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On the Hyperbolic Fly Past Problem as a Velocity Amplifier Using the Elliptic Hohmann Transfer

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We investigate the problem of fly past of a space vehicle traveling in a generalized elliptic Hohmann transfer system between the elliptic orbits of the Earth and Jupiter around the Sun. We consider the four feasible elliptic Hohmann configurations. We begin our treatment by a more precise expression for the hyperbolic excess velocity, because we deal with the elliptic not the circular Hohmann case. We assign the semi–major axes and the eccentricity of the hyperbolic trajectory that lies within the sphere of influence of the Jovian planet. Whence we have a more accurate determination of the elements of the hyperbolic trajectory before the vehicle's departure out of Jupiter's influence sphere to follow its trip to a further outer planet of the local solar system.

Keywords: Astrodynamics, orbital mechanics, Hohmann orbit transfer, fly past problem, sphere of influence

1. Introduction

Recently planetary fly past has been used, so that the space vehicle might reach another planet, in the local solar system of our galaxy. For instance the fly past of Venus by Mariner 10 makes three successive fly pasts of Mercury. The Voyager fly pasts of Jupiter recessed out wards to Saturn and beyond [1]. A.E. Roy considered the problem of fly past by considering the classical circular case of the Hohmann transfer [2], [3]. Whence he began his treatment by expressing the approximate hyperbolic excess velocity in the form

$$V = \sqrt{\frac{\mu}{a_2}}\left[1 - \sqrt{\frac{2}{1+\frac{a_2}{a_1}}}\right]$$

We derived the corresponding formulae of the velocity excess for the generalized Hohmann elliptic case, i.e. when all the three trajectories (initial, transfer and final) are elliptic. Thus we begin our analysis by a more precise formula, but we assume the coplanar version of tackling the problem. We should notice that Jupiter overtakes the space vehicle at an almost tangential orbit, and that the hyperbolic fly past trajectory of the spacecraft is wholly within the influence sphere of planet Jupiter [1].

2. Preliminary Concepts

2.1. *Feasible Configurations*

For the generalized Hohmann elliptic case, we are confronted with four feasible configurations [4], according to the coincidence of peri–apse and apo–apse of the three elliptic trajectories: the terminal orbits and the one transfer orbit, namely:

1. Apo–apse of transfer orbit coincides with apo–apse of final orbit.
2. Apo–apse of transfer orbit coincides with peri–apse of final orbit.
3. Peri–apse of transfer orbit coincides with peri–apse of final orbit.
4. Apo–apse of transfer orbit coincides with apo–apse of final orbit.

2.2. *Change of energy concept*

The most convenient way of handling the problem is to adopt the concept of change of energy. Referring to Fig. 1, we note that

$$\begin{aligned} a_1\left(1-e_1\right) &= a_T\left(1-e_T\right) \\ a_2\left(1+e_2\right) &= a_T\left(1+e_T\right) \end{aligned} \tag{1}$$

From Eq. (1), we find that after little reduction

$$\begin{aligned} a_T &= \frac{a_1\left(1-e_1\right)+a_2\left(1+e_2\right)}{2} \\ e_T &= \frac{-a_1\left(1-e_1\right)+a_2\left(1+e_2\right)}{a_1\left(1-e_1\right)+a_2\left(1+e_2\right)} \end{aligned} \tag{2}$$

It is a well known fact that for the standard circular classical Hohmann system

$$a_T = \frac{a_1+a_2}{2} \qquad e_T = \frac{a_2-a_1}{a_2+a_1} \tag{3}$$

since $e_1 = 0$; $e_2 = 0$.

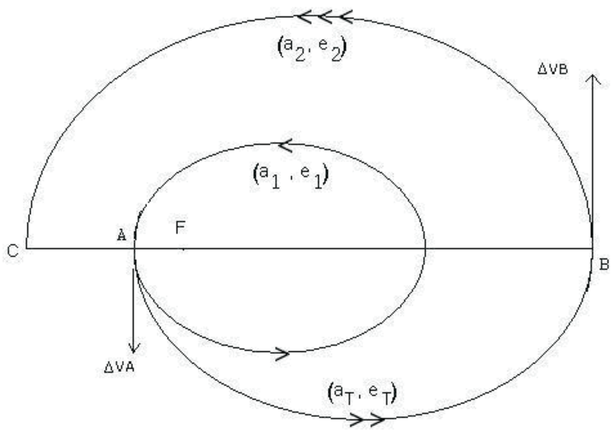

Figure 1

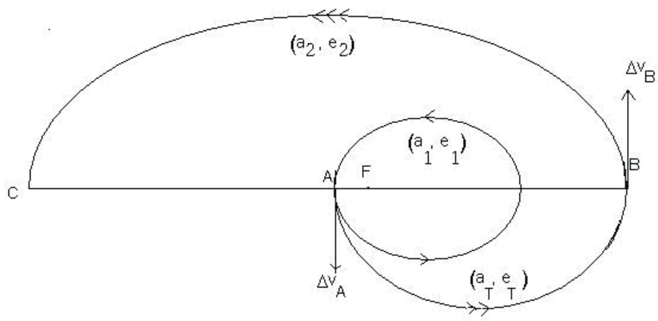

Figure 2

For the total and kinetic energy of the transfer ellipse, we have

$$C_T = \frac{V_T^2}{2} - \frac{\mu}{r_T} \qquad V_T^2 = \mu\left(\frac{2}{r_T} - \frac{1}{a_T}\right) \tag{4}$$

which yield

$$C_T = -\frac{\mu}{2a_T} \tag{5}$$

whence

$$C_T = \frac{-\mu}{a_1\,(1-e_1) + a_2\,(1+e_2)} \tag{6}$$

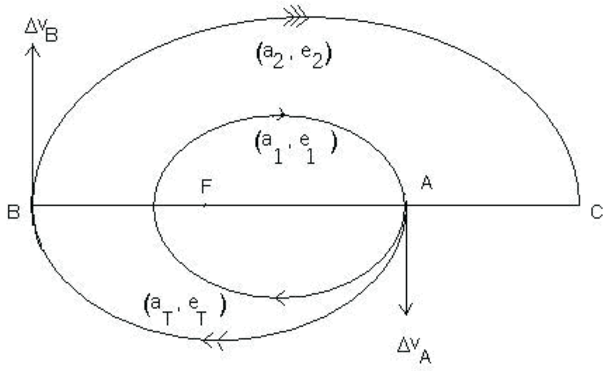

Figure 3

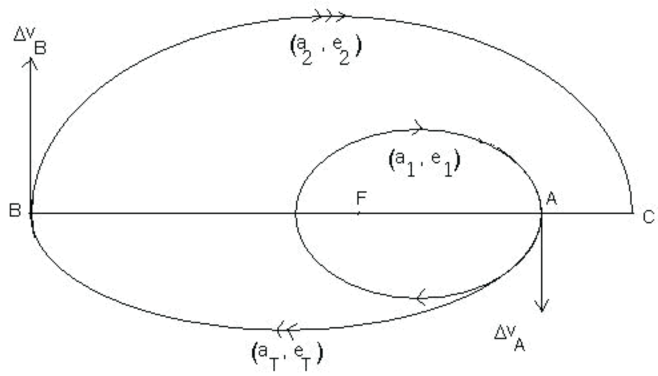

Figure 4

For the energy increment at A we get

$$\Delta C_A = C_T - C_1 \tag{7}$$

But

$$C_1 = \frac{-\mu}{2a_1} \tag{8}$$

So, we acquire

$$\Delta C_A = \frac{\mu}{2a_1}\left[\frac{-a_1\left(1+e_1\right)+a_2\left(1+e_2\right)}{a_1\left(1-e_1\right)+a_2\left(1+e_2\right)}\right] \tag{9}$$

Similarly we can find

$$\Delta C_B = \frac{\mu}{2a_2}\left[\frac{-a_1(1-e_1)+a_2(1-e_2)}{a_1(1-e_1)+a_2(1+e_2)}\right] \tag{10}$$

For the change in kinetic energies at points A& B, we obtain the following equalities

$$\Delta C_A = \frac{1}{2}(V_A+\Delta V_A)^2 - \frac{1}{2}V_A^2 \tag{11}$$

$$\Delta C_B = \frac{1}{2}(V_B+\Delta V_B)^2 - \frac{1}{2}V_B^2 \tag{12}$$

where

$$\begin{aligned}(V_A)_{Per.} &= \left(\frac{\mu}{a_1}\right)^{1/2}\left(\frac{1+e_1}{1-e_1}\right)^{1/2}\\ (V_B)_{Apo.} &= \left(\frac{\mu}{a_2}\right)^{1/2}\left(\frac{1-e_2}{1+e_2}\right)^{1/2}\end{aligned} \tag{13}$$

By the solution of the two second degree equations in $\Delta V_A \& \Delta V_B$, we get

$$\begin{aligned}\Delta V_A &= -V_A \pm \left(V_A^2+2\Delta C_A\right)^{1/2}\\ \Delta V_B &= -V_B \pm \left(V_B^2+2\Delta C_B\right)^{1/2}\end{aligned} \tag{14}$$

Whence we may write for the first configuration,

$$\begin{aligned}\Delta V_{A1} = \left(\frac{\mu}{a_1}\right)^{1/2}\Bigg[&-\left(\frac{1+e_1}{1-e_1}\right)^{1/2}\\ &\pm\left\{\frac{1+e_1}{1-e_1}+\frac{a_2(1+e_2)-a_1(1+e_1)}{a_2(1+e_2)+a_1(1-e_1)}\right\}^{1/2}\Bigg]\end{aligned} \tag{15}$$

$$\begin{aligned}\Delta V_{B1} = \left(\frac{\mu}{a_2}\right)^{1/2}\Bigg[&-\left(\frac{1-e_2}{1+e_2}\right)^{1/2}\\ &\pm\left\{\frac{1-e_2}{1+e_2}+\frac{a_2(1-e_2)-a_1(1-e_1)}{a_2(1+e_2)+a_1(1-e_1)}\right\}^{1/2}\Bigg]\end{aligned} \tag{16}$$

After some reductions, we assign easily a_T , e_T , for the three configurations (II); (III) and (IV) as the following:

$$
\begin{aligned}
a_T &= \frac{a_1(1-e_1)+a_2(1-e_2)}{2} \\
e_T &= \frac{-a_1(1-e_1)+a_2(1-e_2)}{a_1(1-e_1)+a_2(1-e_2)}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
a_T &= \frac{a_1(1+e_1)+a_2(1-e_2)}{2} \\
e_T &= \frac{a_1(1+e_1)-a_2(1-e_2)}{a_1(1+e_1)+a_2(1-e_2)}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
a_T &= \frac{a_1(1+e_1)+a_2(1+e_2)}{2} \\
e_T &= \frac{-a_1(1+e_1)+a_2(1+e_2)}{a_1(1+e_1)+a_2(1+e_2)}
\end{aligned}
\tag{19}
$$

For the first two configurations, the initial impulse is located at the peri–apse of the transfer elliptic orbit, and for the third and fourth configurations, the initial impulse is located at the apo–apse of the elliptic transfer orbit. Needless to say, there is one elliptic transfer orbit for the generalized Hohmann elliptic transfer, but we have two elliptic transfer orbits for the bi–elliptic transfer.

3. Determination of the major axis and eccentricity of the fly past hyperbolic trajectory

For a spacecraft moving in cotangential ellipse between orbits of Earth and Jupiter, for instance, the hyperbolic excess velocity V in the case of the classical Hohmann transfer is given approximately by

$$
V = \Delta V_B = \left(\frac{\mu}{a_2}\right)^{1/2} \left\{ 1 - \left(\frac{2}{1+\frac{a_2}{a_1}}\right)^{1/2} \right\} \tag{20}
$$

In the generalized elliptic Hohmann case, we have for the four configurations

$$
\begin{aligned}
& V_1 = \Delta V_{B1} = \text{Eq. (16)} \\
& V_2 = \Delta V_{B2} = \left(\frac{\mu}{a_2}\right)^{1/2} \left[- \left(\frac{1+e_2}{1-e_2}\right)^{1/2} \right. \\
& \left. \pm \left\{ \frac{1+e_2}{1-e_2} + \frac{a_2\,(1+e_2) - a_1\,(1-e_1)}{a_2\,(1-e_2) + a_1\,(1-e_1)} \right\}^{1/2} \right]
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
& V_3 = \Delta V_{B3} = \left(\frac{\mu}{a_2}\right)^{1/2} \left[- \left(\frac{1+e_2}{1-e_2}\right)^{1/2} \right. \\
& \left. \pm \left\{ \frac{1+e_2}{1-e_2} + \frac{a_2\,(1+e_2) - a_1\,(1+e_1)}{a_2\,(1-e_2) + a_1\,(1+e_1)} \right\}^{1/2} \right]
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
& V_4 = \Delta V_{B4} = \left(\frac{\mu}{a_2}\right)^{1/2} \left[- \left(\frac{1-e_2}{1+e_2}\right)^{1/2} \right. \\
& \left. \pm \left\{ \frac{1-e_2}{1+e_2} + \frac{a_2\,(1-e_2) - a_1\,(1+e_1)}{a_2\,(1+e_2) + a_1\,(1+e_1)} \right\}^{1/2} \right]
\end{aligned}
\tag{23}
$$

Moreover, we may prove that:

$$
\begin{aligned}
& \Delta V_{A1} = \text{Eq. (15)} \\
& \Delta V_{A2} = \left(\frac{\mu}{a_1}\right)^{1/2} \left[- \left(\frac{1+e_1}{1-e_1}\right)^{1/2} \right. \\
& \left. \pm \left\{ \frac{1+e_1}{1-e_1} + \frac{a_2\,(1-e_2) - a_1\,(1+e_1)}{a_2\,(1-e_2) + a_1\,(1-e_1)} \right\}^{1/2} \right]
\end{aligned}
\tag{24}
$$

$$\Delta V_{A3} = \left(\frac{\mu}{a_1}\right)^{1/2} \left[- \left(\frac{1-e_1}{1+e_1}\right)^{1/2} \right.$$
$$\left. \pm \left\{ \frac{1-e_1}{1+e_1} + \frac{a_2\,(1-e_2) - a_1\,(1-e_1)}{a_2\,(1-e_2) + a_1\,(1+e_1)} \right\}^{1/2} \right] \tag{25}$$

$$\Delta V_{A4} = \left(\frac{\mu}{a_1}\right)^{1/2} \left[- \left(\frac{1-e_1}{1+e_1}\right)^{1/2} \right.$$
$$\left. \pm \left\{ \frac{1-e_1}{1+e_1} + \frac{a_2\,(1+e_2) - a_1\,(1-e_1)}{a_2\,(1+e_2) + a_1\,(1+e_1)} \right\}^{1/2} \right] \tag{26}$$

We have the two principal equalities

$$r \approx \left(\frac{m_J}{M_S}\right)^{2/5} a_J \tag{27}$$

$$V^2 = \mu_J \left(\frac{2}{r} + \frac{1}{a}\right) \tag{28}$$

Eq. (27) represents the Jovian sphere of influence and V^2 is the hyperbolic excess velocity Eq. (28) , or in other words the hyperbolic fly past velocity of space vehicle within the sphere of influence of Jupiter.

Consequently we have

$$\begin{aligned} V^2 &= \mu_J \left(\frac{2}{r} + \frac{1}{a}\right) = \frac{2\mu_J}{r} + \frac{\mu_J}{a} \\ \mu_J &= Gm_J \\ \mu &= GM_S \end{aligned} \tag{29}$$

We may write to a first approximation

$$r \approx \left(\frac{m_J}{M_S}\right)^{2/5} a_J \tag{30}$$

Since the orbit of Jupiter is nearly a circular one as well as all the planets of the local solar system.

whence $a_J \approx a_2$ in the formula (30)

$$r = \left(\frac{m_J}{M_S}\right)^{2/5} a_2 \tag{31}$$

where a_2 is the orbital circular radius of Jupiter.

Moreover we may simply write

$M = M_S$ – the mass of the Sun,

$m = m_J$ – the mass of Jupiter.

Whence

$$\frac{\mu_J}{a} = \frac{\mu}{a_2}[\psi]^2 - \frac{2\mu_J}{\left(\frac{m}{M}\right)^{2/5} a_2} \tag{32}$$

where

$$\mu = GM \qquad \mu_J = Gm$$

Finally, We may write, after some reductions

$$a = \frac{Gma_2}{GM[\psi]^2 - 2GM^{2/5}m^{3/5}} \tag{33}$$

Whence we have the following four values of ψ:

$$\begin{gathered}\psi_1 = \left(\frac{a_2}{\mu}\right)^{1/2} \Delta V_{B1} = \left[-\left(\frac{1-e_2}{1+e_2}\right)^{1/2}\right.\\ \left.\pm\left\{\frac{1-e_2}{1+e_2} + \frac{a_2(1-e_2) - a_1(1-e_1)}{a_2(1+e_2) + a_1(1-e_1)}\right\}^{1/2}\right]\end{gathered} \tag{34}$$

$$\begin{gathered}\psi_2 = \left(\frac{a_2}{\mu}\right)^{1/2} \Delta V_{B2} = \left[-\left(\frac{1+e_2}{1-e_2}\right)^{1/2}\right.\\ \left.\pm\left\{\frac{1+e_2}{1-e_2} + \frac{a_2(1+e_2) - a_1(1-e_1)}{a_2(1-e_2) + a_1(1-e_1)}\right\}^{1/2}\right]\end{gathered} \tag{35}$$

$$\begin{gathered}\psi_3 = \left(\frac{a_2}{\mu}\right)^{1/2} \Delta V_{B3} = \left[-\left(\frac{1+e_2}{1-e_2}\right)^{1/2}\right.\\ \left.\pm\left\{\frac{1+e_2}{1-e_2} + \frac{a_2(1+e_2) - a_1(1+e_1)}{a_2(1-e_2) + a_1(1+e_1)}\right\}^{1/2}\right]\end{gathered} \tag{36}$$

$$\begin{gathered}\psi_4 = \left(\frac{a_2}{\mu}\right)^{1/2} \Delta V_{B4} = \left[-\left(\frac{1-e_2}{1+e_2}\right)^{1/2}\right.\\ \left.\pm\left\{\frac{1-e_2}{1+e_2} + \frac{a_2(1-e_2) - a_1(1+e_1)}{a_2(1+e_2) + a_1(1+e_1)}\right\}^{1/2}\right]\end{gathered} \tag{37}$$

We now assigned the first element a of the hyperbolic trajectory of space vehicle, within the sphere of influence of Jupiter. As for the eccentricity of this hyperbola, we may choose its peri Jove distance r_P to be not more than r_J , we may write the relation

$$r_P = a\,(e-1) \tag{38}$$

whence

$$e = \frac{r_P}{a} + 1 \qquad r_P \approx r_J \tag{39}$$

Where e is the second element of the hyperbolic trajectory.

Moreover , we may write from the geometry of the hyperbola

$$b = a\left(e^2 - 1\right)^{1/2} \qquad \tan\psi = \pm b/a$$

From Fig. 5 it is evident that the encounter rotates the direction of the velocity when entering the Jovian sphere of influence by an angle $\theta = \pi - 2\psi$, we may deduce that

$$\tan\theta/2 = \frac{1}{\sqrt{e^2-1}}$$

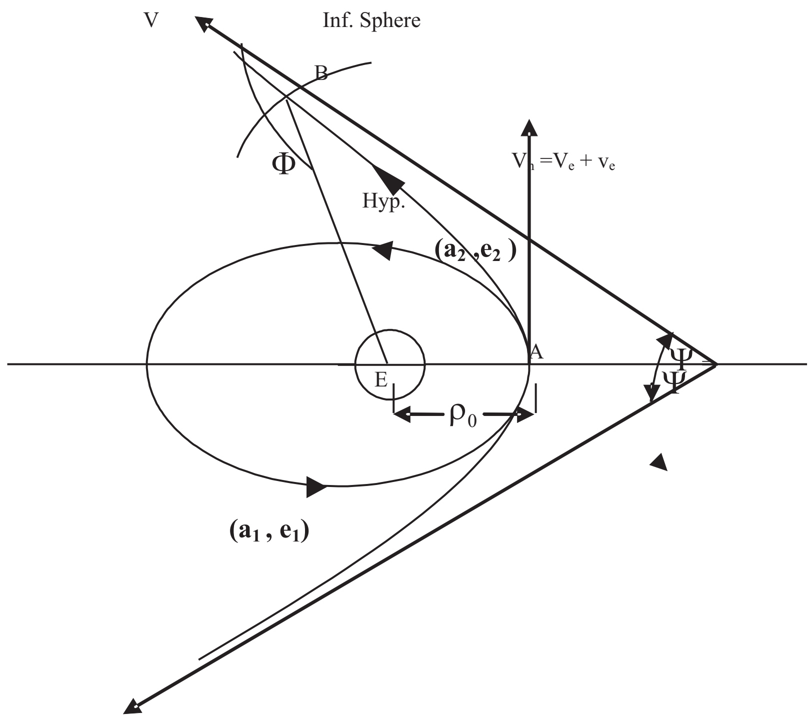

Figure 5 Bonduary of Jupiter sphere of influence

4. Concluding Remarks

A more precise determination of the two elements major axis and eccentricity of the hyperbolic fly past trajectory within Jupiter's sphere of influence is acquired. It is necessary to know the exact two elements in order that the spacecraft might follow its right trip to an outer planet, for instance planet Saturn – an orbit determination problem [5]. The main purpose of this process is to eject the spacecraft in an almost opposite direction outside Jupiter's sphere of influence with a greater speed than that in the transfer Hohmann trajectory so that the space vehicle might start departure to another planet, at a greater distance than Jupiter's from the Sun. A further amplification of speed may be attained by firing the vehicle's engines at peri Jove. It is evident that planet Jupiter has a practical application by its utilization as a velocity amplifier. We start the analysis from the hyperbolic excess velocity $\Delta_{B1,2,3,4}$ relevant to the Hohmann elliptic transfer. There are four values of a, e corresponding to the four feasible configurations.

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Nomenclature

$a_1 = a_E$	semi – major axis of Earth's orbit
$a_2 = a_J$	semi – major axis of Jupiter's orbit
a_T	semi – major axis of transfer orbit
$e_1 = e_E$	eccentricity of Earth's orbit
$e_2 = e_J$	eccentricity of Jupiter's orbit
e_T	eccentricity of transfer orbit
V	hyperbolic excess velocity
r	radius of sphere of influence
a	first element of hyperbolic trajectory of spacecraft entering Jupiter's sphere of influence
e	second element of hyperbolic trajectory of space vehicle entering Jupiter's sphere of influence
$\mu_J = Gm_J$	constant of gravitation w.r.t. Jupiter's motion
m_J	mass of Jupiter
$\mu = GM$	constant of gravitation w.r.t. Sun
M	mass of the Sun
$r_P \quad \sim \quad r_J$	defined from $r_P = a(e-1)$
2Ψ	as shown in Fig. (5). Angle between the two extreme hyperbolic asymptotes
$\Delta V_B = V$	hyperbolic excess velocity when considering elliptic Hohmann transfer
r_J	radius of Jupiter = 0.000477 A.U.

Appendix
The Hyperbolic Escape From The First Body

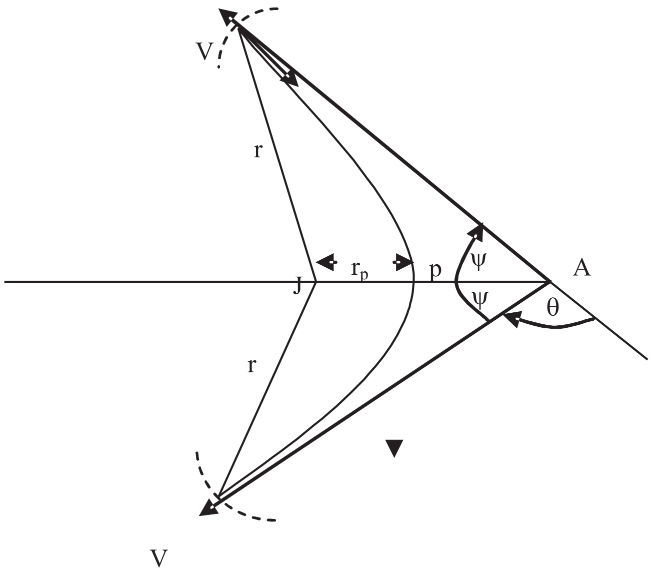

Figure 6

We generalize the parking orbit to be elliptic instead of circular, which will be coplanar with the hyperbolic orbit. Moreover we assume a tangential impulse. The geometry of transfer is shown in Fig. 6. The first body is centered at E. A is the peri–apse of the parking elliptic orbit as shown in Fig. 5, and motors are fired at A. We have the following relationships:

$$\rho_0 = a_1\,(1 - e_1) = a_2\,(e_2 - 1) \tag{40}$$

$$v_e = \Delta v_A = \sqrt{\frac{\mu\,(e_2+1)}{a_2\,(e_2-1)}} - \sqrt{\frac{\mu\,(1+e_1)}{a_1\,(1-e_1)}} = V_h - V_e \tag{41}$$

$$V_h = V_e + \Delta v_A \tag{42}$$

where

$$V_e = \sqrt{\frac{2\mu}{\rho_0}} = \sqrt{\frac{2\mu}{a_1\,(1-e_1)}} = \sqrt{\frac{2\mu}{a_2\,(e_2-1)}} \tag{43}$$

i.e.

$$V_h = \sqrt{\frac{2\mu}{a_1\,(1-e_1)}} + \Delta v_A \tag{44}$$

Now we may write

$$\frac{1}{2}\left[\sqrt{\frac{2\mu}{a_1\left(1-e_1\right)}}+\sqrt{\frac{\mu\left(e_2+1\right)}{a_2\left(e_2-1\right)}}-\sqrt{\frac{\mu\left(1+e_1\right)}{a_1\left(1-e_1\right)}}\right]^2-\frac{\mu}{a_1\left(1-e_1\right)} = \frac{V^2}{2}-\frac{\mu}{\rho} \tag{45}$$

Where ρ is the radius of sphere of influence of the first planet.

From Eq. (45), we find that V the velocity at distance ρ in the hyperbolic orbit around the first planet is given by

$$V^2=\mu\left[\frac{1}{a_1\left(1-e_1\right)}\left\{\left(1+e_1\right)-2\sqrt{2\left(1+e_1\right)}\right\} +\frac{2\sqrt{1+e_2}\left\{\sqrt{2}+\sqrt{1+e_1}\right\}}{\sqrt{a_1a_2\left(1-e_1\right)\left(e_2-1\right)}}+\frac{e_2+1}{a_2\left(e_2-1\right)}+\frac{2}{\rho}\right] \tag{46}$$

V is called the hyperbolic excess and measures the velocity at a point just outside the sphere of influence of mass m and estimates the vehicle's escape velocity from the first field of attraction.

Moreover from the geometry of the hyperbola, we have for the angles ψ , φ.

$$\tan\psi = \pm b/a$$
$$b^2=a^2\left(e^2-1\right)$$
$$\varphi = 90^0+\cos^{-1}\left[\frac{a^2\left(e^2-1\right)}{\rho\left(2a+\rho\right)}\right]^{1/2}$$

Nomenclature (b)

a_1	elliptic semi–major axes
e_1	elliptic eccentricity
a_2	hyperbolic first element
e_2	hyperbolic second element
ρ_0	elliptic peri–apse distance
$v_e=\Delta v_A$	velocity increment
V_h	hyperbolic velocity at distance ρ_0 at A
V_e	velocity of escape (parabolic)
$V=V_\infty$	the velocity of vehicle at a distance when it has just left the effective gravitational field of the central mass
ρ	radius of sphere of influence
μ	constant of gravitation