layers and subjected to compressive radial loads is presented in the paper. The demonstrated method may be applicable to the investigations of buckling problems for plates with the use both the Love–Kirchhoff theory and SDT. The presented study confirms the usually accepted conclusion that the use of the SDT theory reduces buckling loads in the comparison with the classical Love–Kirchhoff plate theory. It is worth to note that the external electric field (piezoelectric actuators) may have stabilizing or destabilizing influence on compressed plate. The length of piezoelectric actuator strongly affects the values of buckling loads, one may observe that with the increase of the length of PZT patch destabilizing effects grows.

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