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Solution of the Gaussian Transfer Orbit Equations of Motion

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This article deals with an orbit transfer problem by the application of only one motor thrust engine impulse at any point (r , v) on the elliptic initial orbit. The terminal orbits are elliptic. We consider the coplanar non-limited duration case. We succeeded to attain an analytical solution for the transfer Lagrange–Gauss modulated equations of motion. We selected the eccentric anomaly to be the independent parameter. We evaluated the integrals that appear in the R.H.S. of the equations of motion for $\mathrm{d}a/\mathrm{d}E$, $\mathrm{d}e/\mathrm{d}E$ and $e\mathrm{d}\omega/\mathrm{d}E$. Accordingly the three elements defining the final orbit are determined from $(a - a_o)$, $(e - e_o)$, $e(\omega - \omega_o)$.

Keywords: Orbital mechanics, orbit transfer, Lagrange–Gauss equations of motion, rocket dynamics

1. Introduction

The problem of orbit transfer is evidently essential for space flight exploration. It is also necessary and very interesting to assign the most economic procedures for a certain orbit transfer or rendezvous. There are numerical as well as analytical procedures for optimization problems. For such requirements the problem is complicated even if we neglect the perturbative acceleration.[1].

In this analysis we assume coplanar orbits without limitation of duration. In our treatment we assume a single point of attraction of a given mass. The space vehicle follows a Keplerian initial orbit. At initial instance $t_1 = 0$ the space vehicle is revolving in an elliptic orbit, whilst at time t_2 the vehicle will be on another assigned elliptic orbit, after the consumption of minimum latent velocity of its propulsion. The rendezvous problem is more complex than the simple transfer one, because this is not the case of unlimited time [2]. We investigate in this article the simple transfer problem, and we shall confine ourselves to the orbital element alternations due to instantaneous motor thrust engines propulsion.

We take the direction of peri–apse of the initial elliptic orbit as the reference of direction, and the sense of rotation is also that of the initial orbit. The initial and final elliptic orbits (terminal orbits) are completely defined by a_1, e_1, a_2, e_2, ϖ_2 respectively. Rendezvous and perturbation influences are discarded in our treatment [2].

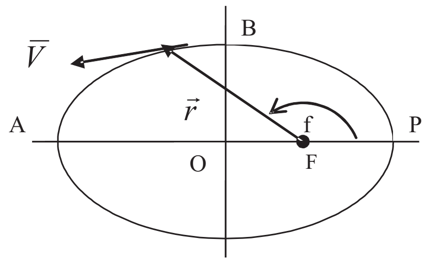

Figure 1 Orbit

$$
\begin{aligned}
&OA = OP = FB = a\\
&OB = b\\
&OF = c\\
&FP = P\\
&FA = A\\
&r = \frac{p}{1 + e\cos f} = a(1 - e\cos E)
\end{aligned}
$$

2. Derivation of Equations of Motion: da/dE, de/dE and edω/dE

In the coplanar case, the classical Lagrangian equations of motion, in the Gaussian form are

$$\frac{da}{dt} = \frac{2a}{nb}[Se\sin f + T(1 + e\cos f)] \tag{1}$$

$$\frac{de}{dt} = \frac{b}{na^2}[S\sin f + T(\cos f + \cos E)] \tag{2}$$

$$\frac{d\omega}{dt} = \frac{b}{nac}[-S\cos f + T(\sin f + \frac{a}{b}\sin E)] \tag{3}$$

We assume S , T to be the radial and transverse components respectively,

$$S = \gamma sin\phi$$
$$T = \gamma cos\phi \tag{4}$$

where

ϕ is the angle between the velocity vector and the transverse component of acceleration.

γ is the acceleration due to the impulse of propulsion defined by

$V_c = \int\limits_0^t |\vec{\gamma}| \, \mathrm{d}t$ is characteristic velocity

The quantities enclosed in the brackets could be expressed in terms of the eccentric anomaly E, using the following formulae (Appendix A)

$$\sin\varphi = \frac{1 + e\cos f}{\sqrt{1 + e^2 + 2e\cos f}} \qquad \cos\varphi = \frac{e\sin f}{\sqrt{1 + e^2 + 2e\cos f}} \tag{5}$$

$$\sin f = \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E} \qquad \cos f = \frac{\cos E - e}{1 - e\cos E} \tag{6}$$

The independent variable t (physical time) may be replaced by the eccentric anomaly E via the relation

$$\mathrm{d}t = \frac{1 - e\cos E}{n}\mathrm{d}E \tag{7}$$

Substituting (4) into (1), (2), (3), we get

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a\gamma}{nb}[e\sin\phi\sin f + (1 + e\cos f)\cos\phi] \tag{8}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{b\gamma}{na^2}[\sin\phi\sin f + (\cos f + \cos E)\cos\phi] \tag{9}$$

$$e\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{b\gamma}{na^2}[-\sin\phi\cos f + (\sin f + \frac{a}{b}\sin E)\cos\phi] \tag{10}$$

The change to the eccentric anomaly E as an independent variable is achieved by substituting (5), (6), (7) into (8), (9), (10) which leads to [3], [4]:

$$\frac{\mathrm{d}a}{\mathrm{d}E} = \frac{4\gamma\sqrt{1 - e^2}}{n^2}\frac{e\sin E}{\sqrt{1 - e^2\cos^2 E}} \tag{11}$$

$$\frac{\mathrm{d}e}{\mathrm{d}E} = \frac{b\gamma}{n^2a^2}\frac{\sin E}{\sqrt{1 - e^2\cos^2 E}}[(1 - e^2\cos^2 E) + 2e\cos E - 2e^2] \tag{12}$$

$$e\frac{\mathrm{d}\omega}{\mathrm{d}E} = \frac{\gamma}{n^2a^2}\frac{1}{\sqrt{1 - e^2\cos^2 E}}[(3e - 2e^3) - \cos E$$
$$-(2e - e^3)\cos^2 E + e^2\cos^3 E] \tag{13}$$

3. Integration (Solution of the Equations of Motion)

The equations of motion (11), (12), can be easily integrated using elementary calculus.

Integration of the equation (11) gives

$$\Delta a = a - a_o = \frac{4\gamma\sqrt{1-e^2}}{n^2}\cos^{-1}(e\cos E) \tag{I}$$

In equation (12) we put the substitution

$$u = e\ cos\ E$$

Thus the equation can be transformed into the simple form

$$\int \mathrm{d}e = \frac{b\gamma}{n^2a^2}\left[-\frac{1}{e}\int\sqrt{1-u^2}\mathrm{d}u - \frac{1}{e}\int\frac{u\mathrm{d}u}{\sqrt{1-u^2}} + 2e\int\frac{\mathrm{d}u}{\sqrt{1-u^2}}\right]$$

which can be integrated to give

$$\Delta e = e - e_o = \frac{\gamma b}{n^2a^2}\left[(\frac{2}{e}-\frac{1}{2})\cos E\sqrt{1-e^2\cos^2 E} - (\frac{1}{2e}-2e)\sin^{-1}(e\cos E)\right] \tag{II}$$

Now we shall consider the third integral on the R.H.S. of (13), which may be written as

$$\begin{aligned}\int \mathrm{d}\omega = \frac{\gamma}{n^2a^2}\Bigg[(4-2e^2-\frac{2}{e^2})\int\frac{\mathrm{d}E}{\sqrt{1-e^2\cos^2 E}} - \frac{1}{e^2}\int\frac{e\cos E}{\sqrt{1-e^2\cos^2 E}}\mathrm{d}E \\ -(1-\frac{2}{e^2})\int\sqrt{1-e^2\cos^2 E}\mathrm{d}E + \frac{1}{e}\int\frac{e^2\cos^3 E}{\sqrt{1-e^2\cos^3 E}}\mathrm{d}E\Bigg]\end{aligned} \tag{14}$$

By the use of the relations in Appendix (B), we can calculate the required integrals in Eq. (13). We obtain

$$\begin{aligned}e\Delta\omega = e(\omega-\omega_0) = \frac{\gamma}{n^2a^2}\Bigg[\frac{4-2e^2-2/e^2}{\sqrt{1-e^2}}EllipticF[E,\frac{-e^2}{1-e^2}] \\ -\frac{1}{e^2}\sinh^{-1}\left(\frac{e\sin E}{\sqrt{1-e^2}}\right) + \frac{2-e^2}{e^2}\sqrt{1-e^2}EllipticE[E,\frac{-e^2}{1-e^2}] \\ -\frac{1}{2e}\sin E\sqrt{1-e^2\cos^2 E} + \frac{1+e^2}{2e^2}\sinh^{-1}\left(\frac{e\sin E}{\sqrt{1-e^2}}\right)\Bigg]\end{aligned} \tag{III}$$

where $EllipticF[\phi, k]$ and $EllipticE[\phi, k]$ are respectively the elliptic integrals of the first and second kind, defined as

$$EllipticF[\phi,\, k] = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2\phi}}\mathrm{d}\phi \tag{15}$$

$$EllipticE[\phi,\, k] = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\mathrm{d}\phi \tag{16}$$

Here k is known as the modulus and ϕ is the amplitude.

Summarizing the results, we can write the variations in the orbital elements due to instantaneous motor thrust engines propulsion as:

$$\Delta a = a - a_o = \frac{4\gamma\sqrt{1-e^2}}{n^2}\cos^{-1}(e\cos E) \tag{17}$$

$$\Delta e = e - e_o = \frac{\gamma b}{n^2 a^2}\left[(\frac{2}{e}-\frac{1}{2})\cos E\sqrt{1-e^2\cos^2 E}\right.$$
$$\left. -(\frac{1}{2e}-2e)\sin^{-1}(e\cos E)\right] \tag{18}$$

$$e\Delta\omega = e(\omega-\omega_0) = \frac{\gamma}{n^2 a^2}\left[-\frac{2(1-e^2)^{3/2}}{e}EllipticF[E,\frac{-e^2}{1-e^2}]\right.$$
$$-\frac{1-e^2}{2e}\sinh^{-1}\left(\frac{e\sin E}{\sqrt{1-e^2}}\right)+\frac{(2-e^2)\sqrt{1-e^2}}{e}EllipticE[E,\frac{-e^2}{1-e^2}] \tag{19}$$
$$\left. -\frac{1}{2}\sin E\sqrt{1-e^2\cos^2 E}\right]$$

4. Conclusion

An instantaneous single motor thrust engine propulsion is induced at a given point on the initial Keplerian elliptic orbit. Consequently the components of propulsion acceleration after the impulse affect the motion of the space vehicle and then the yields an alternation in the orbital elements estimated by the equations for $a - a_o$, $e - e_o$, $e(\omega - \omega_o)$, after the solution of the equations of motion in its Gaussian form. We assume that no change in the plane of the initial orbit occurs. The propulsion is in the positive direction of the tangential velocity vector, which coincides with the proceeding direction. We selected the eccentric anomaly E, as being the independent variable, instead of the physical time t, because it is more convenient to adopt, facilitates the analysis and remove slow convergence difficulties. The solution of the problem is represented by the integration of the three equations for $\mathrm{d}a/\mathrm{d}E$, $\mathrm{d}e/\mathrm{d}E$ and $e\mathrm{d}\omega/\mathrm{d}E$.

Elliptic integrals appear through the implementation of the interaction process. In the Appendix we list the values of the auxiliary quantities involved in the trigonometric calculations and evaluation of the elliptic integrals (of the first and second kinds). The integrations may be carried out numerically. This article is a first time detailed treatment for elliptic – elliptic orbit transfer using the Gaussian form of the equations of motion, when a unique propulsion impulse is operated.

References

[1] **Marec, J.P.**: Transferts orbitaux economiques (orbits elliptiques coplanaires coaxials, dur'ee illimit'ee), *La Recherche A'erospatiale*, no. 105, 11–21, **1965**.

[2] **Marchal, C.**: Transferts optimaux entre orbite elliptiques coplanaires, Dur'ee indiffe'rente, *Astronautica Acta*, Vol 11, No. 6, **1965**).

[3] **Roy, A. E.**: Orbital Motion, *Adam Hilger*, **1982**.

[4] **Levallois, J. J, and Kovalevsky, J.**: Géodésie Générale, Tome IV, *Eyrolles*, **1971**.

[5] **Gradshteyn,I. S. and Ryzhik, I. M.**: Table of integrals, series, and Products, —textitAcademic Press, **1980**.

Auxiliary Formulae and Evaluation of Elliptic Integrals

Appendix (A)

We utilized the formulae for $\cos\phi$, $\sin\phi$, $\cos f$, $\sin f$ Eqs. (5), (6) to derive Eqs. (11), (12), (13). Moreover, we have

$$\begin{aligned} r &= \frac{a(1-e^2)}{1+e\cos f} \\ r &= a(1-e\cos E) \\ \frac{dt}{df} &= \frac{(1-e^2)^{3/2}}{n(1+e\cos f)} \\ \frac{dt}{dE} &= \frac{1-e\cos E}{n} \end{aligned}$$

It is found preferable to choose E as the independent variable, because of reasons mentioned in the text. To change the parameter from $f \to E$ we adopt the two body problem relationships:

$$\begin{aligned} \sin f &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \\ \cos f &= \frac{\cos E - e}{1-e\cos E} \end{aligned}$$

The independent physical time t may be replaced by the eccentric anomaly E, via the relation

$$\mathrm{d}t = \frac{1-e\cos E}{n}\mathrm{d}E$$

We also have the expressions

$$
\begin{aligned}
\sin\varphi\sin f &= \frac{(1-e^2)\sin E}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
\cos\varphi\cos f &= \frac{e\sin E(\cos E-e)}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
\sin\varphi\cos f &= \frac{\sqrt{1-e^2}(\cos E-e)}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
\cos\varphi\sin f &= \frac{e\sqrt{1-e^2}\sin^2 E}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
\cos\varphi\cos E &= \frac{e(1-e\cos E)\sin E\cos E}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
\cos\varphi\sin E &= \frac{e(1-e\cos E)\sin^2 E}{(1-e\cos E)\sqrt{1-e^2\cos^2 E}}\\
1+e^2+2e\cos f &= \frac{(1-e^2)(1+e\cos E)}{(1-e\cos E)}
\end{aligned}
$$

Appendix (B) To derive the equations for $\mathrm{d}a/\mathrm{d}E$, $\mathrm{d}e/\mathrm{d}E$ and $e\mathrm{d}\omega/\mathrm{d}E$ we encounter the following integrals [5]:

$$
\begin{aligned}
&(1)\quad \int\sqrt{1-e^2\cos^2 E}\,\mathrm{d}E = \sqrt{1-e^2}\,EllipticE[E,\frac{-e^2}{1-e^2}]\\
&(2)\quad \int\frac{1}{\sqrt{1-e^2\cos^2 E}}\mathrm{d}E = \frac{1}{\sqrt{1-e^2}}EllipticF[E,\frac{-e^2}{1-e^2}]\\
&(3)\quad \int\frac{\cos E}{\sqrt{1-e^2\cos^2 E}}\mathrm{d}E = \frac{1}{e}\sinh^{-1}\left(\frac{e\sin E}{\sqrt{1-e^2}}\right)\\
&(4)\quad \int\frac{\cos^2 E}{\sqrt{1-e^2\cos^2 E}}\mathrm{d}E = \frac{1}{e^2}\left[\frac{1}{\sqrt{1-e^2}}EllipticF[E,\frac{-e^2}{1-e^2}]\right.\\
&\qquad\left.-\sqrt{1-e^2}\,EllipticE[E,\frac{-e^2}{1-e^2}]\right]\\
&(5)\quad \int\frac{\cos^3 E}{\sqrt{1-e^2\cos^2 E}}\mathrm{d}E = \frac{1}{2e^3}\left[(1+e^2)\sinh^{-1}\left(\frac{e\sin E}{\sqrt{1-e^2}}\right)\right.\\
&\qquad\left.-e\sin E\sqrt{1-e^2\cos^2 E}\right]
\end{aligned}
$$

Nomenclature

$\vec{\gamma}$	acceleration due to the propulsion
$V_c = \int\limits_0^t \lvert\vec{\gamma}\rvert \, \mathrm{d}t$	characteristic velocity = latent velocity
$a = OA = OP = FB$	semi–major axis
e	eccentricity
$b = a\sqrt{1-e^2}$	semi–minor axis
$p = a(1-e^2)$	parameter
$c = ae$	focal distance
ϖ	longitude of peri – apse
f	true anomaly
E	eccentric anomaly
n	mean motion
$\mu = n^2a^3$	gravitational constant
$\vec{r}$	position vector
$\vec{V}$	velocity vector
$\vec{H} = \lvert\, \vec{r} \wedge \vec{V} \Big\rvert$	moment of momentum vector
$H = \left\lvert \vec{H} \right\rvert \quad = nab$	magnitude of the moment of momentum vector
$\xi = -\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{V^2}{2}$	energy
$P = FP = a\ (1 - e)$	peri–apse distance $0 \leq P \leq p \leq$
$A = FA = a\ (1 + e)$	apo–apse distance $\leq b \leq a \leq A$