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Regenerative Model of Cutting Process with Nonlinear Duffing Oscillator

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The paper presents analytical and numerical study of externally forced Duffing oscillator with delayed displacement feedback. By using analytical method, stability lobes for a linear oscillator with time delay is determined and the fundamental resonance of the Duffing oscillator with time delay is calculated by means of the multiple scale method. Next, an influence of delayed displacement feedback on the classical Duffing oscillator is examined and an effect of cubic nonlinearity on the stability of the regenerative model of cutting process is analysed.

Keywords: Nonlinear vibrations, Duffing oscillator, time delay, regenerative chatter

1. Introduction

Recent years have provided numerous studies on time delay systems and their applications in various fields. The delayed state feedback is often used to active control of mechanical vibrations, specially of nonlinear systems [1]. As shown in [2] time delay effect can stabilize the unstable periodic motions in linear oscillators and also can control local bifurcations of nonlinear systems through improving the stability of periodic motion. Therefore, the time delay systems are more and more popular not only to control processes but to model various physical phenomena as well. The delayed state feedback is commonly being made use to model machining processes with, so called, regenerative effect that is very harmful because can produced self–excited chatter vibrations [3–10]. Vibrations become one of the most common limitations for productivity and quality of final surface in milling, turning and boring operations with long flexible tools or highly flexible thin-wall parts. Yao and Chen [11] consider self–excited vibrations of van der Pol–Duffing oscillator with delayed velocity and displacement feedback. In order to eliminate the limit cycle, parametric excitation is applied to the oscillator. Next, parametric excitation is put into a use for stability analysis of the cutting vibrations system [11]. Similarly, a single and two coupled Duffing–van der Pol system with delayed displacement feedback is taken into consideration in the paper [12] where stability and synchronization analysis is performed.

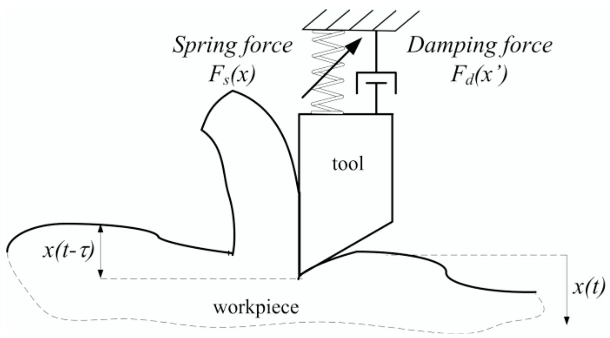

Figure 1 Model of cutting process with regenerative effect (time delay)

This paper deals with a problem of a single Duffing oscillator with time delayed displacement. A cutting process model with regenerative effect and nonlinear stiffness characteristic can be mentioned as an example of (Fig. 1). In the literature Hu et al. [13] scrutinized a harmonically forced Duffing oscillator under linear time delay control. They analysed, together, delayed displacement and velocity feedback by the multiple scale method. Also, Hu and Wang [2] used the multiple scale method to examine a Duffing oscillator but only with delayed velocity feedback. Our work focuses on the influence of Duffing type nonlinearity on behaviour of time delayed displacement system. Thus, the system under consideration, that is nonlinear, external forced Duffing oscillator with delayed displacement, is governed by a second order nonlinear differential equation with shifted argument:

$$x''(t) + \delta x' + \omega^2{}_0 x(t) + \gamma x(t)^3 = \alpha(-\mu x(t) + x(t-\tau)) + f\cos(\lambda t) \qquad (1)$$

where, δ is a viscous damping coefficient, ω_o means natural frequency of a linear system, γ – small coefficient representing nonlinear stiffness α – amplitude of delay, f – amplitude of external force, λ – frequency of external excitation, τ – time delay and μ is a switching parameter: if μ=*1* then term exits and if μ=*0* this term vanishes.

2. Linear and nonlinear model of regenerative chatter

At the beginning let us examine a linear equation with time delay which is popular to describe cutting process with regenerative effect and can be found in various variants in literature [8, 10, 14–17]. Assuming differential equation of motion takes the linear form:

$$x''(t) + \delta x' + \omega^2{}_0 x(t) + \gamma x(t)^3 = \alpha(-x(t) + x(t - \tau)) \tag{2}$$

where α in this case can be treated as cutting width. Since, equation (2) is linear therefore it can be solved analytically. Periodic solutions can be sought in the classical form:

$$\begin{aligned} x(t) &= A\cos(\omega t) \\ x(t-\tau) &= A\cos(\omega t - \omega\tau) \end{aligned} \tag{3}$$

or making use of the Laplace transform method. Then the characteristic equation is obtained as:

$$s^2 = \delta s + \omega_0^2 + \alpha(1 - e^{s\tau}) = 0 \tag{4}$$

For asymptotic stability, the equation (4) must yield roots with negative real parts. Equation (4) is a transcendental algebraic equation on s. The stability depends on δ, ω_o α and the time delay τ. An exact analytical solution of this equation cannot, in general, be obtained in terms of elementary functions. However from (4) we can determine following relationships (*j=1,2,...*) :

$$\begin{aligned} \tau &= \frac{1}{\omega}\left(\arctan\frac{-\delta\omega}{\omega_0^2 + \alpha - \omega^2} + j\pi\right) \\ \alpha &= \frac{\omega_0^4 - \delta^2\omega^2 + 2\omega^2\omega_0^2 - \omega^4}{2(\omega_0^2 - \omega^2)} \end{aligned} \tag{5}$$

On the basis of equations (5) the stability boundary can be computed numerically. For parameters: δ=0.1, ω_o =1 they are drawn as, so called, stability lobs diagram (SLD) in Fig. 2 (continuous line) where angular velocity Ω is defined as:

$$\Omega = \frac{2\pi}{\tau}, \quad j = 0,\, 1,\, 2,\, ..., n \tag{6}$$

Below the stability curve, the solutions are asymptotically stable (decrease to zero), whereas above the stability lobs the solutions tend to infinity. The system is unconditionally asymptotic stable, regardless of Ω when α is less then α_{cr} which can be calculated as a minimum of function $\alpha(\omega)$ provided that:

$$\frac{d\alpha}{d\omega} \tag{7}$$

Thus, ω_{cr} is:

$$\omega_{cr} = \sqrt{\delta\omega_0 + \omega_0^2} \tag{8}$$

and α_{cr} is expressed as:

$$\alpha_{cr} = \frac{1}{2}\delta(\delta + 2\omega_0) \tag{9}$$

For δ=0.1, ω_o =1 the demand α_{cr} =0.105.

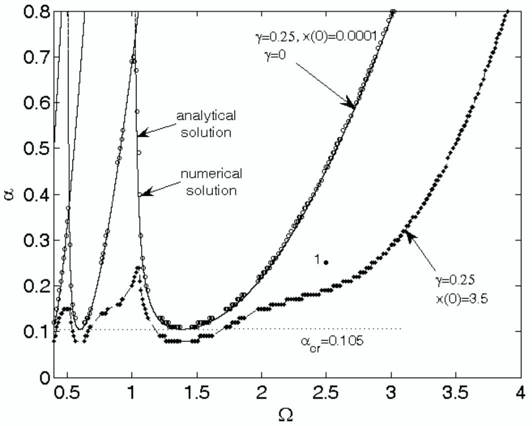

Figure 2 Stability lobs diagram for linear and nonlinear case

The points on the analytical curve (Fig. 2) are the result of numerical integration of differential equation (2). The situation changes when nonlinearity is added to the system. For instance when γ=0.25 (f is still equal to 0) the equation (1) is solved numerically in two cases, first starting form small initial condition $x(0)$=0.0001 and next from bigger one $x(0)$=3.5. The former case gives similar results to the linear model (continuous line in Fig. 2) but in the latest, the unstable region is wider (dashed line in Fig. 2). Additionally, Fig. 3 demonstrates asymptotic stable solution starting from small initial conditions (blue trajectory on the phase space) or the limit cycle for big initial conditions (black trajectory).

Vibrations amplitude, that is the level of limit cycle depends, of course, on time delay in the way presented in Fig. 4, where the displacement amplitude versus Ω is drawn for α=0.2.

The cutting width α is a second parameter which influences vibrations in regenerative cutting model. In the case of a linear system (γ=0) the increase of α outside the stability limit, determined by lobs in SLD (Fig. 2), produces amplitude escape to infinity. While for nonlinear system (γ=0.25) at α =0.110 and Ω =1.3 the supercritical Hopf bifurcation occurs during increasing α but when α decreases, the jump of amplitude is visible at α=0.175. Note, that these bifurcation diagrams shows displacement obtained for Poincaré section of x for x'=0, therefore two branches are observable in Fig. 5 and Fig. 7. The limit cycle with periodic solution is presented on Poincaré sections in Fig. 6 a. whereas, for α=1.0 more complex, quasi–periodic motion is obtained (Fig.6 b).

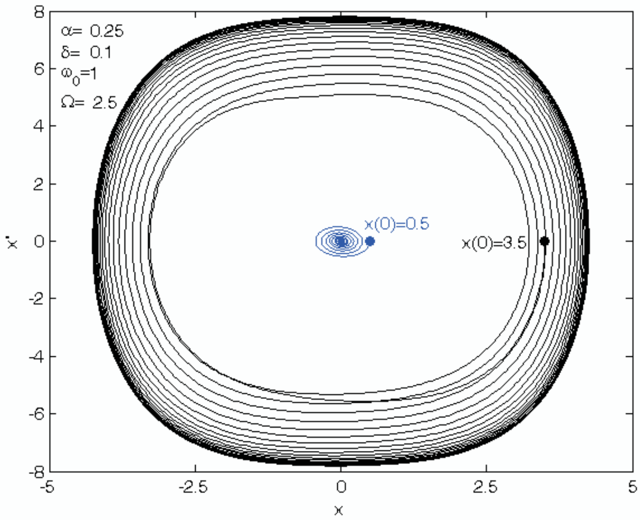

Figure 3 Phase space for nonlinear system for initial condition x(0)=00.5 (blue) and x(0)=3.5 (black)

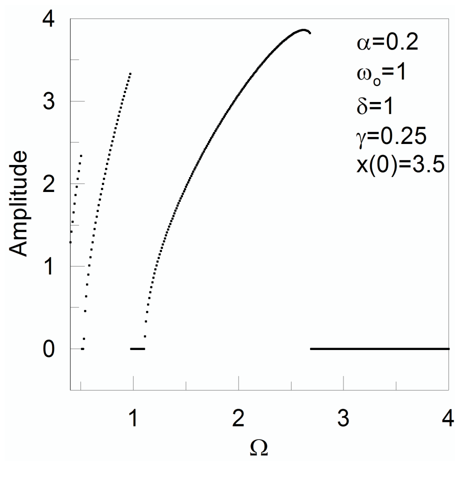

Figure 4 Dependence of displacement amplitude vs. angular velocity Ω

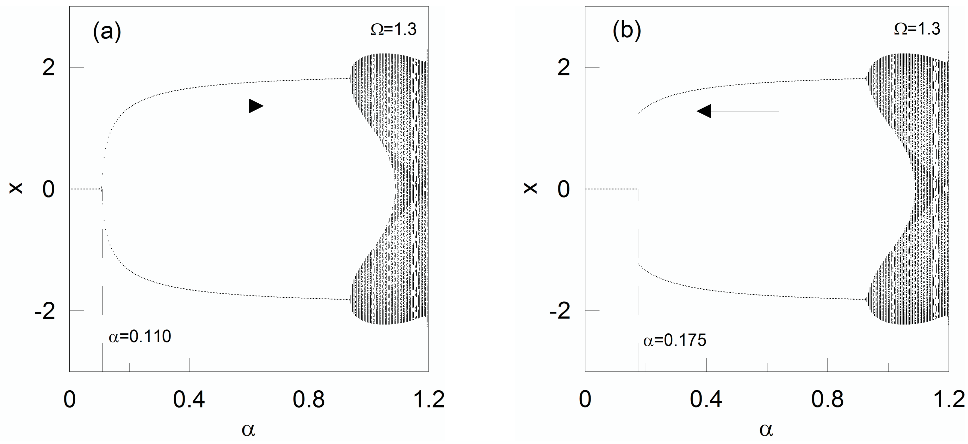

Figure 5 Bifurcation diagram x(α) for Ω=1.3; increasing α(a), decreasing α(b)

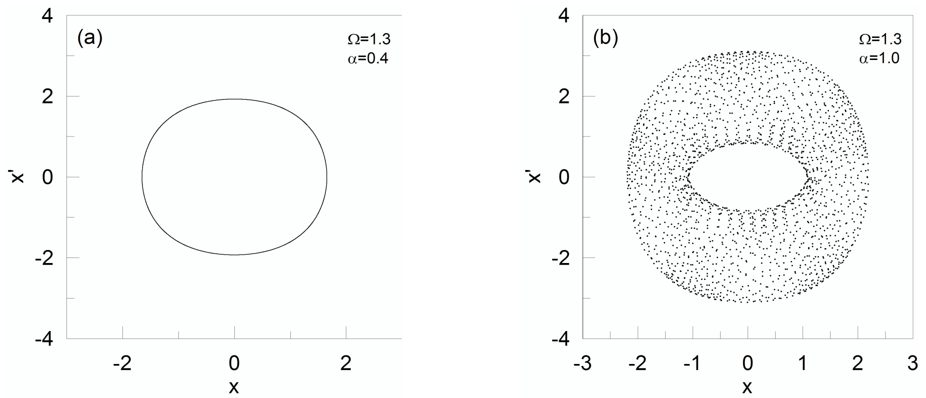

Figure 6 Poincar maps for Ω=1.3 and α=0.4 (a), α=1.0 (b)

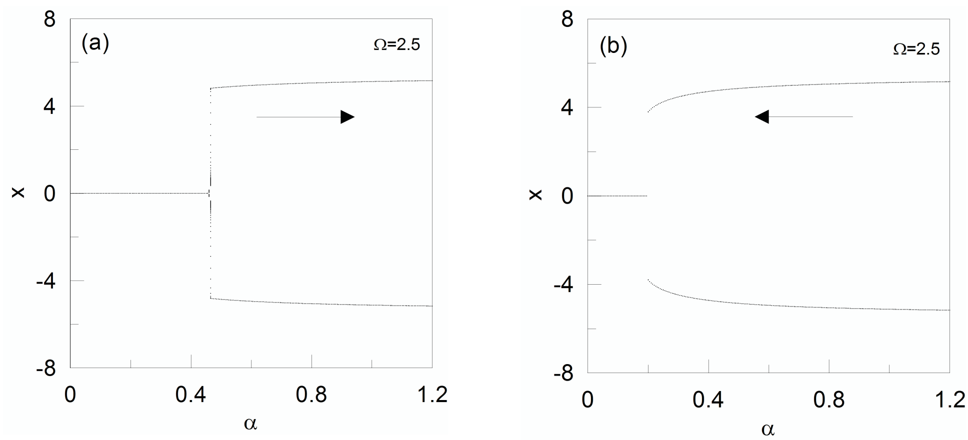

Figure 7 Bifurcation diagram x(α) for Ω=2.5; increasing α (a), decreasing α (b)

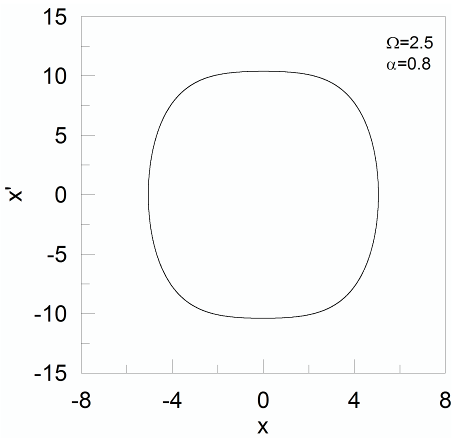

Figure 8 Poincar maps for Ω=2.5 and α=0.8

Looking at bifurcation diagrams obtained for Ω =2.5 (Fig. 7), one can notice a transition from stable zero solution to periodic motion (Fig. 8) through subcritical Hopf bifurcation.

Thus, nonlinearity of the Duffing type gets narrow the stable zero solution region but it also limits vibrations amplitude to a limit cycle that is a positive aspect from practical point of view.

3. Duffing oscillator with time delay

In this section the externally forced Duffing oscillator is investigated in variants with and without time delayed displacement. The former is classical Duffing oscillator, the latter can be treated as a example of a regeneration model of a cutting process with a control system, modelled as external excitation with f amplitude and frequency λ. At the beginning, the system described by Eq. (1) is solved analytically with the help of the multiple scale method [18]. A fast scale and a slow scale of time are introduced, then a solution in the first order approximation is sought in the form:

$$\begin{aligned} x(t) &= x_0(T_0, T_1) + \epsilon x_1(T_0, T_1) \\ x(t-\tau) &= x_\tau = x_{0\tau}(T_0, T_1) + \epsilon x_{1\tau}(T_0, T_1) \end{aligned} \tag{10}$$

It is assumed that:

$$\delta = \epsilon\tilde{\delta}, \;\; \omega_0^2 = 1, \;\; \gamma = \epsilon\tilde{\gamma}, \;\; \alpha = \epsilon\tilde{\alpha}, \;\; \lambda = 1 + \epsilon\delta, \;\; f = \epsilon\tilde{f} \tag{11}$$

where ε is a formal small parameter, and σ is detuning parameter around the natural frequency $\omega_0 = 1$. Next, in order to facilitate notation, the tilde is omitted. By using the chain rule, the time derivative is transformed according to the expressions:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} \tag{12}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + \epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon \frac{\partial^2}{\partial T_1 \partial T_0} + ... = \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + ... \tag{13}$$

then

$$\begin{aligned} &\frac{\partial^2 x(t)}{\partial T_0^2} + 2\epsilon \frac{\partial^2 x(t)}{\partial T_0 \partial T_1} + \epsilon\delta \frac{\partial x(t)}{\partial T_0} + x(t) + \epsilon\gamma x(t)^3 \\ &= \epsilon\alpha[-\mu x(t) + x(t-\tau)] + \epsilon f \cos(1 - \epsilon\sigma)t \end{aligned} \tag{14}$$

Expanding derivatives of the equation (14)

$$\frac{\partial x(t)}{\partial T_0} = \frac{\partial x_0}{\partial T_0} + \epsilon \frac{\partial x_1}{\partial T_0} \tag{15}$$

$$\frac{\partial^2 x(t)}{\partial T_0^2} = \frac{\partial^2 x_0}{\partial T_0^2} + \epsilon \frac{\partial^2 x_1}{\partial T_0^2} \tag{16}$$

$$\frac{\partial^2 x(t)}{\partial T_0 \partial T_1} = \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + \epsilon \frac{\partial^2 x_1}{\partial T_0 \partial T_1} \tag{17}$$

one gets:

$$\begin{aligned} &\frac{\partial^2 x_0}{\partial T_0^2} + \epsilon \frac{\partial^2 x_1}{\partial T_0^2} + 2\epsilon \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + \epsilon\delta \frac{\partial x_0}{\partial T_0} + x_0 + \epsilon x_1 + \epsilon\gamma x_0^3 \\ &= \epsilon\alpha(-\mu x_0 + x_{0\tau}) + \epsilon f \cos(T_0 + \sigma T_1) \end{aligned} \tag{18}$$

Equating coefficients of powers yields

$$\epsilon^0: \quad \frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{19}$$

$$\begin{aligned}\epsilon^1: \quad & \frac{\partial^2 x_1}{\partial T_0^2} + 2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} + \delta \frac{\partial x_0}{\partial T_0} + x_1 + \gamma x_0^3 + \mu\alpha x_0 - \alpha x_{0\tau} \\ & -f\cos(T_0 - \sigma T_1) = 0\end{aligned} \tag{20}$$

The general solution of (20) is

$$x_0(T_0, T_1) = A(T_1)e^{iT_0} + \bar{A}(T_1)ie^{-iT_0} \tag{21}$$

$$x_{0\tau}(T_0, T_1) = A(T_1)e^{i(T_0-\tau)} + \bar{A}(T_1)e^{-i(T_0-\tau)} \tag{22}$$

where is complex conjugate of . Substituting equations (21) and (22) into equation (20) and expanding derivatives we get:

$$\frac{\partial x_0}{\partial T_0} = A(T_1)ie^{iT_0} - \bar{A}(T_1)ie^{-iT_0} \tag{23}$$

$$\frac{\partial^2 x_0}{\partial T_0 \partial T_1} = A'(T_1 i)e^{iT_0} - \bar{A}'(T_1)ie^{-iT_0} \tag{24}$$

$$f\cos(T_0 - \sigma T_1) = \frac{1}{2}f[e^{i(T_0+\sigma T_1)} - e^{-i(T_0+\sigma T_1)}] \tag{25}$$

and then the equation is obtained:

$$\begin{aligned}&\frac{\partial x_1}{\partial T_0^2} + 2\left[A'(T_1)ie^{iT_0} - \bar{A}'(T_1)ie^{-iT_0}\right] + \delta\left[A(T_1)ie^{iT_0} - \bar{A}(T_1)ie^{-iT_0}\right] + x_1 \\ &+\gamma\left[A(T_1)e^{iT_0} + \bar{A}(T_1)e^{-iT_0}\right]^3 + \mu\alpha\left[A(T_1)e^{iT_0} + \bar{A}(T_1)e^{-iT_0}\right] \\ &-\alpha\left[A(T_1)e^{i(T_0-\tau)} + \bar{A}(T_1)e^{-i(T_0-\tau)}\right] - \frac{1}{2}f[e^{i(T_0+\sigma T_1)} - e^{-i(T_0+\sigma T_1)}] = 0\end{aligned} \tag{26}$$

Ordering equation (26) we get its final form

$$\begin{aligned}&\frac{\partial x_1}{\partial T_0^2} + x_1 + \gamma A(T_1)^3 e^{3iT_0} + \gamma\bar{A}(T_1)^3 e^{-3iT_0} + e^{iT_0}[-\frac{1}{2}fe^{i\sigma T_1} - \alpha A(T_1)e^{-i\tau} \\ &+i\delta A(T_1) + \mu\alpha A(T_1) + 3\gamma A(T_1)^2\bar{A}(T_1) + 2iA'(T_1)] + e^{-iT_0}[-\frac{1}{2}fe^{-i\sigma T_1} \\ &-\alpha\bar{A}(T_1)e^{i\tau} - i\delta\bar{A}(T_1) + \mu\alpha\bar{A}(T_1) + 3\gamma\bar{A}(T_1)^2 A(T_1) - 2i\bar{A}'(T_1)] = 0\end{aligned} \tag{27}$$

Eliminating from equation (27) the terms and , that lead to secular terms, we have

$$\frac{\partial x_1}{\partial T_0^2} + x_1 + \gamma A(T_1)^3 e^{3iT_0} + \gamma\bar{A}(T_1)^3 e^{-3iT_0} = 0 \tag{28}$$

Solving (28) for :

$$x_1(T_0, T_1) = B(T_1)e^{3iT_0} + \bar{B}(T_1)e^{-3iT_0} \tag{29}$$

$$x_{1\tau}(T_0, T_1) = B(T_1)e^{3i(T_0-\tau)} + \bar{B}(T_1)e^{-3i(T_0-\tau)} \tag{30}$$

where

$$B(T_1) = \frac{1}{8}\gamma A(T_1)^3 \tag{31}$$

$$\bar{B}(T_1) = \frac{1}{8}\gamma \bar{A}(T_1)^3 \tag{32}$$

we obtain

$$x_1(T_0, T_1) = \frac{1}{8}\gamma A(T_1)e^{3iT_0} + \frac{1}{8}\gamma \bar{A}(T_1)e^{-3iT_0} \tag{33}$$

$$x_{1\tau}(T_0, T_1) = \frac{1}{8}\gamma A(T_1)e^{3i(T_0-\tau)} + \frac{1}{8}\gamma \bar{A}(T_1)e^{-3i(T_0-\tau)} \tag{34}$$

Substituting (21), (22) and (33), (34) into (10), the approximate solutions are as follows:

$$\begin{aligned} x_0(T_0, T_1) &= A(T_1)e^{iT_0} + \bar{A}(T_1)ie^{-iT_0} \\ &+\epsilon\left(\frac{1}{8}\gamma A(T_1)e^{3i(T_0-\tau)} + \frac{1}{8}\gamma \bar{A}(T_1)e^{-3i(T_0-\tau)}\right) \end{aligned} \tag{35}$$

$$\begin{aligned} x_{0\tau}(T_0, T_1) &= A(T_1)e^{i(T_0-\tau)} + \bar{A}(T_1)e^{-i(T_0-\tau)} \\ &+\epsilon\left(\frac{1}{8}\gamma A(T_1)e^{3i(T_0-\tau)} + \frac{1}{8}\gamma \bar{A}(T_1)e^{-3i(T_0-\tau)}\right) \end{aligned} \tag{36}$$

and can be calculated from equations:

$$\begin{aligned} &-\frac{1}{2}fe^{i\sigma T_1} - \alpha A(T_1)e^{-i\tau} + i\delta A(T_1) + \mu\alpha A(T_1) + 3\gamma A(T_1)^2\bar{A}(T_1) \\ &+2iA'(T_1)] = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} &-\frac{1}{2}fe^{-i\sigma T_1} - \alpha \bar{A}(T_1)e^{i\tau} - i\delta \bar{A}(T_1) + \mu\alpha \bar{A}(T_1) + 3\gamma \bar{A}(T_1)^2 A(T_1) \\ &-2i\bar{A}'(T_1)] = 0 \end{aligned} \tag{38}$$

Introduction into equation (37) the polar form of the complex amplitude:

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)} \tag{39}$$

$$\bar{A}(T_1) = \frac{1}{2}a(T_1)e^{-i\beta(T_1)} \tag{40}$$

$$A'(T_1) = \frac{1}{2}a'(T_1)e^{i\beta(T_1)} + \frac{1}{2}ia(T_1)\beta'(T_1)e^{i\beta(T_1)} \tag{41}$$

results in

$$-\frac{1}{2}fe^{i\sigma T_1}-\frac{1}{2}\alpha a(T_1)e^{i\beta(T_1)-i\tau}+\frac{1}{2}i\delta a(T_1)e^{i\beta(T_1)}$$
$$+\frac{1}{2}\mu\alpha a(T_1)e^{i\beta(T_1)}+3\gamma\left[\frac{1}{2}a(T_1)e^{i\beta(T_1)}\right]^2\frac{1}{2}a(T_1)e^{-i\beta(T_1)} \tag{42}$$
$$+2i\left[\frac{1}{2}a'(T_1)e^{i\beta(T_1)}+\frac{1}{2}ia(T_1)\beta'(T_1)e^{i\beta(T_1)}\right]=0$$

Then recalling

$$e^{i\tau}=\cos\tau-i\sin\tau \tag{43}$$
$$e^{i\sigma T_1-i\beta(T_1)}=\cos\varphi(T_1)-i\sin\varphi(T_1) \tag{44}$$
$$\beta'(T_1)=\sigma-\varphi'(T_1) \tag{45}$$

the normal form is obtained

$$\frac{1}{2}i\delta a(T_1)+\frac{1}{2}\mu\alpha a(T_1)-\sigma a(T_1)+\frac{3}{8}\gamma a(T_1)^3-\frac{1}{2}\alpha a(T_1)\cos\tau-\frac{1}{2}f\cos\varphi(T_1)$$
$$+\frac{1}{2}i\alpha a(T_1)\sin\tau-\frac{1}{2}f\sin\varphi(T_1)+ia'(T_1)+a(T_1)\varphi'(T_1) \tag{46}$$

Separating real and imaginary parts, the two, so called, modulation equations are found:

$$\frac{1}{2}\delta a(T_1)+\frac{1}{2}\alpha a(T_1)\sin\tau-\frac{1}{2}f\sin\varphi(T_1)+a'(T_1)=0 \tag{47}$$

$$\frac{1}{2}\mu\alpha a(T_1)-\sigma a(T_1)+\frac{3}{8}\gamma a(T_1)^3-\frac{1}{2}\alpha a(T_1)\cos\tau$$
$$-\frac{1}{2}f\cos\varphi(T_1)+a(T_1)\varphi'(T_1)=0 \tag{48}$$

For a steady state and , then

$$\sin\varphi(T_1)=\frac{a(T_1)[\delta+\alpha\sin\tau]}{f} \tag{49}$$
$$\cos\varphi(T_1)=\frac{a(T_1)[4\mu\alpha-8\sigma+3\gamma a(T_1)^3-4\alpha\cos\tau]}{4f} \tag{50}$$

Using simple trigonometric manipulations, the frequency response relation between and , and between and is obtained:

$$a(T_1)^2[16(\delta+\alpha\sin\tau)^2+(4\mu\alpha-8\sigma+3\gamma a(T_1)^3$$
$$-4\alpha\cos\tau)^2]-16f^2=0 \tag{51}$$

$$\tan\varphi(T_1)=\frac{4(\delta+\alpha\sin\tau)}{4\mu\alpha-8\sigma+3\gamma a(T_1)^3-4\alpha\cos\tau} \tag{52}$$

First, it is assumed that there is no time delay feedback α=0 in the system and f=0.2, μ=0, then analytical resonance curve obtained from equation (52) and from numerical simulations are similar as presented in Fig. 9.

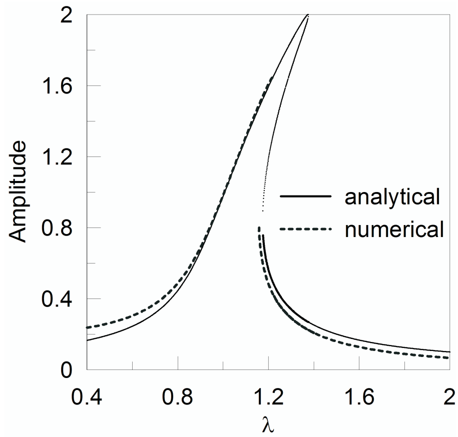

Figure 9 Amplitude–frequency characteristic for classical externally forced Duffing oscillator

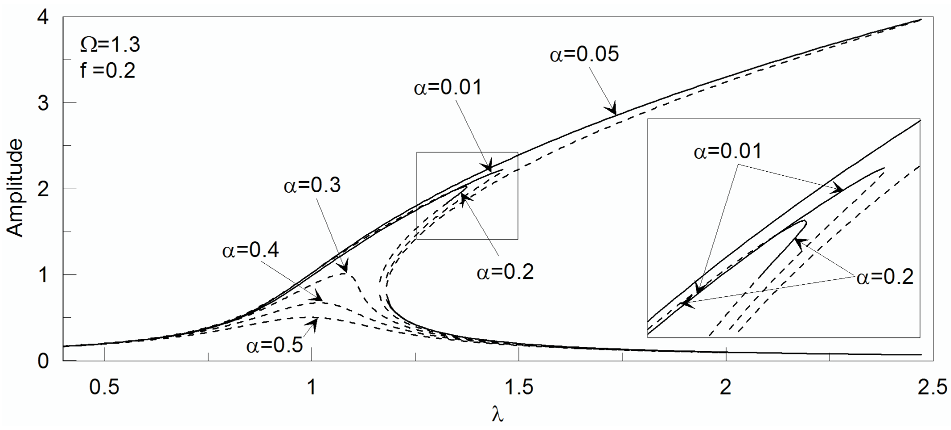

Figure 10 Amplitude–frequency characteristic for externally forced Duffing oscillator with delayed displacement feedback. The influence of delay amplitude α

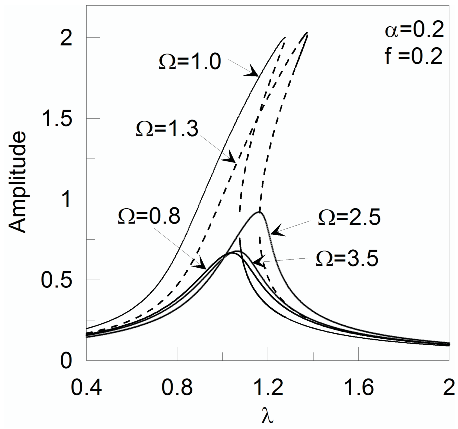

Figure 11 Amplitude-frequency characteristic for externally forced Duffing oscillator with delayed displacement feedback. The influence of delay Ω

Now, influence of the time delay level α on the displacement amplitude is shown in Fig. 10. It is turned out that the bigger α results in smaller vibrations but the solutions are unstable. The system loses stability at about $\alpha = 0.2$, therefore the time delayed feedback is so important to control the Duffing oscillator in a proper way. On the other hand, looking at cutting process when time delay effect is unavoidable because of regenerative effect. For vibrating system that contains nonlinearity of Duffing type and external excitation, big stable vibrations occur only at relatively small width of cut (α).

Finally, the effect of a time delay change, expressed by Ω, is analysed near the fundamental resonance of the system around $\omega_0 = 1$and for α =0.2 (Fig. 11). Parameter Ω=0.8, Ω =1.0, Ω =1.3 and Ω =2.5 correspond to unstable region in SLD (Fig. 2) while Ω=3.5 represent stable region. Generally, the change of Ω shifts the primary resonance curve but the shift direction depends on the analysed point position on SLD. Interestingly, the unstable curve in Fig.11 obtained for Ω =1.3 can be stabilised by changing slightly the frequency Ω of external excitation.

4. Conclusions

The nonlinearity of Duffing type and external excitation influence the behaviour of a classical linear model of regenerative cutting. Stability zones obtained on the basis of a linear model with time delay get small after adding Duffing type nonlinearities. It means that the regions of the asymptotic stable zero solutions are narrower. On the other hand, in a case of a linear system the unstable solutions run away to infinity while for the nonlinear model the solutions tend to limit cycle. That is better alternative from practical point of view because vibrations amplitude is limited during a real cutting process.

Duffing oscillator with external excitation and time delay can be a good method of chatter vibrations suppression. That is especially important, since an increase of delay amplitude (α), which is in fact represents a cutting width, limits vibrations and favours stability loss of harmonic solution, comparing to the externally forced Duffing oscillator without time delay.

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